The lifting method II

Iván Angiono

Workshop: Hopf algebras and Tensor categories. August 2020

CONICET
UNC Universidad Nacional de Córdoba
C I E M

This talk is based on the articles

AAGMV **Lifting via cocycle deformation**. *J. Pure Appl. Alg. (2014)*, with N. Andruskiewitsch, A. García Iglesias, A. Masuoka and C. Vay;

This talk is based on the articles

- AAGMV **Lifting via cocycle deformation**. *J. Pure Appl. Alg. (2014)*, with N. Andruskiewitsch, A. García Iglesias, A. Masuoka and C. Vay;
- AAG **Liftings of Nichols algebras of diagonal type I. Cartan type A**. *Int. Math. Res. Notices (2016)*, with Nicolás Andruskiewitsch and A. García Iglesias;

This talk is based on the articles

- AAGMV **Lifting via cocycle deformation**. *J. Pure Appl. Alg. (2014)*, with N. Andruskiewitsch, A. García Iglesias, A. Masuoka and C. Vay;
- AAG **Liftings of Nichols algebras of diagonal type I. Cartan type A**. *Int. Math. Res. Notices (2016)*, with Nicolás Andruskiewitsch and A. García Iglesias;
- AG **Liftings of Nichols algebras of diagonal type II. All liftings are cocycle deformations**. *Selecta Math. (2019)*, with A. García Iglesias.

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)
- $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, $V = \oplus \Bbbk_{g_i}^{\chi_i} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (we fix the infinitesimal braiding $\rightsquigarrow L_1$)

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)
- $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, $V = \oplus \Bbbk_{g_i}^{\chi_i} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (we fix the infinitesimal braiding $\rightsquigarrow L_1$)
- $\mathfrak{q} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where $\chi_j(i) = q_{ij}$, the diagram is

$$\overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ}.$$

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)
- $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, $V = \oplus \Bbbk_{g_i}^{\chi_i} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (we fix the infinitesimal braiding $\rightsquigarrow L_1$)
- $\mathfrak{q} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where $\chi_j(i) = q_{ij}$, the diagram is
  $$\overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ}.$$
- $\mathcal{B}_{\mathfrak{q}} = T(V)/\langle x_i^2, i = 1, 2, 3;\ x_{13},\ [x_{123}, x_2]_c;\ x_{12}^2, x_{23}^2, x_{123}^2 \rangle$.
  $\rightsquigarrow T(V)\#\Bbbk\Gamma, \mathcal{B}_{\mathfrak{q}}\#\Bbbk\Gamma$

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)
- $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, $V = \oplus \Bbbk_{g_i}^{\chi_i} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (we fix the infinitesimal braiding $\rightsquigarrow L_1$)
- $\mathfrak{q} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where $\chi_j(i) = q_{ij}$, the diagram is

$$\overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ}.$$

- $\mathcal{B}_\mathfrak{q} = T(V)/\langle x_i^2, i = 1, 2, 3;\ x_{13},\ [x_{123}, x_2]_c;\ x_{12}^2, x_{23}^2, x_{123}^2\rangle$.
  $\rightsquigarrow T(V)\#\Bbbk\Gamma, \mathcal{B}_\mathfrak{q}\#\Bbbk\Gamma$

Question

How to compute all liftings L?

Warm up: Cartan A_3, $q = -1$

- Γ finite abelian group (we fix the coradical L_0)
- $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, $i = 1, 2, 3$, $V = \oplus \Bbbk_{g_i}^{\chi_i} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (we fix the infinitesimal braiding $\rightsquigarrow L_1$)
- $\mathfrak{q} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where $\chi_j(i) = q_{ij}$, the diagram is

$$\overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ} \xrightarrow{\quad -1 \quad} \overset{-1}{\circ}.$$

- $\mathcal{B}_{\mathfrak{q}} = T(V)/\langle x_i^2, i = 1, 2, 3;\ x_{13},\ [x_{123}, x_2]_c;\ x_{12}^2, x_{23}^2, x_{123}^2 \rangle$.
$\rightsquigarrow T(V)\#\Bbbk\Gamma, \mathcal{B}_{\mathfrak{q}}\#\Bbbk\Gamma$

Question

How to compute all liftings L?

- Generation in degree one: L lifting, $\operatorname{gr} L \simeq \mathcal{B}_{\mathfrak{q}}\#\Bbbk\Gamma$.
$\exists \pi : T(V)\#\Bbbk\Gamma \twoheadrightarrow L$ a *lifting map* (Andruskiewitsch-Vay):
π identifies $L_0 = \Bbbk\Gamma$, $L_1 = V\#\Bbbk\Gamma \oplus \Bbbk\Gamma$.

