Lecture 3

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Goals

- Fermionic topological phases in 1+1d and spin-TQFT
- Classification of fermionic SRE phases in 1+1d
- Topological phases in 1d and bosonization
- SRE phases in higher dimensions

Fermions and superalgebras

A superalgebra is the same as a $\mathbb{Z}_2$-graded algebra, or the same as a $\mathbb{Z}_2$-equivariant algebra.

What is different is the rule for forming the product of two superalgebras $A\widehat{\otimes}A'$:

$$(a\widehat{\otimes}a')\cdot(b\widehat{\otimes}b') = (-1)^{|a'||b|}(a\cdot b)\widehat{\otimes}(a'\cdot b'), \quad a,b\in A, \quad a',b'\in A'.$$

Here $|a|\in\{0,1\}$ is the $\mathbb{Z}_2$-parity of the element a.

The ordinary product $\otimes$ of $\mathbb{Z}_2$-graded algebras has no such minus sign.

If A and A' describe observables in two fermionic systems, stacking them gives a fermionic system with the algebra $A\widehat{\otimes}A'$.

We should devise the rules for TQFTs describing fermionic phases accordingly.

Stacking bosonic TQFTs

Let us examine bosonic stacking first. Let A and A' be semi-simple algebras to be used in the state-sum construction. Let e_i and f_α be basis elements of A and A', respectively. Then:

$$C_{i\alpha,i'\alpha',i''\alpha''} = C_{ii'i''}C_{\alpha\alpha'\alpha''}, \quad \eta_{i\alpha,i'\alpha'} = \eta_{ii'}\eta_{\alpha\alpha'}.$$

Let L (resp. L') be a labeling of edges for every vertex by elements of A

(resp. A').
Let $Z(A, L)$ (resp. $Z(A', L')$) be the contribution of a particular labeling to the state-sum for A (resp. A'). The partition function for $A \otimes A'$ is

$$Z_{A\otimes A'} = \sum_{L,L'} Z(A, L)Z(A', L') = Z_A Z_{A'}.$$

This is what we expected.

Stacking fermionic TQFTs

On the other hand, $Z_{A \widehat{\otimes} A'} = \pm Z(A, L) Z(A', L')$, so to get $Z_{A \widehat{\otimes} A'} = Z_A Z_{A'}$ we will need to modify our "Feynman rules" with signs:

$$Z^f(A) = \sum_L (-1)^{q(L)} Z(A, L)$$

for some $\mathbb{Z}_2$-valued function $q(L)$.
Let $\epsilon(L)$ be the "projection" of L to a $\mathbb{Z}_2$-labeling which remembers only the fermionic parity of each basis vector. The signs depend only on $\epsilon(L)$, not L itself.

It is easy to see that $\epsilon(L)$ is a 1-cycle, and moreover

$$Z_{A \widehat{\otimes} A'} = (-1)^{\langle \epsilon(L), \epsilon(L') \rangle} Z(A, L) Z(A', L'),$$

where $\langle , \rangle$ denotes the intersection pairing of two elements of $H_1(\Sigma, \mathbb{Z}_2)$.

What condition on $q(L)$ gives the correct stacking?

Quadratic refinements

Let K be an abelian group, and $b : K \times K \to \mathbb{Z}_2$ be a bilinear function which is skew, $b(x,x) = 0$. A function $q : K \to \mathbb{Z}_2$ is called a quadratic refinement of b if $q(0) = 0$ and

$$q(x+y) - q(x) - q(y) = b(x,y), \quad \forall x, y \in K.$$

Now let $K = H_1(\Sigma, \mathbb{Z}_2)$, b be the intersection paring, and $q(L) = q(\epsilon(L))$ be a quadratic refinement of b. Then:

$$\begin{aligned} Z^f(A \widehat{\otimes} A') &= \sum_{L,L'} (-1)^{q(\epsilon(L)+\epsilon(L'))+b(\epsilon(L),\epsilon(L'))} Z(A,L) Z(A',L') \\ &= \sum_{L,L'} (-1)^{q(\epsilon(L))+q(\epsilon(L'))} Z(A,L) Z(A',L') = Z^f(A) Z^f(A'). \end{aligned}$$

Atiyah-Johnson theorem

- Quadratic refinements always exist
- The difference of any two is a linear function on $H_1(\Sigma, \mathbb{Z}_2)$, i.e. an element of $H^1(\Sigma, \mathbb{Z}_2)$.

