Chern-Simons theory

M closed 3-mfd
A $SU(N)$ connection on M.

$$S(A) = \int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

From $$c_2 = \frac{1}{8\pi^2} \int_W \mathrm{tr}\, F_A^2,$$

W closed 4-mfd, $F_A = dA + A \wedge A$

if $A' = g^{-1} A g + g^{-1}(dg)$

then $S(A') = S(A) + 8\pi^2 m$,
$m \in \mathbb{Z}$.

If $A \mapsto A + \delta A$ then

$$\delta S = \int_M \mathrm{tr}(F_A \wedge \delta A)$$

and flat connections are stationary for S.

The C-S partition function

path integral (over gauge orbits)

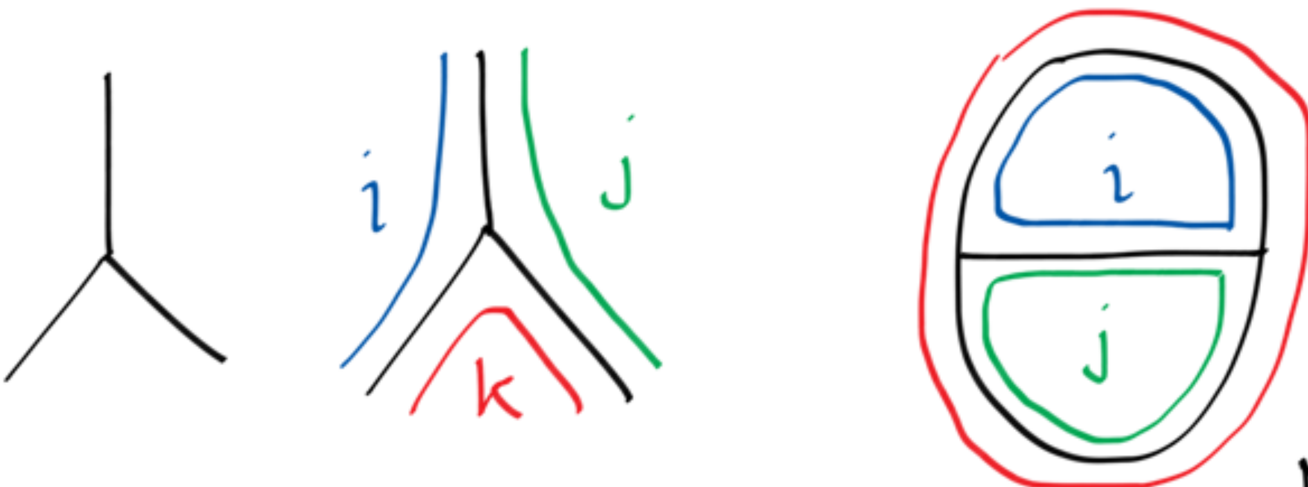
$$Z_{CS}(M) = \int \mathcal{D}A \; e^{\frac{ik}{4\pi} S(A)}$$

k integer $\left(\approx 1/\hbar\right)$.

Feynman calculus gives a perturbation expansion of Z_{CS} near the flat connections in $1/k$. The interaction term is:

$$A^i_j \wedge A^j_k \wedge A^k_i$$

gives repeated use of the same vertex and fattening the Feynman diagrams we can keep track of contributions:



N^3/k

Diagrams with h boundary components and r loops contribute

$$N^h k^{1-r} = N^h \left(\frac{1}{k}\right)^{-\chi}.$$

HOMFLY

- $V(\bigcirc) = (Q^{1/2} - Q^{-1/2}) / (q^{1/2} - q^{-1/2})$

- $Q^{1/2} V(\;) - Q^{-1/2} V(\;) = (q^{1/2} - q^{-1/2}) V(\;)$

Witten showed:

$$V(K) = \langle U(K) \rangle =$$

$$= Z_{cs}(S^3)^{-1} \int \mathcal{D}A \, e^{\frac{ik}{4\pi} S(A)} \operatorname{tr} U(K)$$

