Geometric quotients of S^3

Degree 0 Reeb orbits correspond to free hmtpy classes of geodesics and for index reasons

$$HQ_0 \cong \mathbb{C}[\gamma_1 \cdots \gamma_m]$$

Large N duality

$$T^*M \longrightarrow X_M$$

$t_0, t_1, \ldots, t_m$

Kähler param.

For $K \subset M$, $L_K \subset T^*M$ moved into X_M

$$\mathcal{A}_K = CE(\Lambda_K) =$$

$$\mathbb{C}[e^{\pm x}, e^{\pm p}, \gamma_1, \dots, \gamma_m]\langle \underline{0}, \mathbb{1}, \dots \rangle$$

In X_M each γ_j has a potential

$$\varepsilon(\gamma_j) = \sum C^j_k Q^k$$

and exactly as before

$$p = \frac{\partial W_K}{\partial x}$$

gives a parameterization of V_K.

Example, the line in projective 3-space

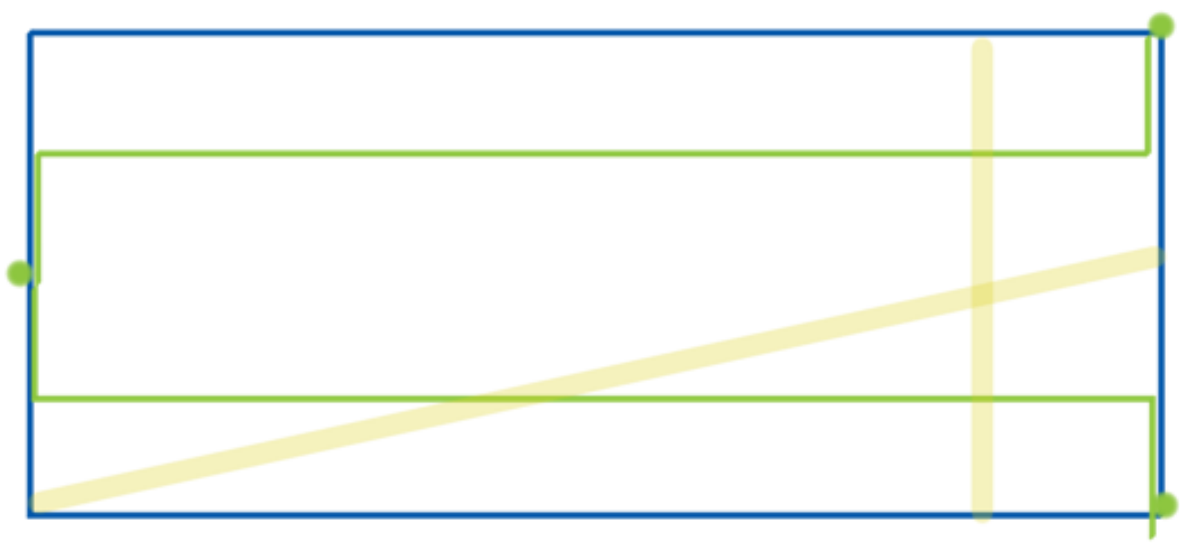
$$X_{\mathbb{RP}^3} = \mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1.$$

$$A_{\mathbb{RP}^1} = e^{x} - e^{-x} - e^{-p-x} + Qe^{p+x} + \gamma$$

which gives the standard mirror

$$A_{\mathbb{RP}^1} = uv$$

of local $\mathbb{CP}^1 \times \mathbb{CP}^1$.



Legendrian SFT - higher genus curves from infinty

$K \subset S^3$; link $L_K \subset X$.

Structure of the theory

There is an

<u>SFT - potential</u>

$$F = F(e^x, Q, g_s)$$

that counts rigid curves on L_K with bounding chains etc.

$$F = F^0 + F^1 + F^2$$

F^0 :

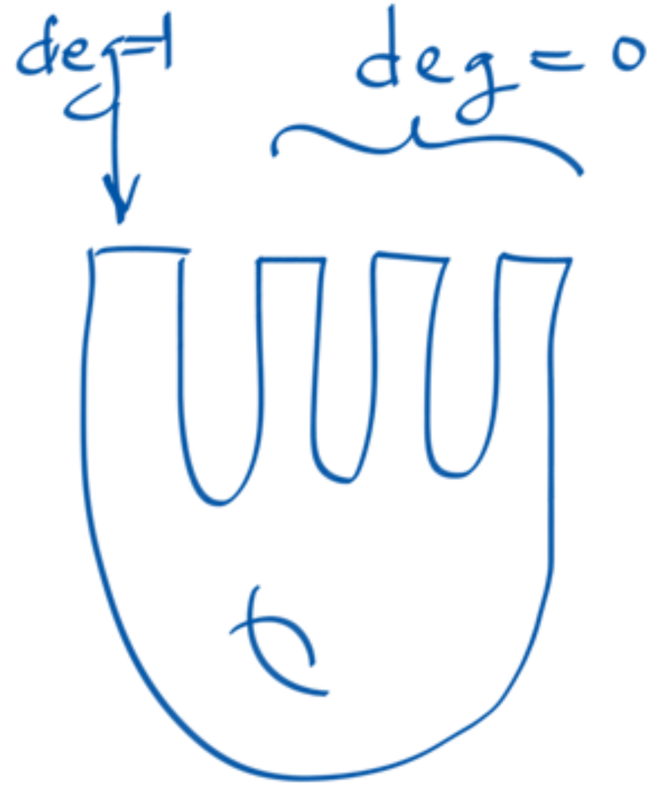
F^j :