Thus there are as many quadratic refinements as there are elements in $H^1(\Sigma, \mathbb{Z}_2)$, i.e. $2^{b_1(\Sigma)}$.

So what geometric data do we need to specify to get such a q?

Aityah-Johnson: quadratic refinements of the intersection form on $H_1(\Sigma, \mathbb{Z}_2)$ naturally correspond to spin structures on Σ.

Topological spin-statistics relation: to get the fermionic stacking rule, need a spin structure on space-time!

Spin structures

A spin structure on an orientable n-manifold is a lift of a certain principal $SO(n)$ bundle (the orthonormal frame bundle) to a principal $Spin(n)$ bundle.

Any two spin structures differ by a ± 1 on double overlaps, and the cocycle rule is obeyed on triple overlaps.

To get a quadratic function on $S(\Sigma)$, consider $\dim \ker D$ modulo 2. This is $Arf(\eta)$, the Arf invariant of the spin structure $\eta \in S(\Sigma)$.

Let

$$q_\eta(\alpha) = Arf(\eta + \alpha) - Arf(\eta), \quad \alpha \in H^1(\Sigma, \mathbb{Z}_2).$$

Atiyah showed that this is a quadratic refinement of the intersection form. (Johnson used a different, more topological definition of a spin structure rather than the Dirac operator).

Spin-TQFTs

We are not done yet, because the factor $(-1)^{q_\eta(\epsilon(L))}$ is not local in any obvious sense. Can we extend this recipe to a fully-fledged spin-TQFT?

The axioms must be modified. Now M and M' are closed 1d spin manifolds, and Σ is a spin bordism from M to M'.

A spin structure on a circle is the same as an element s of $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$. The non-bounding (Ramond) spin structure is 0, the bounding (Neveu-Schwarz) spin stucture is 1. Restriction of spin structure from Σ to S^1 is $s = q(S^1) + 1$.

The space of states of a spin TQFT is $\mathbb{Z}_2$-graded by $1 - s$: $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$. There is another $\mathbb{Z}_2$-grading given by two cylinder bordisms (with different spin structures). Pair of pants bordism gives $\mathcal{A}$ the structure of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra. In a unitary spin-TQFT $\mathcal{A}$ is semi-simple.

State-sum construction of 1+1d spin-TQFTs

One starts with a semi-simple $\mathbb{Z}_2$-graded algebra A. This $\mathbb{Z}_2$-grading represents fermion number. A is the space of states of the spin-TQFT on an interval (with some boundary conditions). The "Feynman rules" are the same, but along with a sum over L there is a Grassmann integral over pairs of fermionic variables $\theta, \bar{\theta}$.

Such a pair lives on every edge e with $\epsilon(e) = 1$. One needs to attach one variable to each end of such an edge.

The Grassmann integrand is a product over all vertices. The contribution of every vertex is a product of all Grassmann variables living on edges issuing from the vertex.

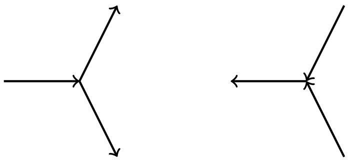
Since each Grassmann variable enters the integrand exactly once, the integral is ± 1.

Branching structure

This construction was proposed by Gu and Wen. Is it well-defined?

- Need to choose orientation on each edge
- Need to choose an order on edges issuing from every vertex (not just cyclic order)

Both can be accomplished by picking a branching structure: an orientation of edges such that no vertex is a sink or a source.



The Gu-Wen integral as a quadratic refinement

The connection of the Gu-Wen integral with spin structures was explained by D. Gaiotto and AK (2015).