Cartan A_3, $q = -1$

- $\mathcal{H}_0 = T(V)\#\Bbbk\Gamma = \mathcal{A}_0 = \mathcal{L}_0$, $\mathcal{B}_0 = T(V) = \mathcal{E}_0$.

Cartan A_3, $q=-1$

- $\mathcal{H}_0 = T(V)\#\Bbbk\Gamma = \mathcal{A}_0 = \mathcal{L}_0$, $\mathcal{B}_0 = T(V) = \mathcal{E}_0$.
- In $\mathcal{H}_0$ we have skew primitives:

$$\Delta(x_i^2) = x_i^2 \otimes 1 + g_i^2 \otimes x_i^2, \quad \Delta(x_{13}) = x_{13} \otimes 1 + g_1 g_3 \otimes x_{13} :$$

$$\Longrightarrow \quad \pi(x_i^2) = \lambda_i(1 - g_i^2), \pi(x_{13}) = \lambda_{13}(1 - g_1 g_3), \lambda_i, \lambda_{13} \in \Bbbk.$$

(need a computation to prove it is a trivial skew-primitive).

Cartan A_3, $q=-1$

- $\mathcal{H}_0 = T(V)\#\Bbbk\Gamma = \mathcal{A}_0 = \mathcal{L}_0$, $\mathcal{B}_0 = T(V) = \mathcal{E}_0$.
- In $\mathcal{H}_0$ we have skew primitives:

$$\Delta(x_i^2) = x_i^2 \otimes 1 + g_i^2 \otimes x_i^2, \quad \Delta(x_{13}) = x_{13} \otimes 1 + g_1 g_3 \otimes x_{13} :$$
$$\implies \pi(x_i^2) = \lambda_i(1 - g_i^2), \pi(x_{13}) = \lambda_{13}(1 - g_1 g_3), \lambda_i, \lambda_{13} \in \Bbbk.$$

(need a computation to prove it is a trivial skew-primitive).

- As $g x_i^2 = \chi_i(g)^2 x_i^2 g$, $g x_{13} = \chi_1\chi_3(g)^2 x_{13} g$, $g(1-h) = (1-h)g$ (Γ abelian):

$$\lambda_i = 0 \text{ if } \chi_i^2 \neq \epsilon, \qquad \lambda_{13} = 0 \text{ if } \chi_1\chi_3 \neq \epsilon.$$

Cartan A_3, $q=-1$

- $\mathcal{H}_0 = T(V)\#\Bbbk\Gamma = \mathcal{A}_0 = \mathcal{L}_0$, $\mathcal{B}_0 = T(V) = \mathcal{E}_0$.
- In $\mathcal{H}_0$ we have skew primitives:

$$\Delta(x_i^2) = x_i^2 \otimes 1 + g_i^2 \otimes x_i^2, \quad \Delta(x_{13}) = x_{13} \otimes 1 + g_1g_3 \otimes x_{13}:$$
$$\implies \pi(x_i^2) = \lambda_i(1-g_i^2), \pi(x_{13}) = \lambda_{13}(1-g_1g_3), \lambda_i, \lambda_{13} \in \Bbbk.$$

(need a computation to prove it is a trivial skew-primitive).

- As $gx_i^2 = \chi_i(g)^2 x_i^2 g$, $gx_{13} = \chi_1\chi_3(g)^2 x_{13} g$, $g(1-h) = (1-h)g$ (Γ abelian):

$$\lambda_i = 0 \text{ if } \chi_i^2 \neq \epsilon, \qquad \lambda_{13} = 0 \text{ if } \chi_1\chi_3 \neq \epsilon.$$

- $\mathcal{L}_1(\lambda_i, \lambda_{13}) = \mathcal{L}_0 / \langle x_i^2 - \lambda_i(1-g_i^2), x_{13} - \lambda_{13}(1-g_1g_3) \rangle$.

$$\implies \exists \quad \pi : \mathcal{L}_1(\lambda_i, \lambda_{13}) \twoheadrightarrow L.$$

- Set $r_{1232} = [x_{123}, x_2]_c$, compute $\Delta\langle[x_{123}, x_2]_c\rangle$ in $\mathcal{H}_0$ and project: r_{1232} not skew primitve, but $\widetilde{r}_{1232} = r_{1232} - 4q_{12}\lambda_{13}\lambda_2(1-g_2^2)g_1g_3$ is so. $\implies \pi(\widetilde{r}_{1232}) = \lambda_{1232}(1 - g_1g_2^2g_3)$, $\lambda_{1232} \in \Bbbk$. $\lambda_{1232} = 0$ if $\chi_1\chi_2^2\chi_3 \neq \epsilon$.