There is similarly an SFT-Hamiltonian that counts 1-parameter families of curves at infinity

$$H = H(e^x, e^p, Q, g_s)$$

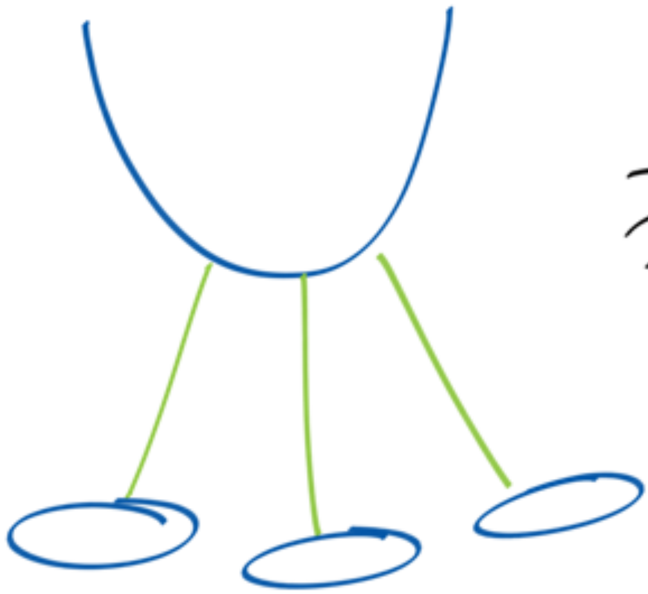
The boundary of 1-dim moduli space then gives

$$e^{-F} H e^F = 0 \quad (\text{or } He^F = 0)$$

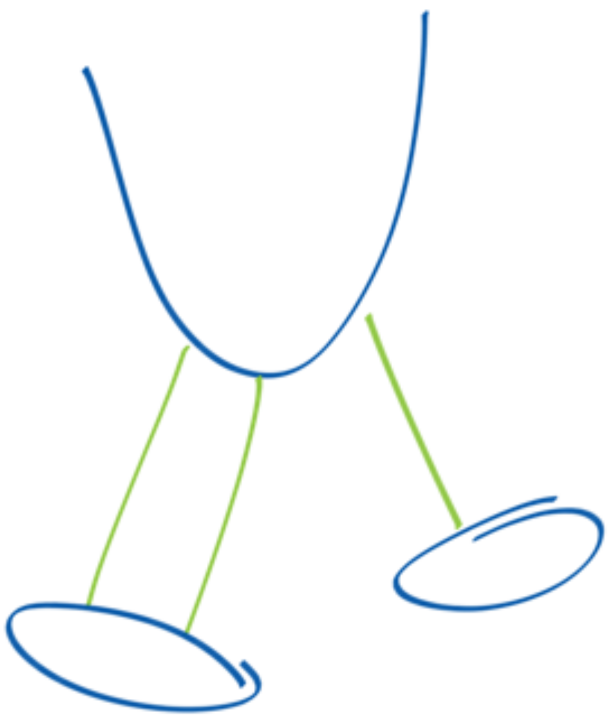


We next note that the analogue of $p = \frac{\partial W_K}{\partial x}$ is

$$p = g_s \frac{\partial}{\partial x}$$



$$\frac{1}{3!}\left(\frac{\partial F}{\partial x}\right)^3 e^F$$



$$\frac{1}{3!}\left(\frac{\partial^2 F}{\partial x^2}\,\frac{\partial F}{\partial x}\right)$$

etc.

Note next that

$$\Psi_K(X) = e^{F_0}.$$

Eliminating Reeb chords in the non-commutative setting $e^p e^x = e^{g_s} e^x e^p$

gives operator equation

$$\hat{A}_K(e^x, e^p, Q)\, \Psi_K(x) = 0$$

(the recursion for colored HOMFLY).

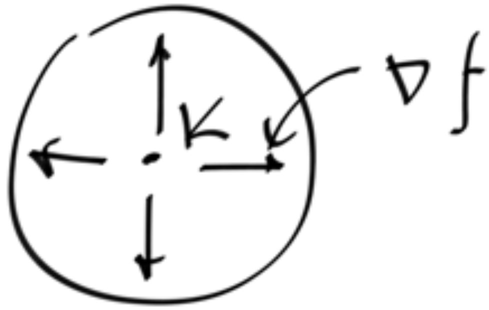
Sketch of construction of SFT

Write $a_1, \ldots, a_n$ for $\deg = 0$ chords

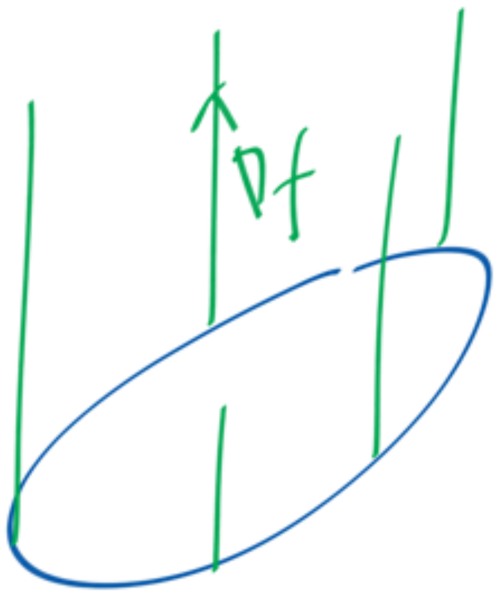
$b_1, \ldots, b_m$ for $\deg = 1$.