where $U(K)$ is the holonomy of A along K and

$$q = \exp\left(\frac{2\pi i}{N+k}\right), \quad Q = q^N$$

A-model topological string on T^*M

Consider T^*M with N branes along $M \subset T^*M$ and a.c.s. J. The theory is defined by a path integral over

$$\Phi = (\phi, \chi, \psi)$$

- $\phi : (\Sigma, \partial\Sigma) \to (T^*M, M)$, Σ Riemann surface
- χ vector field along ϕ (super partner of ϕ)
- ψ section of $T^{0,1}(T^*M) \otimes T^{*1,0}\Sigma \oplus T^{1,0}(T^*M) \otimes T^{*0,1}\Sigma$ (super partner of $d\phi$).

The Lagrangian is

$$L = 2t\int_{\Sigma} d^2z \left(\frac{1}{2} G_{ij}\, \partial_z \phi^i \partial_{\bar z} \phi^j + i\, G_{I\bar J}\, \psi^I_{\bar z} D_z \chi^{\bar J} + i\, G_{\bar I J}\, \psi^{\bar I}_z D_{\bar z} \chi^I \right.$$
$$\left. - R_{I\bar I J\bar J}\, \psi^I_{\bar z} \psi^{\bar I}_z \chi^J \chi^{\bar J} \right)$$

where $G = \omega(J\cdot, \cdot)$ and

$$D_\alpha \chi^i = \partial_\alpha \chi^i + \partial_\alpha \phi^j \Gamma^i_{jk} \chi^k$$

The path integral is

$$Z = \int \mathcal{D}\Phi \; e^{L(\Phi)}$$

Z localizes on holomorphic curves and we get a closed form on $\mathcal{M}_{g,h}$ corresponding to the surface Σ. Integration the gives the free energy $F_{g,h}$ for surfaces of genus g with h boundary components.

Since T^*M is a CY 3-fold

$$\dim(\mathcal{M}_{g,h}) = (n-3)(2-2g-h) + 2c_1 = 0.$$

and

$$Z_{GW}(N, g_s) = \exp\left(\sum F_{g,h} N^h g_s^{2g+h-2}\right).$$

g_s = string coupling $= e^{\phi_0}$, ϕ_0 energy of dilaton field.

Holomorphic maps into (T*M,M) are all constant and the manifold M of constant maps is degenerate. Using string field theory, Witten computed the contribution the constants showing

$$Z_{GW}\left(N, g_s = \frac{2\pi i}{k+N}\right) = Z_{CS}(N,k)$$

in line with the fattened Feynman diagrams.

Including knots

$M = S^3$; $K \subset S^3$ Knot

add one brane on L_K
get

$$Z_{GW}(L_K ; N, g_s, x)$$

where e^x is the $U(1)$ holonomy around the generator of $H_1(L_K)$.

An analogue of the string field theory argument shows that integrating out constant strings in $L_K \cap S^3 = K$ corresponds to inserting

$$\det\left(1 - e^{-x}\, U(K)\right)^{-1}$$

into the path integral

$$\det\left(1 - e^{-x}\, U(K)\right) =$$

$$= \sum_k \mathrm{tr}_{S_k}\, U(K)\, e^{-kx}$$

$$S_k = \underbrace{\square\square\square\square\square\square}_{k}$$ k^{th} symmetric