- Set $r_{1232} = [x_{123}, x_2]_c$, compute $\Delta\langle[x_{123}, x_2]_c\rangle$ in $\mathcal{H}_0$ and project: r_{1232} not skew primitve, but $\widetilde{r}_{1232} = r_{1232} - 4q_{12}\lambda_{13}\lambda_2(1 - g_2^2)g_1g_3$ is so.
  $\implies \pi(\widetilde{r}_{1232}) = \lambda_{1232}(1 - g_1g_2^2g_3)$, $\lambda_{1232} \in \mathbb{k}$.
  $\lambda_{1232} = 0$ if $\chi_1\chi_2^2\chi_3 \neq \epsilon$.
- $\mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) = \mathcal{L}_1/\langle\widetilde{r}_{1232} - \lambda_{1232}(1 - g_1g_2^2g_3)\rangle$.

$$\implies \exists \quad \pi : \mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) \twoheadrightarrow L.$$

- Set $r_{1232} = [x_{123}, x_2]_c$, compute $\Delta\langle[x_{123}, x_2]_c\rangle$ in $\mathcal{H}_0$ and project: r_{1232} not skew primitve, but $\widetilde{r}_{1232} = r_{1232} - 4q_{12}\lambda_{13}\lambda_2(1 - g_2^2)g_1g_3$ is so.
  $\implies \pi(\widetilde{r}_{1232}) = \lambda_{1232}(1 - g_1g_2^2g_3)$, $\lambda_{1232} \in \Bbbk$.
  $\lambda_{1232} = 0$ if $\chi_1\chi_2^2\chi_3 \neq \epsilon$.
- $\mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) = \mathcal{L}_1/\langle\widetilde{r}_{1232} - \lambda_{1232}(1 - g_1g_2^2g_3)\rangle$.

$$\implies \exists \quad \pi : \mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) \twoheadrightarrow L.$$

- Similarly, $\exists$ skew primitives $\widetilde{r}_{12}$, $\widetilde{r}_{23}$, $\widetilde{r}_{123}$ (deformations of x_{12}^2, x_{23}^2, x_{123}^2), set scalars λ_{12}, λ_{23}, λ_{123},

- Set $r_{1232} = [x_{123}, x_2]_c$, compute $\Delta\langle [x_{123}, x_2]_c \rangle$ in $\mathcal{H}_0$ and project: r_{1232} not skew primitve, but $\widetilde{r}_{1232} = r_{1232} - 4q_{12}\lambda_{13}\lambda_2(1 - g_2^2)g_1g_3$ is so. $\implies \pi(\widetilde{r}_{1232}) = \lambda_{1232}(1 - g_1g_2^2g_3)$, $\lambda_{1232} \in \Bbbk$. $\lambda_{1232} = 0$ if $\chi_1\chi_2^2\chi_3 \neq \epsilon$.
- $\mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) = \mathcal{L}_1 / \langle \widetilde{r}_{1232} - \lambda_{1232}(1 - g_1g_2^2g_3) \rangle$.

$$\implies \exists \quad \pi : \mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) \twoheadrightarrow L.$$

- Similarly, $\exists$ skew primitives $\widetilde{r}_{12}$, $\widetilde{r}_{23}$, $\widetilde{r}_{123}$ (deformations of x_{12}^2, x_{23}^2, x_{123}^2), set scalars λ_{12}, λ_{23}, λ_{123},
- $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}_2 / \langle \widetilde{r}_\beta - \lambda_\beta(1 - g_\beta) \rangle \implies \exists \quad \pi : \mathcal{L}_3(\boldsymbol{\lambda}) \twoheadrightarrow L.$

- Set $r_{1232} = [x_{123}, x_2]_c$, compute $\Delta\langle [x_{123}, x_2]_c\rangle$ in $\mathcal{H}_0$ and project: r_{1232} not skew primitve, but $\widetilde{r}_{1232} = r_{1232} - 4q_{12}\lambda_{13}\lambda_2(1-g_2^2)g_1g_3$ is so.
  $\implies \pi(\widetilde{r}_{1232}) = \lambda_{1232}(1-g_1g_2^2g_3)$, $\lambda_{1232} \in \Bbbk$.
  $\lambda_{1232} = 0$ if $\chi_1\chi_2^2\chi_3 \neq \epsilon$.
- $\mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) = \mathcal{L}_1/\langle \widetilde{r}_{1232} - \lambda_{1232}(1-g_1g_2^2g_3)\rangle$.