Additional data.

- A Morse function which is a small perturbation of f



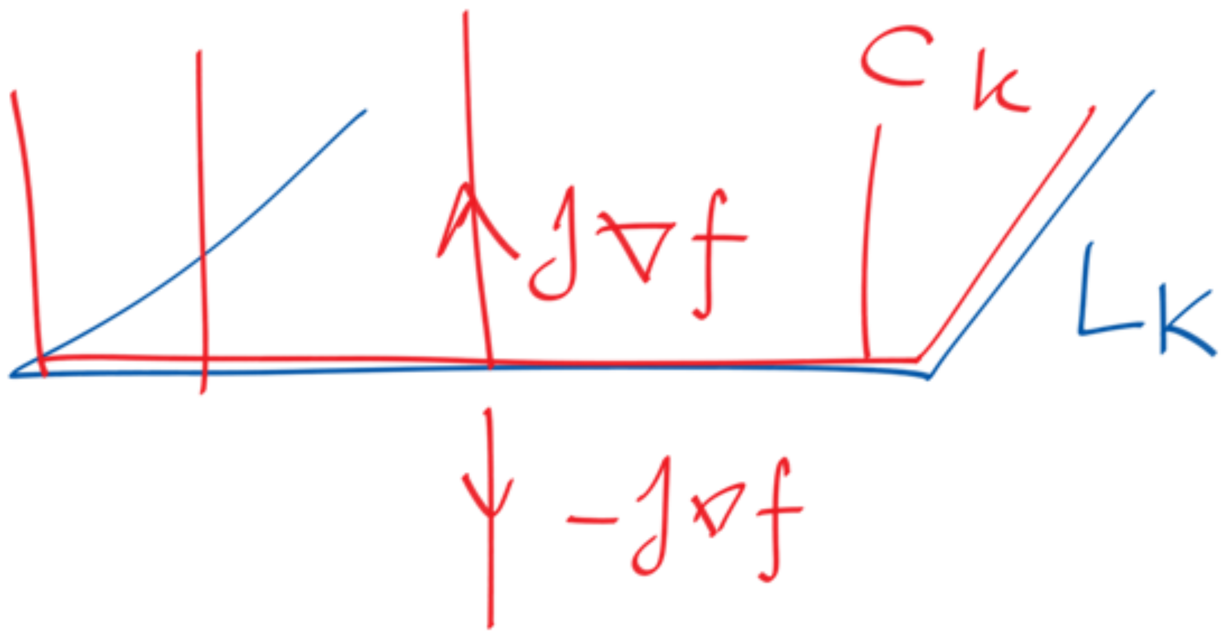
For generic curves ∇f gives bounding chains



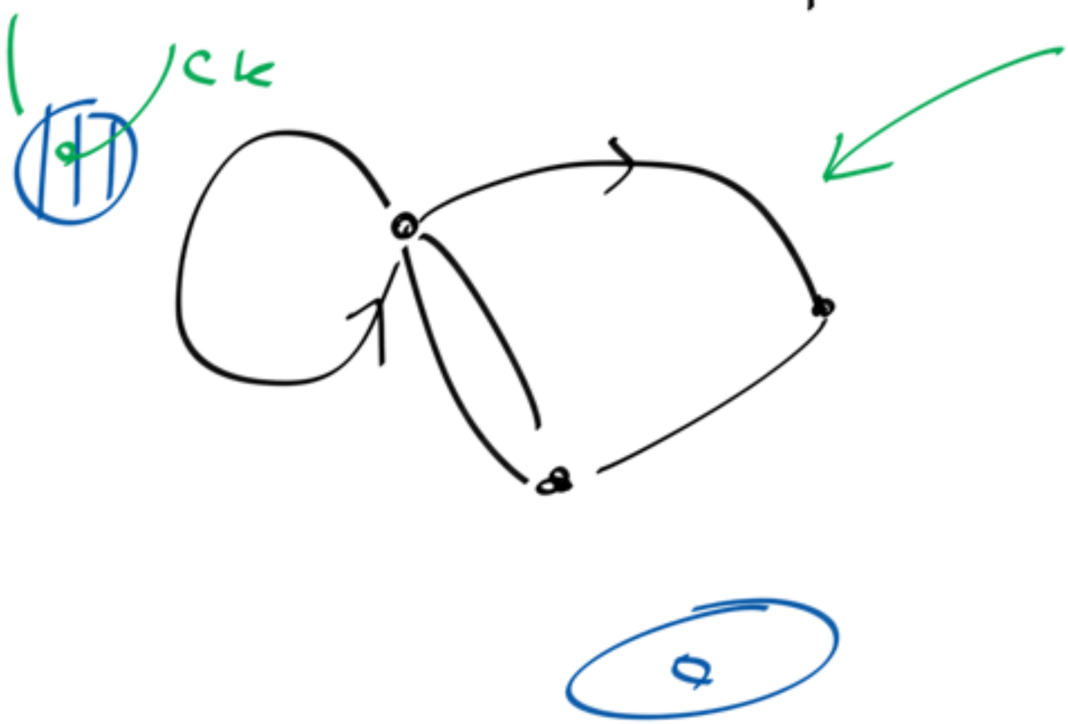
$\sigma_u = \mathrm{flow}(\partial u) - p \cdot \mathrm{unstable}(s)$