The Gu-Wen integral is a quadratic function of the 1-cycle ϵ, and the corresponding bilinear form is the intersection form. But differs from $q_\eta(\epsilon)$ in three respects:

- Depends on the branching structure
- Does not depend on spin structure
- Depends on ϵ itself, not just its homology class in $H_1(\Sigma, \mathbb{Z}_2)$.

This can be fixed by multiplying the Gu-Wen integral by an additional sign which is a linear function of ϵ, and depends both on spin structure and the branching structure.

Discrete spin structure

This extra factor is

$$(-1)^{\eta(\epsilon)},$$

where η is any 1-cochain satisfying $\delta\eta = w_2$, where w_2 is some special 2-cocycle constructed using orientation and branching structure.

w_2 is a particular representative of the 2nd Stiefel-Whitney class of Σ (the obstruction to having spin structure). It is natural to expect that any trivialization of this obstruction can be interpreted as a spin structure. The above construction makes this concrete.

Other approaches to defining spin structures combinatorially were given by Reshetikhin-Cimasoni and Novak-Runkel.

Simple superalgebras and invertible spin-TQFTs

A simple superalgebra has the form $End(U)$ (where U is a $\mathbb{Z}_2$-graded vector space) or $\mathrm{Cl}(1) \otimes End(U)$ (where U is a purely even vector space).

One can show that tensoring with $End(U)$ does not affect the deformation class of the TQFT. Hence any indecomposable TQFT is either equivalent to the trivial one ($A = \mathbb{C}$) or to $A = \mathrm{Cl}(1)$.

Note that $\mathrm{Cl}(1) \otimes \mathrm{Cl}(1) = \mathrm{Cl}(2) = End(\mathbb{C}^{1|1})$. Hence the $A = \mathrm{Cl}(1)$ is an invertible spin-TQFT and is its own inverse.

We conclude that $Inv_1^f = \mathbb{Z}_2$.

The nontrivial fermionic SRE phase

The TQFT for $A = \mathrm{Cl}(1)$ has a one-dimensional space of states both for $s = 0$ (RR) and $s = 1$ (NS). $\mathcal{A}_0$ is odd, while $\mathcal{A}_1$ is even.

In contrast, for $A = \mathbb{C}$ both $\mathcal{A}_1$ and $\mathcal{A}_0$ are even.

The partition function for $A = \mathbb{C}$ is

$$Z^f(\Sigma, \eta) = 2^{-b_1(\Sigma)/2} \sum_{[\epsilon] \in H_1(\Sigma, \mathbb{Z}_2)} (-1)^{q(\epsilon)} = (-1)^{Arf(\eta)}.$$

Relation with the Majorana chain

The $A = \mathrm{Cl}(1)$ TQFT corresponds to the Majorana spin chain. Indeed, A is the space of states an interval, and it is doubly degenerate, with one bosonic and one fermionic state.

Equivalently, Majorana chain in the continuum limit becomes a massive Majorana fermion.

A massless Majorana fermion on a circle has no zero modes for NS (anti-periodic) spin structure, and one zero mode for Ramond (periodic) spin structure. Thus the R ground state is doubly degenerate.

Adding a mass term splits the two ground states in the R sector.

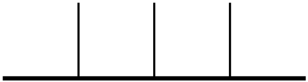
For $m < 0$ the R sector ground state is fermionic, for $m > 0$ it is bosonic. The NS ground state is always bosonic.

The Majorana chain corresponds to $m < 0$, while the trivial fermionic SRE phase corresponds to $m > 0$.

Branes for spin-TQFTs

To construct branes (boundary conditions) for a spin-TQFT based on an algebra A, we need a $\mathbb{Z}_2$-graded module M over A.

The required modification of the Feynman rule is essentially the same as in the bosonic case.



Each brane/module M gives rise to a particular TQFT state $\phi_M \in \mathcal{A}$. One just considers an annulus on whose interior boundary the boundary condition M is inserted.

In fact, we get two such states: one for NS spin structure, and one for Ramond spin structure.