$$\implies \exists \quad \pi : \mathcal{L}_2(\lambda_i, \lambda_{13}, \lambda_{1232}) \twoheadrightarrow L.$$

- Similarly, $\exists$ skew primitives $\widetilde{r}_{12}$, $\widetilde{r}_{23}$, $\widetilde{r}_{123}$ (deformations of x_{12}^2, x_{23}^2, x_{123}^2), set scalars λ_{12}, λ_{23}, λ_{123},
- $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}_2/\langle \widetilde{r}_\beta - \lambda_\beta(1-g_\beta)\rangle \implies \exists \quad \pi : \mathcal{L}_3(\boldsymbol{\lambda}) \twoheadrightarrow L$.

Question

dim $\mathcal{L}_3(\boldsymbol{\lambda})$? or first... $\mathcal{L}_3(\boldsymbol{\lambda}) \neq 0$?

Solution: $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}_3, \mathcal{H}_3)$ for $\mathcal{H}_3 = \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, $\mathcal{A}_3(\boldsymbol{\lambda}) = \mathcal{E}_3(\boldsymbol{\lambda}) \# \Bbbk\Gamma$ a Galois object.

- Same for other liftings, from r skew primitive relation in an *intermediate quotient* of $\mathcal{B}_{\mathfrak{q}}$, we get modified relation $\widetilde{r}$, skew primitive in the *intermediate deformation*.

Solution: $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}_3, \mathcal{H}_3)$ for $\mathcal{H}_3 = \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, $\mathcal{A}_3(\boldsymbol{\lambda}) = \mathcal{E}_3(\boldsymbol{\lambda}) \# \Bbbk\Gamma$ a Galois object.

- Same for other liftings, from r skew primitive relation in an *intermediate quotient* of $\mathcal{B}_{\mathfrak{q}}$, we get modified relation $\widetilde{r}$, skew primitive in the *intermediate deformation*.
- Andruskiewitsch-Schneider classification (2010): $|\Gamma|$ coprime with 210, $\operatorname{gr} H = \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, $\mathcal{B}_{\mathfrak{q}} \simeq \mathfrak{u}_q^+(\mathfrak{g})$; quantum Serre relations not deformed *inside* each connected component.

Solution: $\mathcal{L}_3(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}_3, \mathcal{H}_3)$ for $\mathcal{H}_3 = \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, $\mathcal{A}_3(\boldsymbol{\lambda}) = \mathcal{E}_3(\boldsymbol{\lambda}) \# \Bbbk\Gamma$ a Galois object.

- Same for other liftings, from r skew primitive relation in an *intermediate quotient* of $\mathcal{B}_{\mathfrak{q}}$, we get modified relation $\widetilde{r}$, skew primitive in the *intermediate deformation*.
- Andruskiewitsch-Schneider classification (2010): $|\Gamma|$ coprime with 210, $\operatorname{gr} H = \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$, $\mathcal{B}_{\mathfrak{q}} \simeq \mathfrak{u}_q^+(\mathfrak{g})$; quantum Serre relations not deformed *inside* each connected component.
- Masuoka (2011): H is a cocycle deformation of $\operatorname{gr} H$.

The strategy

1. Stratify a minimal set $\mathcal{G}$ of defining relations of $\mathcal{B}_{\mathfrak{q}}$ as follows: $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_\ell$ so that for each $r \in \mathcal{G}_t$,

$$\Delta(\bar{r}) = \bar{r} \otimes 1 + 1 \otimes \bar{r} \quad \text{in} \quad \mathfrak{B}_t := T(V)/\langle \cup_{i=0}^{t-1} \mathcal{G}_i \rangle.$$

The strategy

1. Stratify a minimal set $\mathcal{G}$ of defining relations of $\mathcal{B}_{\mathfrak{q}}$ as follows: $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_\ell$ so that for each $r \in \mathcal{G}_t$,

$$\Delta(\bar{r}) = \bar{r} \otimes 1 + 1 \otimes \bar{r} \quad \text{in} \quad \mathfrak{B}_t := T(V)/\langle \cup_{i=0}^{t-1} \mathcal{G}_i \rangle.$$

2. Fix $\Lambda = \{\boldsymbol{\lambda} = (\lambda_r) | \lambda_r = 0 \text{ if } \chi_r \neq \epsilon \text{ or } g_r = 1\} \subseteq \Bbbk^{\mathcal{G}}$.