- A 4-chain c_K $\quad \partial c_K = 2[L_K]$

and

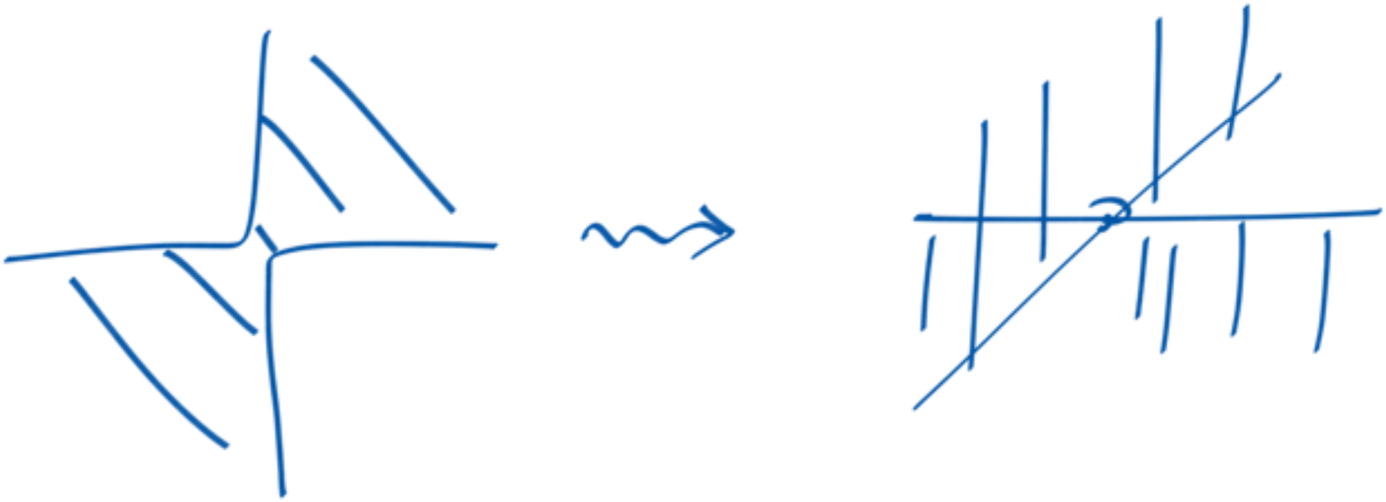


The bounding cochains removes boundary splitting as before. The GW-pot counts graphs



Intersection pts with boundin
chains weighted by 1/2.