Fermionic Matrix Product States

Turzillo, You and AK, 2016

The state-sum construction automatically produces a generalized MPS form for both sectors. In the NS sector:

$$\sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[T(e_{i_1})T(e_{i_2})\ldots T(e_{i_N})\right]|i_1 i_2 \ldots i_N\rangle.$$

In the R sector:

$$\sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[(-1)^F T(e_{i_1})T(e_{i_2})\ldots T(e_{i_N})\right]|i_1 i_2 \ldots i_N\rangle.$$

But this gives only bosonic states.

Generalized Fermionic MPS

More generally, by inserting a local operator $X \in End(M)$ on the inner circle one gets:

$$\sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[PXT(e_{i_1})T(e_{i_2})\ldots T(e_{i_N})\right]|i_1 i_2 \ldots i_N\rangle.$$

where $P = 1$ or $P = (-1)^F$.

If X is an odd element of $End(M)$ (anti-commutes with $(-1)^F$), the MPS is odd as well.

The cylinder projectors (both R and NS) project to states where X supercommutes with all $T(e_i)$:

$$XT(e_i) = (-1)^{|e_i||X|}T(e_i)X, \quad \forall i.$$

This implies that only the R sector can have fermionic states.

G-equivariant spin-TQFTs

In general, $\mathbb{Z}_2^F$ is a central subgroup of the symmetry group $\hat{G}$, but $\hat{G}$ need not be a product $\mathbb{Z}_2^F \times G$.

We will limit ourselves to the split case $\hat{G} \simeq \mathbb{Z}_2^F \times G$, since otherwise there is no way to separate spin structure from the G gauge field.

The state-sum construction is a combination of the equivariant bosonic state-sum and the fermionic state-sum. The starting point is a semi-simple $\mathbb{Z}_2$-graded G-equivariant algebra A.

Such algebras are classified by a subgroup $K \subset G$ and triples $(\alpha, \beta, \gamma) \in H^2(K, U(1)) \times H^1(K, \mathbb{Z}_2) \times \mathbb{Z}_2$.

K is the unbroken subgroup of G. SRE phases (invertible G-equivariant spin-TQFTs) correspond to the case $K = G$. Let's focus on SRE phases.

G-equivariant $\mathbb{Z}_2$-graded central simple algebras

Case 1: $\gamma = 0$. Let U be a $\mathbb{Z}_2$-graded vector space which carries a projective representation Q_g of G with a 2-cocycle $\alpha \in H^2(G, U(1))$. G acts on $A = End(U)$ by

$$R_g : a \mapsto Q_g a Q_g^{-1}.$$

Further, while R_g is even for all g, Q_g is may be odd. Let $\beta(g) = 0$ if Q_g is odd, and $\beta(g) = 0$ otherwise. $\beta : G \to \mathbb{Z}_2$ is a homomorphism.

Case 2: $\gamma = 1$. Let U be a (purely even) vector space which carries a projective representation Q_g of G. G acts on $A = \mathrm{Cl}(1) \otimes End(U)$ by:

$$1 \otimes a \mapsto 1 \otimes Q_g a Q_g^{-1}, \quad \Gamma \otimes 1 \mapsto (-1)^{\beta(g)} \Gamma \otimes 1,$$

where $a \in End(U)$ and Γ is the generator of $\mathrm{Cl}(1)$, $\Gamma^2 = 1$.

Group law for fermionic SRE phases with symmetry $G \times \mathbb{Z}_2^F$

The supertensor product of two algebras of the above sort is again an algebra of the same sort. This gives a group law on the set of fermionic SRE phases.

The computation is a bit involved (Turzillo, You, AK), but the result is simple:

$$(\alpha, \beta, \gamma) + (\alpha', \beta', \gamma') = (\alpha + \alpha' + \frac{1}{2}\beta \cup \beta', \beta + \beta', \gamma + \gamma'),$$

Here

$$[\alpha] \in H^2(G, \mathbb{R}/\mathbb{Z}), \quad \beta \in H^1(G, \mathbb{Z}_2), \quad \gamma \in H^0(G, \mathbb{Z}_2) = \mathbb{Z}_2.$$

and

$$(\beta \cup \beta')(g_1, g_2) = \beta(g_1)\beta(g_2).$$

Oriented and spin bordisms of BG

An element of $\Omega_n^{SO}(BG)$ is an equivalence class of closed oriented n-manifolds together with a map $f : X \to BG$.