The strategy

1. Stratify a minimal set $\mathcal{G}$ of defining relations of $\mathcal{B}_{\mathfrak{q}}$ as follows: $\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_\ell$ so that for each $r \in \mathcal{G}_t$,

$$\Delta(\bar{r}) = \bar{r} \otimes 1 + 1 \otimes \bar{r} \quad \text{in} \quad \mathfrak{B}_t := T(V)/\langle \cup_{i=0}^{t-1} \mathcal{G}_i \rangle.$$

2. Fix $\Lambda = \{\boldsymbol{\lambda} = (\lambda_r) | \lambda_r = 0 \text{ if } \chi_r \neq \epsilon \text{ or } g_r = 1\} \subseteq \Bbbk^{\mathcal{G}}$.
3. $\mathcal{E}_0(\boldsymbol{\lambda}) = T(V)$, $\mathcal{A}_0 = \mathcal{E}_0 \# \Bbbk\Gamma$. Recursively, if $\mathcal{E}_{t-1} \neq 0$, then $\mathcal{A}_{t-1}$ is a cleft object. Fix a section $\gamma_{t-1} : \mathcal{H}_{t-1} \to \mathcal{A}_{t-1}$ with *nice properties*: restricts to $\gamma_{t-1} : \mathcal{B}_{t-1} \to \mathcal{E}_{t-1}$. The coaction gives also an algebra map $\rho : \mathcal{E}_{t-1} \to \mathcal{E}_{t-1} \otimes \mathcal{B}_{t-1}$. For $r \in \mathcal{G}_{t-1}$ let $\widehat{r} = \gamma_{t-1}(r)$: $\rho(\widehat{r}) = \widehat{r} \otimes 1 + 1 \otimes r$.

$$\mathcal{E}_t(\boldsymbol{\lambda}) = \mathcal{E}_{t-1}(\boldsymbol{\lambda})/\langle \widehat{r} - \lambda_r, r \in \mathcal{G}_{t-1} \rangle \neq 0, \quad \mathcal{A}_t(\boldsymbol{\lambda}) = \mathcal{E}_t(\boldsymbol{\lambda}) \# \Bbbk\Gamma.$$

The family of liftings

- By a result of Schauenburg, for each $\mathcal{A}(\boldsymbol{\lambda})$ there exists a Hopf algebra $\mathcal{L}(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$ such that $\mathcal{A}(\boldsymbol{\lambda})$ is a $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{H})$-Galois object.

The family of liftings

- By a result of Schauenburg, for each $\mathcal{A}(\boldsymbol{\lambda})$ there exists a Hopf algebra $\mathcal{L}(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$ such that $\mathcal{A}(\boldsymbol{\lambda})$ is a $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{H})$-Galois object.
- By [AAGMV], $\operatorname{gr} \mathcal{L}(\boldsymbol{\lambda}) = \mathcal{H}$ (that is, we construct a family of liftings of $\mathcal{H}$).

The family of liftings

- By a result of Schauenburg, for each $\mathcal{A}(\boldsymbol{\lambda})$ there exists a Hopf algebra $\mathcal{L}(\boldsymbol{\lambda}) = \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$ such that $\mathcal{A}(\boldsymbol{\lambda})$ is a $(\mathcal{L}(\boldsymbol{\lambda}), \mathcal{H})$-Galois object.
- By [AAGMV], $\operatorname{gr} \mathcal{L}(\boldsymbol{\lambda}) = \mathcal{H}$ (that is, we construct a family of liftings of $\mathcal{H}$).
- We may also construct $\mathcal{L}(\boldsymbol{\lambda})$ recursively as quotients: $\mathcal{L}_t = \mathcal{L}(\mathcal{A}_t, \mathcal{H})$ is a quotient of $\mathcal{L}_{t-1}$, and $\mathcal{L}(\boldsymbol{\lambda})$ is the last step of this descending chain of Hopf algebras.

This setup is depicted in the following *snapshot* from [AAGMV,p.696]:

$$
\begin{tikzcd}
\mathcal{H}_0 \arrow[r, "\gamma_0 = \mathrm{id}"] \arrow[d, two heads, "\pi_{k-1}"] \arrow[dd, two heads, bend right, "\pi_k"'] \arrow[d, phantom, shift right=4ex, "\equiv"] & \mathcal{A}_0 \arrow[r, squiggly] \arrow[d, two heads, "\tau_{k-1}"'] \arrow[dd, two heads, bend left, "\tau_k"] \arrow[d, phantom, shift left=4ex, "\equiv"] & L(\mathcal{A}_0, \mathcal{H}_0) \simeq \mathcal{H}_0 \arrow[d] \arrow[dd, bend left] \arrow[d, phantom, shift left=8ex, "\equiv"] \\
\mathcal{H}_{k-1} \arrow[r, "\gamma_{k-1}"] \arrow[d, two heads, "\pi'_k"] & \mathcal{A}_{k-1} \arrow[r, squiggly] \arrow[d, two heads, "\tau'_k"'] & L(\mathcal{A}_{k-1}, \mathcal{H}_{k-1}) \arrow[d] \\
\mathcal{H}_k \arrow[r, "\gamma_k"] & \mathcal{A}_k \arrow[r, squiggly] & L(\mathcal{A}_k, \mathcal{H}_k)
\end{tikzcd}
$$

Cleft objects for quotients

Gunther: description of cleft objects of $\mathcal{H}_t = \mathcal{H}_{t-1}/\langle \mathcal{G}_{t-1} \rangle$, using cleft objects $\mathcal{A}_{t-1}$ of $\mathcal{H}_{t-1}$. Let $\pi_t : \mathcal{H}_{t-1} \twoheadrightarrow \mathcal{H}_t$.

1. If you are able to compute $X_t := {}^{\mathrm{co}\,\pi_t}\,\mathcal{H}_{t-1}$ and the set $\mathrm{Alg}_{\mathcal{H}_{t-1}}^{\mathcal{H}_{t-1}}(X_t, \mathcal{A}_{t-1})$, then define for each f

$$\mathcal{A}_t(f) := \mathcal{A}_{t-1}/\langle f(X_t^+) \rangle.$$

Gunther's results have technical assumptions solved in [AAGMV].

Cleft objects for quotients

Gunther: description of cleft objects of $\mathcal{H}_t = \mathcal{H}_{t-1}/\langle \mathcal{G}_{t-1} \rangle$, using cleft objects $\mathcal{A}_{t-1}$ of $\mathcal{H}_{t-1}$. Let $\pi_t : \mathcal{H}_{t-1} \twoheadrightarrow \mathcal{H}_t$.

1. If you are able to compute $X_t :=^{\mathrm{co}\,\pi_t} \mathcal{H}_{t-1}$ and the set $\mathrm{Alg}_{\mathcal{H}_{t-1}}^{\mathcal{H}_{t-1}}(X_t, \mathcal{A}_{t-1})$, then define for each f

$$\mathcal{A}_t(f) := \mathcal{A}_{t-1}/\langle f(X_t^+) \rangle.$$

2. No? Take $Y_t \subseteq^{\mathrm{co}\,\pi_t} \mathcal{H}_{t-1}$ such that $\mathcal{H}_t = \mathcal{H}_{t-1}/\langle Y_t^+ \rangle$, $\mathrm{Alg}^{\mathcal{H}_{t-1}}(Y_t, \mathcal{A}_{t-1})$, define for each f

$$\mathcal{A}_t(f) := \mathcal{A}_{t-1}/\langle f(Y_t^+) \rangle \qquad \text{if the quotient is } \neq 0.$$

Gunther's results have technical assumptions solved in [AAGMV].

Example ($t = \ell + 1$)

Hence we apply (1).

Example ($t = \ell + 1$)

- $\mathcal{B}_{\ell+1} = \mathcal{B}_{\mathfrak{q}}$, $\mathcal{B}_{\ell} = \widetilde{\mathcal{B}}_{\mathfrak{q}}$ the *distinguished pre-Nichols algebra*,

Hence we apply (1).

Example ($t = \ell + 1$)

- $\mathcal{B}_{\ell+1} = \mathcal{B}_{\mathfrak{q}}$, $\mathcal{B}_{\ell} = \widetilde{\mathcal{B}}_{\mathfrak{q}}$ the *distinguished pre-Nichols algebra*,
- $X_{\ell+1} = Z_{\mathfrak{q}}$ a q-polynomial algebra, generated by (some) $x_{\beta}^{N_{\beta}}$, and is a Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$,

Hence we apply (1).

Example ($t = \ell + 1$)

- $\mathcal{B}_{\ell+1} = \mathcal{B}_{\mathfrak{q}}$, $\mathcal{B}_{\ell} = \widetilde{\mathcal{B}}_{\mathfrak{q}}$ the *distinguished pre-Nichols algebra*,
- $X_{\ell+1} = Z_{\mathfrak{q}}$ a q-polynomial algebra, generated by (some) $x_{\beta}^{N_{\beta}}$, and is a Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$,
- $f \in \mathrm{Alg}_{\mathcal{H}_{\ell}}^{\mathcal{H}_{\ell}}(X_{\ell+1}, \mathcal{A}_{\ell})$ is given by $f(x_{\beta}^{N_{\beta}}) = \lambda_{\beta} \in \Bbbk$ (with the desired condition on these scalars).