Need to check invariance

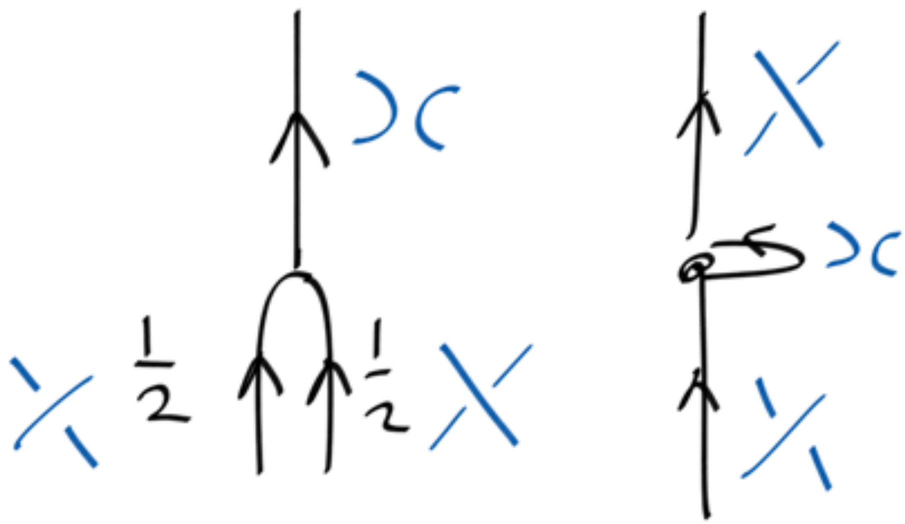


So count

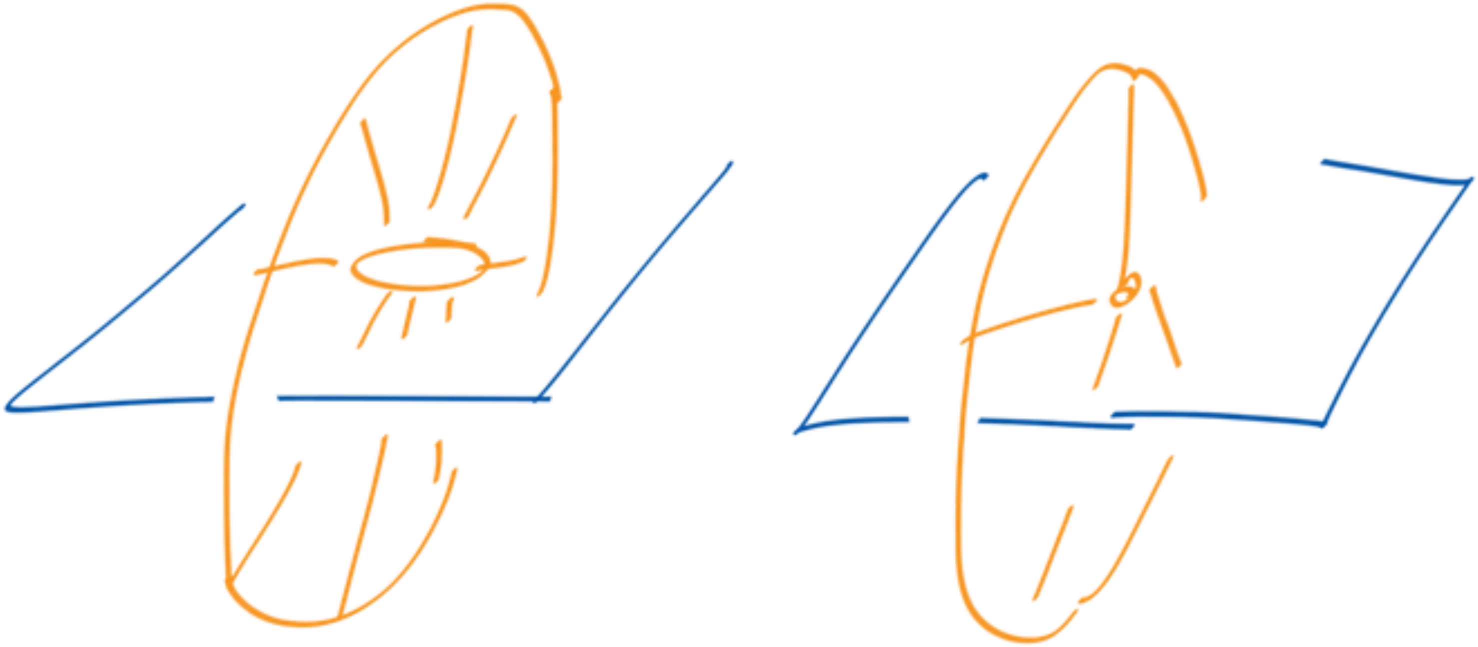
is

$e^{\frac{1}{2} g_s} - e^{-\frac{1}{2} g_s}$

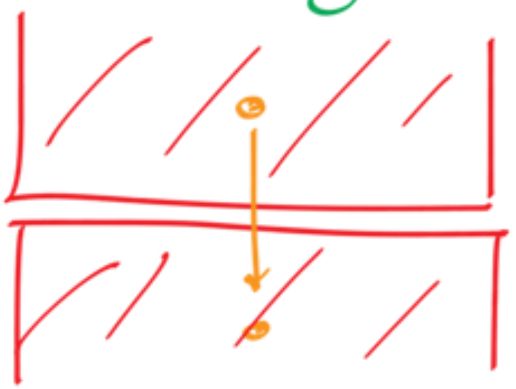
= 



m odd   m even

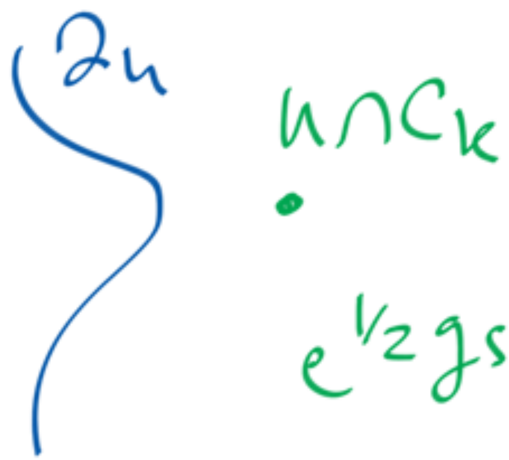


gluing $\qquad e^{\frac{1}{2}g_s} - e^{-\frac{1}{2}g_s}$



C_K

$e^{\frac{1}{2}g_s}$



$e^{\frac{1}{2}g_s}$

Then

$$F = \sum C_{\chi,m,\ell}\, g_s^{-\chi} e^{mx} Q^{\ell} a_{i_1} \cdots a_{i_r}$$

and

$$H(b_j) =$$

$$\sum B_{\chi,m,\ell,k,I,J}\, g_s^{-\chi} e^{\ell x} e^{mp} Q^{k} a_{i_1} \cdots a_{i_m} g_s^{\ell} \frac{\partial}{\partial a_{j_1}} \cdots \frac{\partial}{\partial a_{j_r}}$$

then if $\quad p = g_s \dfrac{\partial}{\partial x}$

$$e^{-F} H(b_j) e^{F}$$

counts ends of oriented 1-dim space so

$$H(b_j)\, e^{F} = 0 .$$

eliminating $\partial/\partial a_j$ gives

$$\hat{A}_K\, e^{F_0} = \hat{A}_K \Psi_K = 0 .$$

Examples

Unknot: no higher genus curves

$$\hat{A}_n = (1 - e^x - e^p - Q e^x e^p)\Psi_n = 0.$$

Hopf link:

$$H(c_{11}) = \left(1 - e^{x_1} - e^{p_1} + Q e^{x_1} e^{p_1}\right) + g_s^2 \partial_{a_{12}} \partial_{a_{21}} + \mathcal{O}(a)$$

$$H(c_{22}) = \left(1 - e^{x_2} - e^{p_2} + Q e^{x_2} e^{p_2}\right) + Q e^{x_2} e^{p_2} g_s^2 \partial_{a_{12}} \partial_{a_{21}} + \mathcal{O}(a)$$