Two such pairs (X, f) and (X', f') are equivalent if there exists a compact oriented $(n+1)$-dimensional manifold Y with boundary $X' \sqcup \bar{X}$ and a map $h : Y \to BG$ which extends both f and f'.

$\Omega_n^{Spin}(BG)$ is defined similarly, but with compatible spin structure on Y, X and X'.

The Freed-Hopkins theorem implies that deformation classes of unitary invertible G-equivariant spin-TQFTs are in 1-1 correspondence with the Poincare-dual of the torsion of $\Omega_{d+1}^{Spin}(BG)$.

Spin-cobordism of BG for $n = 2$

One can identify the dual of $\Omega_2^{Spin}(BG)$ with the triples (α, β, γ) as follows:

$$(X, f) \mapsto \exp(2\pi i \int_X f^*\alpha)(-1)^{q_\eta(f^*\beta)}(-1)^{Arf(\eta)}.$$

The physical meaning of this map is the TQFT partition function for a spin 2-manifold (X, η) equipped with a G gauge field $f : X \to BG$.

It is easy to check that the partition function corresponding to

$$(\alpha + \alpha' + \frac{1}{2}\beta \cup \beta', \beta + \beta', \gamma + \gamma')$$

is the product of partition functions corresponding to (α, β, γ) and $(\alpha', \beta', \gamma')$.

Including time-reversing symmetries

More generally, $\hat{G}$ (an extension of G by $\mathbb{Z}_2^F$) is equipped with a homomorphism $\sigma : \hat{G} \to \mathbb{Z}_2$ indicating which elements of $\hat{G}$ act anti-unitarily.

A suitable equivariant algebra can be used to construct Matrix Product States with this symmetry and (conjecturally) a fully-fledged TQFT.

The corresponding geometric structure on Σ is involved, so let me describe the result only for $G = \hat{G}/\mathbb{Z}_2^F = \mathbb{Z}_2^T$. That is, a single bosonic symmetry T which acts anti-unitarily.

Case 1: $\hat{G} = \mathbb{Z}_4$. $T^2 = (-1)^F$. This corresponds to Σ equipped with a Pin^+ structure. Fermionic SRE phases are classified by $\mathbb{Z}_2$.

Case 2: $\hat{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$. $T^2 = 1$. This corresponds to Σ equipped with a Pin^- structure. Fermionic SRE phases are classified by $\mathbb{Z}_8$.

Bosonization in 1d

It is well-known that every 1d fermionic system is equivalent to a system of bosonic spins on the same 1d lattice.

Crucially, the equivalence maps local bosonic observables to local $\mathbb{Z}_2$-invariant observables.

The map (the Jordan-Wigner transformation) maps fermionic operators $c_j, c_j^\dagger$ to functions of bosonic spin operators X_j, Y_j, Z_j:

$$c_j^\dagger = \frac{1}{2}(X_j + iY_j)(-1)^{\sum_{k=1}^{j-1} \frac{1}{2}(1+Z_k)},$$
$$c_j = \frac{1}{2}(X_j - iY_j)(-1)^{\sum_{k=1}^{j-1} \frac{1}{2}(1+Z_k)}.$$

This map is invertible, and the inverse maps a system of spins with a $\mathbb{Z}_2$ symmetry to a system of fermions.

Example: the Majorana chain

$$H^f = \sum_j H_j = \sum_j \left(\mu c_j^\dagger c_j + t c_j c_{j+1} + h.c. \right).$$

(this is slightly different from the Hamiltonian I wrote before, but is equivalent to it in some range of μ and t).