Hence we apply (1).

Example ($t = \ell + 1$)

- $\mathcal{B}_{\ell+1} = \mathcal{B}_{\mathfrak{q}}$, $\mathcal{B}_{\ell} = \widetilde{\mathcal{B}}_{\mathfrak{q}}$ the *distinguished pre-Nichols algebra*,
- $X_{\ell+1} = Z_{\mathfrak{q}}$ a q-polynomial algebra, generated by (some) $x_{\beta}^{N_{\beta}}$, and is a Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$,
- $f \in \mathrm{Alg}_{\mathcal{H}_\ell}^{\mathcal{H}_\ell}(X_{\ell+1}, \mathcal{A}_\ell)$ is given by $f(x_{\beta}^{N_{\beta}}) = \lambda_{\beta} \in \Bbbk$ (with the desired condition on these scalars).

Hence we apply (1).

Remark

Otherwise $X_t =^{\mathrm{co}\,\pi_t} \mathcal{H}_{t-1}$ even more difficult to compute (and also the algebra maps), we deal with the *non-zero condition* for the quotient in (2).

We may assume for first steps of the stratification that $|\mathcal{G}_t| = 1$ (up tp refine the stratification).

Lemma (AG)

If $\mathcal{E}_t(\lambda_0, \cdots, \lambda_{t-2}, 0) \neq 0$ *and* $\mathcal{E}_t(0, \cdots, 0, \lambda_{t-1}) \neq 0$*, then* $\mathcal{E}_t(\lambda_0, \cdots, \lambda_{t-2}, \lambda_{t-1}) \neq 0$*.*

Sketch of proof: Use that $\mathcal{E}_{t-1}(0, \cdots, 0) = \mathcal{B}_{t-1}$ and the factorization:

$$\begin{tikzcd}
\mathcal{E}_{t-1}(\lambda_0, \cdots, \lambda_{t-2}) \arrow[r, "\rho"] \arrow[d] & \mathcal{E}_{t-1}(\lambda_0, \cdots, \lambda_{t-2}) \otimes \mathcal{B}_{t-1} \arrow[d] \\
\mathcal{E}_t(\lambda_0, \cdots, \lambda_{t-2}, \lambda_{t-1}) \arrow[r, dotted] & \mathcal{E}_t(\lambda_0, \cdots, \lambda_{t-2}, 0) \otimes \mathcal{E}_t(0, \cdots, 0, \lambda_{t-1}).
\end{tikzcd}$$

Remark

Remark

- Last result may be used to reduce the non-zero condition for each connected component of the Dynkin diagram.

Remark

- Last result may be used to reduce the non-zero condition for each connected component of the Dynkin diagram.
- Also, it essentially reduces to check non-zero condition for each quotient with only one deformed relation (moreover reduction to support of the relation).

Remark

- Last result may be used to reduce the non-zero condition for each connected component of the Dynkin diagram.
- Also, it essentially reduces to check non-zero condition for each quotient with only one deformed relation (moreover reduction to support of the relation).
- First steps we check $\mathcal{E}_t \neq 0$ by hand, other steps using GAP.

Remark

- Last result may be used to reduce the non-zero condition for each connected component of the Dynkin diagram.
- Also, it essentially reduces to check non-zero condition for each quotient with only one deformed relation (moreover reduction to support of the relation).
- First steps we check $\mathcal{E}_t \neq 0$ by hand, other steps using GAP.

Theorem (AG)

$\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ *for all* $\boldsymbol{\lambda} \in \Lambda$.

Theorem (AAG)

If $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ for all $\boldsymbol{\lambda} \in \Lambda$, then every lifting is $L \simeq \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$.

Sketch of proof:

Theorem (AAG)

If $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ for all $\boldsymbol{\lambda} \in \Lambda$, then every lifting is $L \simeq \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$.

Sketch of proof:

- Generation in degree one says that $\operatorname{gr} L \simeq \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$.
  $\exists \pi : T(V)\#\Bbbk\Gamma \twoheadrightarrow L$ a *lifting map* (Andruskiewitsch-Vay):
  π identifies $L_0 = \Bbbk\Gamma$, $L_1 = V\#\Bbbk\Gamma \oplus \Bbbk\Gamma$.

Theorem (AAG)

If $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ for all $\boldsymbol{\lambda} \in \Lambda$, then every lifting is $L \simeq \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$.