$$H(c_{12}) = \left(e^{p_2} e^{-p_1} - Q e^{x_2} e^{p_2}\right) g_s \partial_{a_{12}} + g_s^{-1} \left(e^{-g_s} - 1\right)\left(1 - e^{x_2}\right) a_{21}$$
$$+ \mathcal{O}(a^2)$$

$$H(c_{21}) = \left(Q e^{x_1} e^{p_1} - e^{g_s} e^{p_1} e^{-p_2}\right) g_s \partial_{a_{21}}$$
$$+ g_s^{-1}\left(e^{g_s}(e^{g_s} - 1) - e^{2g_s}(e^{g_s} - 1) e^{x_1}\right) e^{p_1} e^{-p_2} + \left(e^{g_s} - 1\right) Q e^{x_1} e^{p_1} g_s^2 \partial_{a_{12}} \partial_{a_{21}} \big) a_{12}$$
$$+ \mathcal{O}(a^2)$$

Change var's

$$e^{x_1'} = e^{g_s} e^{x_1}, \quad e^{p_1'} = e^{g_s} e^{p_1}$$

$$e^{x_2'} = Q^{-1} e^{-x_2}, \quad e^{p_2'} = e^{-g_s} Q^{-1} e^{-p_2}$$

$$Q' = e^{g_s} Q, \quad g_s' = -g_s$$

$$\hat{A}_1 = (e^{x_1} - e^{x_2}) + (e^{p_1} - e^{p_2}) - Q(e^{x_1}e^{p_1} - e^{x_2}e^{p_2})$$

$$\hat{A}_2 = (1 - e^{-g_s}e^{x_1} - e^{p_1} + Qe^{x_1}e^{p_1})(e^{x_1} - e^{p_2})$$

$$\hat{A}_3 = (1 - e^{-g_s}e^{x_2} - e^{p_2} + Qe^{x_2}e^{p_2})(e^{x_2} - e^{p_1})$$

in agreement with HOMFLY.

Similarly for the trefoil

$$\hat{A}_T =$$

$$Q^3(e^{g_s}e^p - Q)(e^{2p} - e^{g_s}Q)$$

$$+\Big(e^{3g_s}(e^{g_s}Q - e^{2p})(Q - e^{2p})(Q - e^{g_s}e^{2p})$$

$$+ e^{2g_s}Qe^{2p}(e^{g_s}Q - e^{2p})(Q - e^{g_s}e^{p})$$

$$- e^{2g_s}Q^2e^{p}(Q - e^{g_s}e^{2p})(e^{g_s} - e^{p})\Big)e^{x}$$

$$- e^{2g_s}e^{3p}(e^{g_s} - e^{p})(Q - e^{g_s}e^{2p})e^{2x}$$

Reconstructing the wave function

$W_K(X)$ obtained by solving alg eqn, hence analytic.

For $Q=1$ we use the curve counting map

$$CH^{lin}(\Lambda_K) \oplus C_*(K) \longrightarrow \mathrm{Cone}\left(C_*(\Omega(K,K),K) \to C_*(K)\right)$$

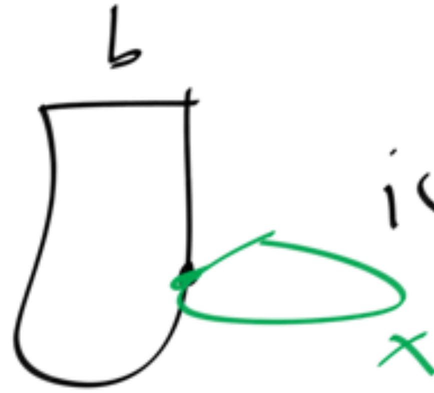
to show that for general pt in V_K

$$\mathrm{rk}\left(CH_0^{lin}\right) = 0$$

$$\mathrm{rk}\left(CH_1^{lin}\right) = 1$$

$$\mathrm{rk}\left(CH_2^{lin}\right) = 1$$

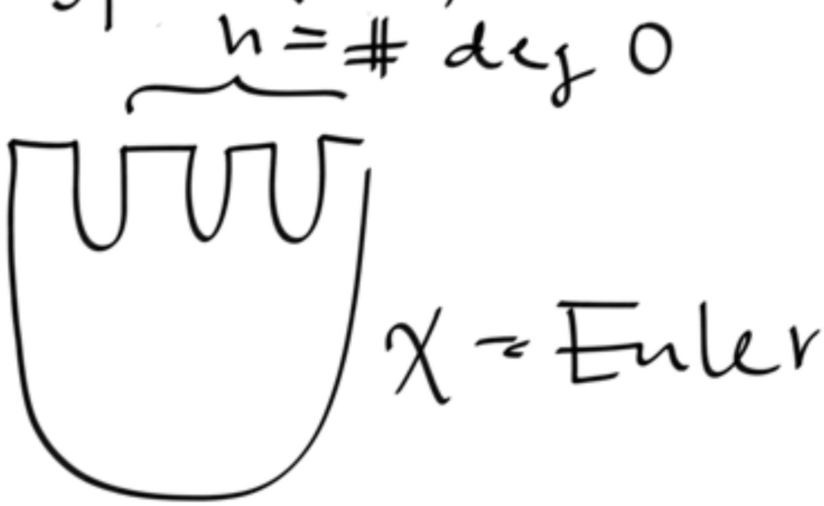
Moreover, if b generates CH_1^{lin} then the count of curves



is generically non-zero.