Since each term H_j is a fermion bilinear, its image under the Jordan-Wigner transformation is a local bosonic expression:

$$H^b = \sum_j (h Z_j + J X_j X_{j+1}) + const.,$$

where h and J are some linear functions of μ and t.

Thus the Majorana chain is mapped to the quantum Ising chain.

The bosonic system has a $\mathbb{Z}_2$ symmetry generated by $\prod_j Z_j$. The fermionic system has $\mathbb{Z}_2^F$.

Topological bosonization

Passing to the IR limit and assuming a gap, bosonization becomes a 1-1 map between spin-TQFTs and $\mathbb{Z}_2$-equivariant TQFTs.

Indeed, we saw that the state-sum construction of both kinds of theories starts with a $\mathbb{Z}_2$-graded algebra A.

One write down an explicit relation between the partition functions (D. Gaiotto and AK, 2015):

$$Z^f(\eta) \sim \sum_{\epsilon \in H^1(\Sigma, \mathbb{Z}_2)} Z^b(\epsilon)(-1)^{q_\eta(\epsilon)},$$

$$Z^b(\epsilon) \sim \sum_{\eta \in S(\Sigma)} Z^f(\eta)(-1)^{q_\eta(\epsilon)},$$

Monoidal structure and bosonization

The bosonization map does not preserve the monoidal structure on the set of TQFTs (stacking).

In particular, an invertible (SRE) fermionic phase need not map to an invertible (SRE) bosonic phase with a $\mathbb{Z}_2$ symmetry.

For example, $A = \mathrm{Cl}(1)$ corresponds to an invertible fermionic phase, because $\mathrm{Cl}(1)\widehat{\otimes}\mathrm{Cl}(1) = \mathrm{Cl}(2)$.

But $A = \mathrm{Cl}(1)$ regarded as a $\mathbb{Z}_2$-equivariant algebra is simply the algebra of functions on $\mathbb{Z}_2$. It corresponds to a phase with a spontaneously broken $\mathbb{Z}_2$. This is not an SRE phase with a $\mathbb{Z}_2$ symmetry.

SRE phases in higher dimensions

Approaches to classification:

- Hand-waving physical arguments
- Appeal to the Freed-Hopkins theorem
- Construct the state-sum explicitly

I will start with the hand-waving arguments, but we will see they can miss some phases in higher dimensions.

The hand-waving approach

Explained in Thorngren and AK, 2017.

Instead of studying the boundary behavior, let us take a bulk approach.

When G is preserved, the SRE bulk is locally "featureless" (no nontrivial topological defects). Let's imagine G is spontaneously broken, but the symmetry breaking scale is low compared to all UV scales.

Now the bulk is not featureless: there are domain walls labeled by $g \in G$. Away from the domain walls we have the trivial product state.

All properties of the SRE phase are encoded in the properties of the domain walls.

Domain walls in 1d bosonic SRE phases

Domain walls are particles. They cannot carry any internal quantum numbers, because even upon turning on a G gauge field the ground state is unique.

Let's turn on a G gauge field. Thus creates a network of domain walls in 2d space-time.

Network rules:

- Each oriented edge labeled by $g \in G$; changing orientation replaces $g \mapsto g^{-1}$.
- At a vertex the product all group elements is 1 (if all edges are incoming).

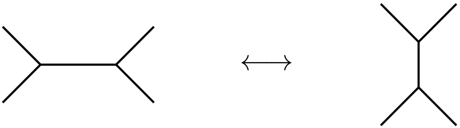
From domain walls to 2-cocycles

Each edge contributes a factor $\exp(-m(g)L)$ goes away in the limit of vanishing $m(g)$. The interesting factors which remain in this limit come from vertices.

Generically only 3-valent vertices. so we need to specify a function $\alpha : G \times G \to U(1)$.

The 2-cocycles constraint can be explained in two different ways.

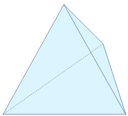
The 1st way: Require the partition function to be unchanged under Pachner moves of the domain wall network. The move



immediately gives the 2-cocycle condition.