Sketch of proof:

- Generation in degree one says that $\operatorname{gr} L \simeq \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$.
  $\exists \pi : T(V) \# \Bbbk\Gamma \twoheadrightarrow L$ a *lifting map* (Andruskiewitsch-Vay):
  π identifies $L_0 = \Bbbk\Gamma$, $L_1 = V \# \Bbbk\Gamma \oplus \Bbbk\Gamma$.
- By induction we prove $\exists \pi : \mathcal{L}_t(\boldsymbol{\lambda}) \twoheadrightarrow L$. If so, for the last step we have an isomorphism since $\dim \mathcal{L}_t = \dim \mathcal{H}_t = \dim L$.

Theorem (AAG)

If $\mathcal{E}(\boldsymbol{\lambda}) \neq 0$ for all $\boldsymbol{\lambda} \in \Lambda$, then every lifting is $L \simeq \mathcal{L}(\mathcal{A}(\boldsymbol{\lambda}), \mathcal{H})$.

Sketch of proof:

- Generation in degree one says that $\operatorname{gr} L \simeq \mathcal{B}_{\mathfrak{q}} \# \Bbbk\Gamma$.
  $\exists \pi : T(V) \# \Bbbk\Gamma \twoheadrightarrow L$ a *lifting map* (Andruskiewitsch-Vay):
  π identifies $L_0 = \Bbbk\Gamma$, $L_1 = V \# \Bbbk\Gamma \oplus \Bbbk\Gamma$.
- By induction we prove $\exists \pi : \mathcal{L}_t(\boldsymbol{\lambda}) \twoheadrightarrow L$. If so, for the last step we have an isomorphism since $\dim \mathcal{L}_t = \dim \mathcal{H}_t = \dim L$.
- For the inductive step fix $r \in \mathcal{G}_t \rightsquigarrow \widetilde{r} \in \mathcal{L}_{t-1}$ skew-primitive ($\mathcal{H}_{t-1} \simeq \mathcal{L}_{t-1}$ as coalgebras) $\implies \pi(\widetilde{r}) \in L_1$.
  By A.-Kochetov-Mastnak, $\operatorname{Hom}_\Gamma^\Gamma(\Bbbk r, V) = 0$, and this implies $\pi(\widetilde{r}) \in L_0$. Hence $\pi(\widetilde{r}) = \lambda_r(1 - g_r)$.

Classification

Theorem (AAG,AG)

Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathbf{q}}(\boldsymbol{\lambda})$.

Classification

Theorem (AAG,AG)

Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathbf{q}}(\boldsymbol{\lambda})$.

- *isoclasses are parametrized by equivalence classes in $\boldsymbol{\Lambda}/\sim$.*

Classification

Theorem (AAG,AG)

Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathbf{q}}(\boldsymbol{\lambda})$.

- *isoclasses are parametrized by equivalence classes in $\boldsymbol{\Lambda}/\sim$.*
- *A is a cocycle deformation of $\operatorname{gr} A$.*

Classification

Theorem (AAG,AG)

Let A be a finite-dimensional pointed Hopf algebra with abelian group of group-likes.
Then there is $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $A \simeq \mathfrak{u}_{\mathbf{q}}(\boldsymbol{\lambda})$.

- *isoclasses are parametrized by equivalence classes in $\boldsymbol{\Lambda}/\sim$.*
- *A is a cocycle deformation of $\operatorname{gr} A$.*

Remark

$\exists$ algorithm to compute $\mathfrak{u}_{\mathbf{q}}(\boldsymbol{\lambda})$ explicitly, based on results of Schauemburg and AAGMV. That is, an algorithm to compute $\widetilde{r}$ recursively.

Consequences

Cleft objects are not only a *tool* for the computation of liftings, we also obtain that the categories of comodules of H and $\operatorname{gr} H$ are tensor equivalent. Applications?

- Reduces properties of cohomology rings to the graded Hopf algebras (better for some needed computations).

Consequences

Cleft objects are not only a *tool* for the computation of liftings, we also obtain that the categories of comodules of H and $\operatorname{gr} H$ are tensor equivalent. Applications?

- Reduces properties of cohomology rings to the graded Hopf algebras (better for some needed computations).
- For Generalized Lifting Method (Andruskiewitsch-Cuadra), when the coradical is not a subalgebra we take the Hopf coradical (subalgebra generated by the coradical). If it is basic, then (Andruskiewitsch-A.) we describe finite-dimensional Nichols algebras.

Muchas gracias

Danke schön

Thanks!