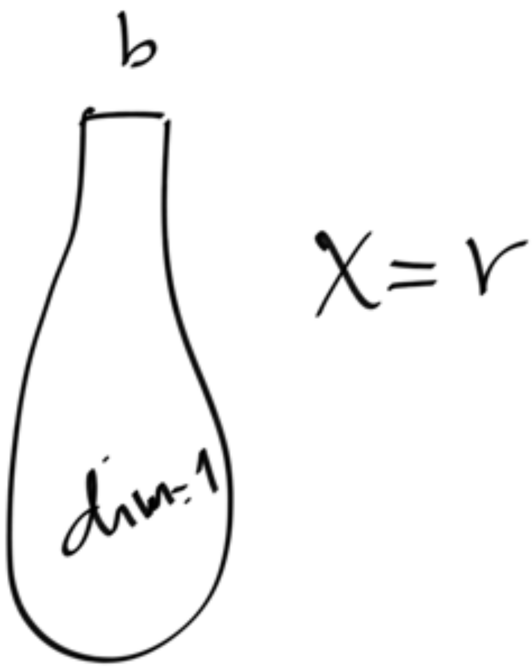
Our inductive procedure starts from ∞ where there are only disks and formal higher genus curves

Type (n, χ)

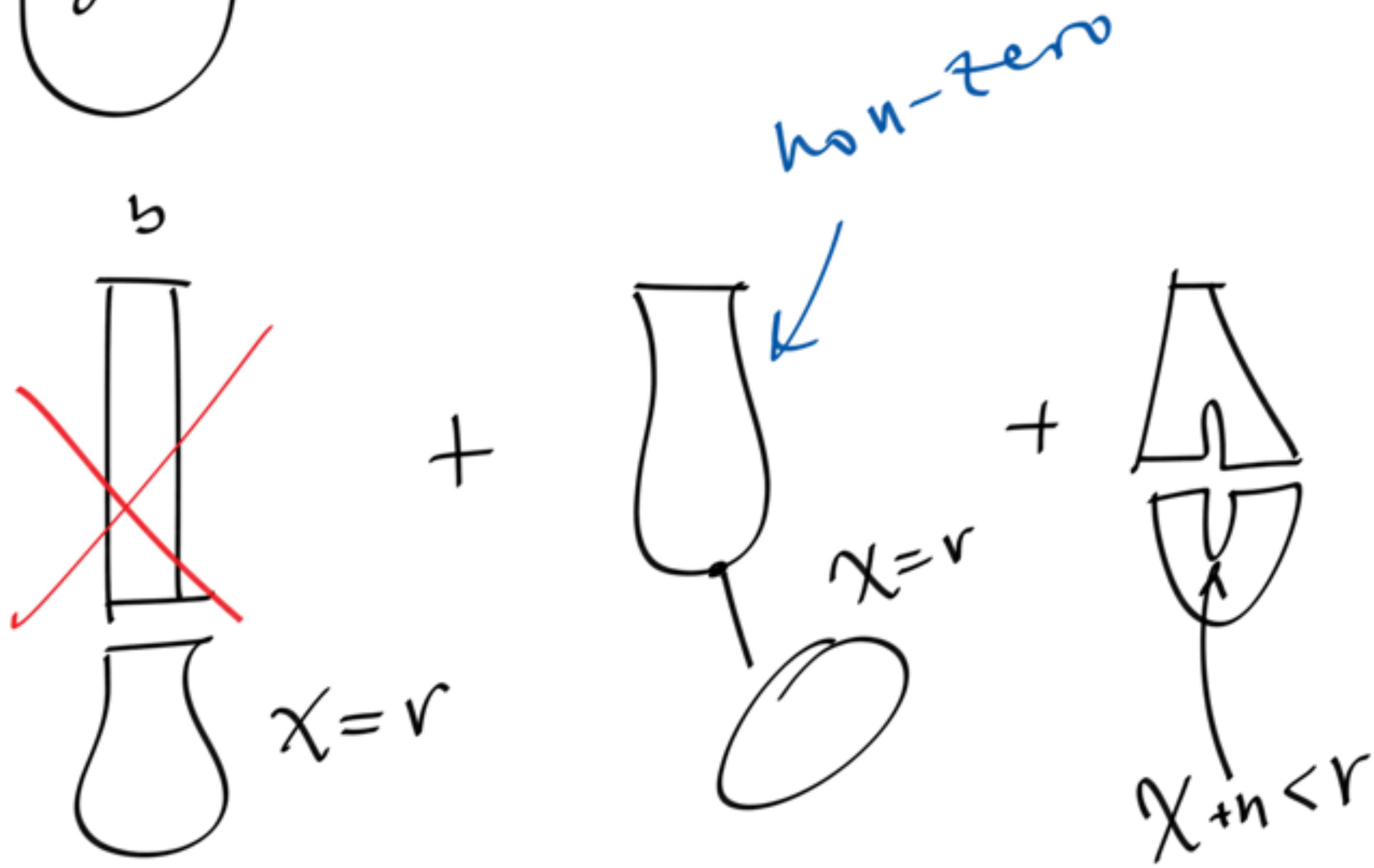


Assume counts of all type (n, χ) with $-\chi + n < r$ are known.

Pick b generator of CH^{lin}_1



boundary



So

$B(e^x, Q) \cdot F_0^r$ = earlier

$B \neq 0$.

pick b_j

$= 1$

gives result.