From domain wall to 2-cocycles, cont.

The 2nd way: Consider the surface of a tetrahedron regarded as a trivalent graph on S^2:



Each edge is labeled by $g \in G$, with a condition at each vertex. One can choose three labels arbitrarily, the rest are then fixed.

Since S^2 is simply-connected, there are no nontrivial G-bundle on it. Hence the partition function must be trivial. This gives one 4-term equation for α, which in fact is the 2-cocycle condition.

The partition function

2-cocycle differing by a coboundary are equivalent, because of cancellations between neighboring vertices. Hence properties of the domain walls are encoded in $[\alpha] \in H^2(G, U(1))$.

The partition function is now readily evaluated for an arbitrary network (i.e. arbitrary gauge field $f : \Sigma \to BG$), and one finds

$$Z(\Sigma, f) = \exp(2\pi i \int_\Sigma f^*\alpha).$$

This can be thought of as an element of the dual of the oriented bordism group $\Omega_2^{SO}(BG)$.

Domain walls in 1d fermionic SRE phases

Here there are several additional complications:

- There is spin-structure dependence
- Even in the absence of symmetry, there is a nontrivial SRE phase (the Majorana chain)
- Domain walls may be fermions

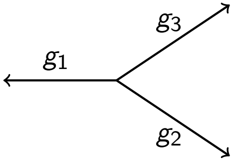
There first two can be accounted for by deleting a point in Σ and taking the spin structure which does not extend to the point. The fermion parity of this impurity gives the parameter $\gamma \in \mathbb{Z}_2$.

The fermion parity of the domain walls is encoded in a function $\beta : G \to \mathbb{Z}_2$. This function is a homomorphism because the fermion parity must be conserved when two domain walls merge.

So we get our parameters (α, β, γ).

Domain walls in 2d bosonic SRE phases

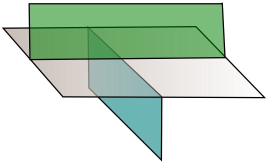
Now domain walls are strings labeled by $g \in G$. At a fixed time, they have triple junctions (generically):



$$g_1 g_2 g_3 = 1$$

The junctions are particle-like. Neither strings nor particles have any internal dynamics.

Worldlines of these particle-like junctions meet at points in space-time:



From domain walls to 3-cocycles

At each such points, 6 domain walls meet. Each wall is labeled by an element of G, and there are 3 independent labels.

Each such point is assigned an element of $U(1)$, so we get a 3-cochain $\alpha \in C^3(G, U(1))$.

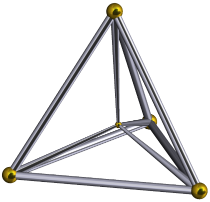
Again, two ways to see that it is a 3-cocycle.

1st way: Note first that the above picture is dual to a tetrahedron. Duality maps each domain wall to an edge.
A network of domain walls in 3d space-time is dual to a triangulation.

Basic reconnections of the domain walls are dual to 3d Pachner moves. Imposing invariance of the partition function under Pachner moves gives the 3-cocycle condition.

From domain walls to 3-cocycles, cont.

2nd way: Consider a 4-simplex regarded as a triangulation of S^3:



2-simplices in this picture are domain wall worldsheets and are labeled by elements of G. The partition function on S^3 is a product over 5 vertices of the 4-simplex.

Since S^3 is simply-connected, the partition function must be 1. This imposes the 3-cocycle condition on α. In the end, we get that 2d bosonic SRE phases are classified by $H^3(G, U(1))$.

Group cohomology vs. oriented bordism

Continuing in this way, we get that bosonic SRE phases in d spatial dimensions are classified by $H^{d+1}(G, U(1))$. This was first proposed by Chen-Gu-Liu-Wen.

But this essentially assumes that domain walls, their junctions, etc, do not carry any nontrivial invertible TQFTs. Only codimension $(d+1)$ junctions matter then.

We know this is not true in general, because invertible TQFTs do exist: they are classified by the dual of $\Omega_n^{SO}(pt) = \Omega_n^{SO}$.