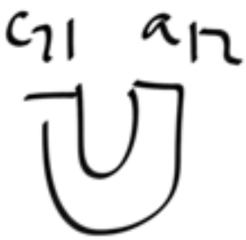
Example, the annulus amplitude for the Hopf link

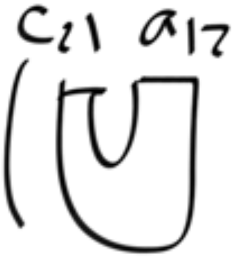
Real chords

$\deg = 0 \qquad a_{12} \quad a_{21}$

$\deg = 1 \qquad c_{11} \quad c_{12} \quad c_{21} \quad c_{22}$

$\qquad\qquad b_{12} \quad b_{21}$

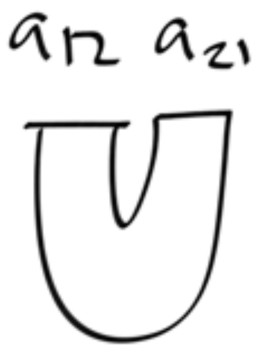
Moduli spaces		Count
(I)		$1-e^{x_1}-e^{p_1}+Qe^{x_1}e^{p_1}$
(II)		1
(III)		$(1-e^{x_1})e^{p_1}e^{-p_2}$
(IV)		$(Qe^{x_1}e^{p_1}-e^{p_1}e^{-p_2})$

$\partial\Big($ 

$\Big) =$ 

$\cup$ (III)

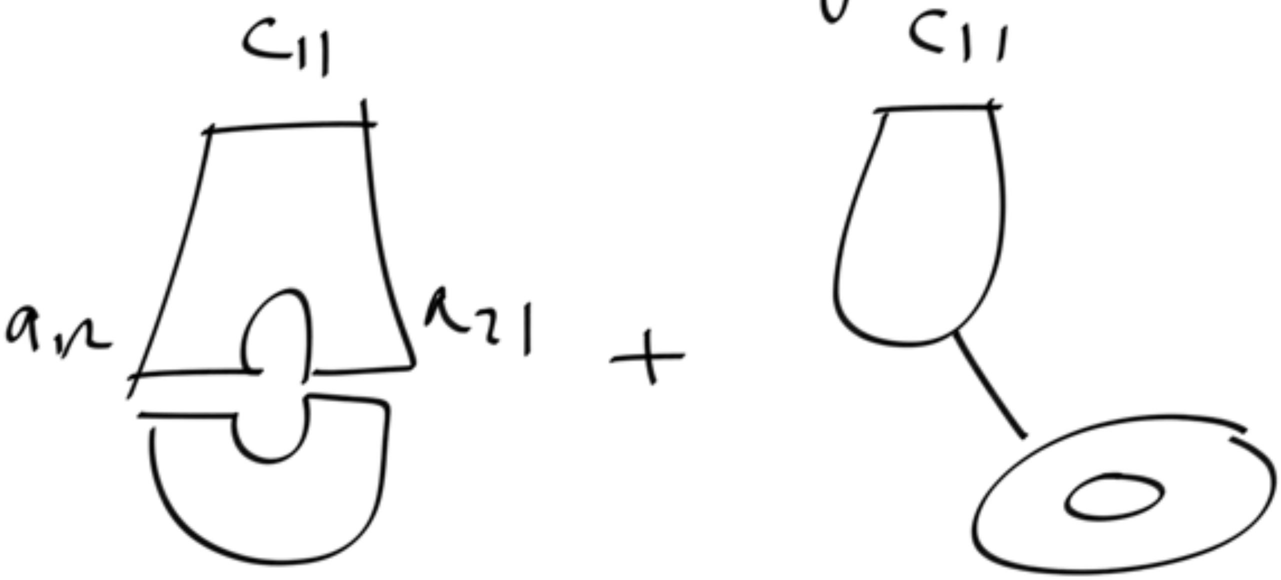
Thus (III) & (IV) gives



$$= \frac{(1-e^{x_1})\, e^{p_1} e^{-p_2}}{Q e^{x_1} e^{p_1} - e^{p_1} e^{-p_2}} \quad \text{(V)}$$

Now look at

The boundary is



(I), (III), (V) then gives with B annulus ampl:

$$\frac{\partial B}{\partial x_1} = \frac{1}{Q e^{p_1} e^{x_1} - e^{p_1}} \, \frac{(1-e^{x_1}) e^{-p_2}}{Q e^{x_1} - e^{-p_2}}$$

Using $1 - e^{x_1} = e^{p_1} - Qe^{x_1}e^{p_1}$ we get

$$\frac{\partial B}{\partial x_1} = \frac{-e^{-p_2}}{Qe^{x_1} - e^{-p_2}} =$$

$$= \frac{Q^{-1}e^{-x_1}e^{-p_2}}{1 - Q^{-1}e^{-x_1}e^{-p_2}}$$

Let $Q^{-1}e^{-x_1} \longmapsto e^{x_1}$

$$\frac{\partial B}{\partial x_1} = \frac{-e^{x_1}/e^{p_1}}{1 - e^{x_1}/e^{p_1}}$$

$$\Rightarrow$$

$$\frac{\partial B}{\partial x_1 \partial x_2} = \frac{\partial}{\partial e^{x_1}} \frac{\partial}{\partial e^{x_2}} \log\left(1 - e^{x_1}/e^{p_2}\right)$$