This group is trivial for $n = 1, 2, 3, 4$, but not for $n = 5$.

Thus until we have junctions of dimension less than 5 only, group cohomology is the right answer.

Thermal Hall conductivity

An extra complication: we neglected quasi-TQFTs which exist for $d = 2$.

Their partition functions have a nontrivial metric dependence via

$$S_{CS}^{grav} = c \int_X \mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right),$$

where ω is the Levi-Civita connection.

For SRE phases, the coefficient c must be integral, in suitable units.

The physical meaning of c is that it determines the thermal Hall effect (flow of energy perpendicular to the temperature gradient) at low temperature:

$$\kappa_T \sim cT^2$$

Including c

The effect of c is very simple in low dimensions:

For $d = 2$, bosonic SRE phases are really classified by $H^3(G, U(1)) \times \mathbb{Z}$.

For $d = 3$, domain walls are 2-dimensional, and could carry an invertible SRE with a nonzero c. Then we have a function $\delta : G \to \mathbb{Z}$.

This function must be a homomorphism, to ensure compatibility with the fusion of domain walls.

But if G is finite, $Hom(G, \mathbb{Z}) = 0$. Hence SRE phases are still classified by $H^4(G, U(1))$.

Domain walls in 2d fermionic SRE phases

The fermionic case is more interesting, because $\Omega_n^{Spin}(pt)$ is nontrivial already for $n = 1, 2$:

$$\Omega_1^{Spin} = \mathbb{Z}_2, \quad \Omega_2^{Spin} = \mathbb{Z}_2.$$

That is, there exist nontrivial fermionic SREs for $d = 0$ and $d = 1$, and they may live on domain walls and various junctions.

For $d = 1$, domain walls may be fermions. In a sense, they may carry nontrivial $d = 0$ SRE phases.

For $d = 2$, domain wall junctions (which are particle-like) may again be fermions. Thus we get a function $\beta : G \times G \to \mathbb{Z}_2$.

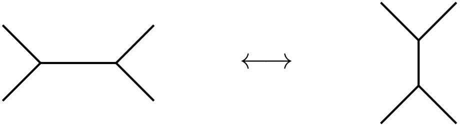
Domain walls themselves are strings, and may be Majorana chains. Thus we get a function $\gamma : G \to \mathbb{Z}_2$.

As in the bosonic case, points in space-time where six domain walls meet give us a function $\alpha : G \times G \times G \to U(1)$.

Conditions on cochains (α, β, γ)

The 1-cochain $\gamma : G \to \mathbb{Z}_2$ must be a 1-cocycle, i.e. a homomorphism $G \to \mathbb{Z}_2$. This is required by compatibility with the domain wall fusion.

The 2-cochain $\beta : G \times G \to \mathbb{Z}_2$ must be a 2-cocycle in order for the fermion number of a domain wall network to be unchanged under a Pachner move:



The 3-cochain α satisfies

$$\delta\alpha = \frac{1}{2}\beta \cup \beta.$$

This is required for the partition function of a 4-simplex to be trivial.

Remarks on SRE phases and the Freed-Hopkins theorem

- The above examples show that knowing Inv_n and Inv_n^f for $n \leq d$ allows one to understand $Inv_n(G)$ and $Inv_n^f(G)$ for $n \leq d$.
- The Freed-Hopkins theorem relates unitary invertible bosonic and fermionic (quasi)-TQFTs to oriented and spin bordism, respectively.
- But in general it is far from clear how to construct a lattice system corresponding to a particular deformation class of TQFTs

For example, consider $d = 3$ bosonic phases with symmetry $\mathbb{Z}_2^T$. FH theorem suggests that there are two generators of Inv_3, with partition functions

$$Z_1 = \exp(\pi i \int_X w_2^2), \quad Z_2 = \exp(\pi i \int_X w_1^4).$$

Here w_i are Stiefel-Whitney classes of X. How does one construct 3+1d lattice models whose effective actions are given by the expressions in the exponent?

I know the answer for these two cases, but not in general.