Thus

$$\Psi_K(x) := Z_{GW}(L_K) / Z_{GW}$$

$$= \sum_k \langle tr_{S_k} U(K) \rangle e^{-kx} =$$

$$= \sum_k H_k(K) \, e^{-kx}$$

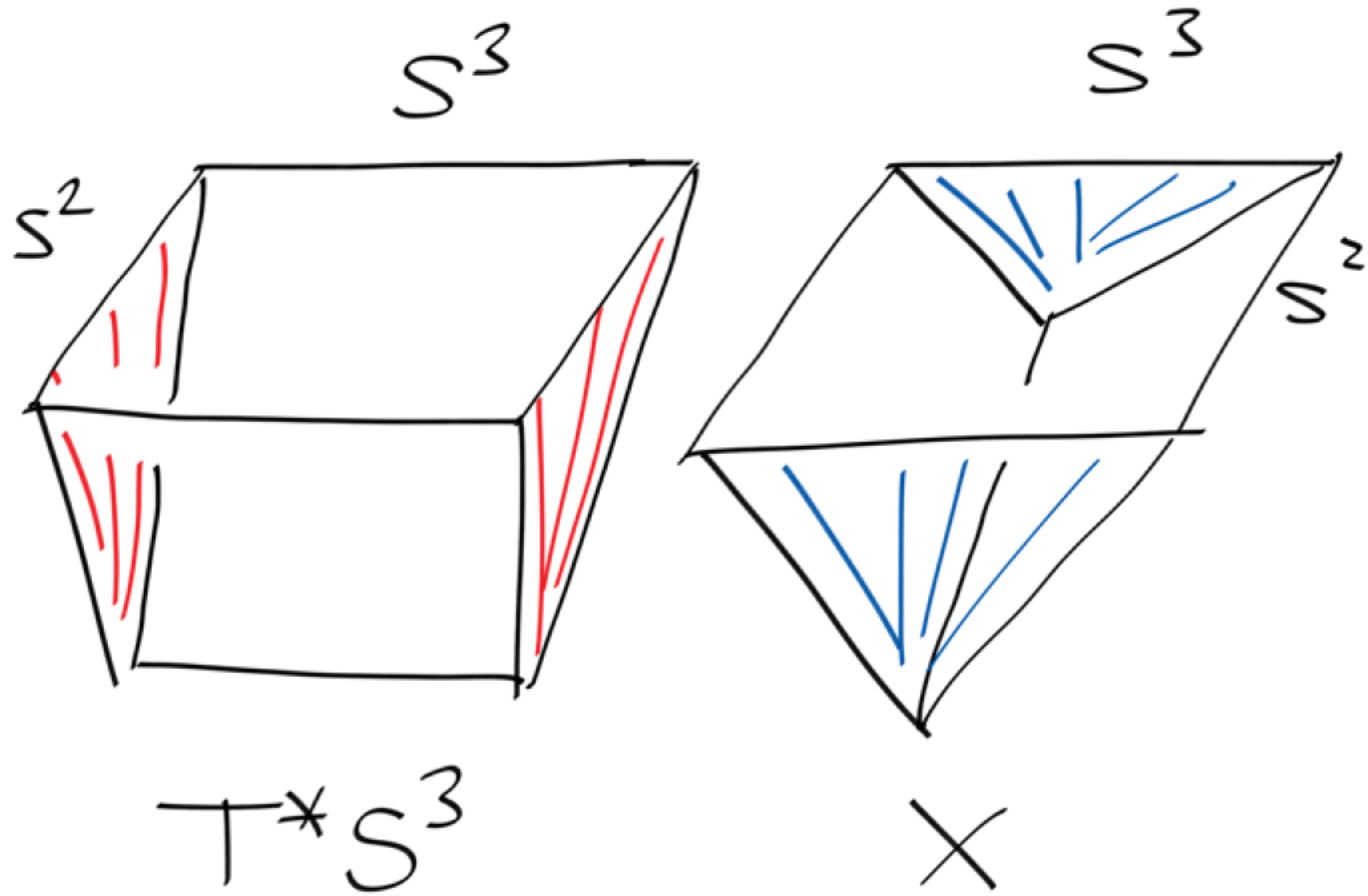
H_k colored HOMFLY polynomial in $q = e^{g_s}$ and $Q = q^N$

<u>Note</u> $P(K) = H_1(K)$.

Large N transition

T^*S^3 is a quadric in $\mathbb{C}^4$, a resolution of a cone.

$X = \mathcal{O}(-1)^{\oplus 2} \to \mathbb{CP}^1$ is also a resolution.



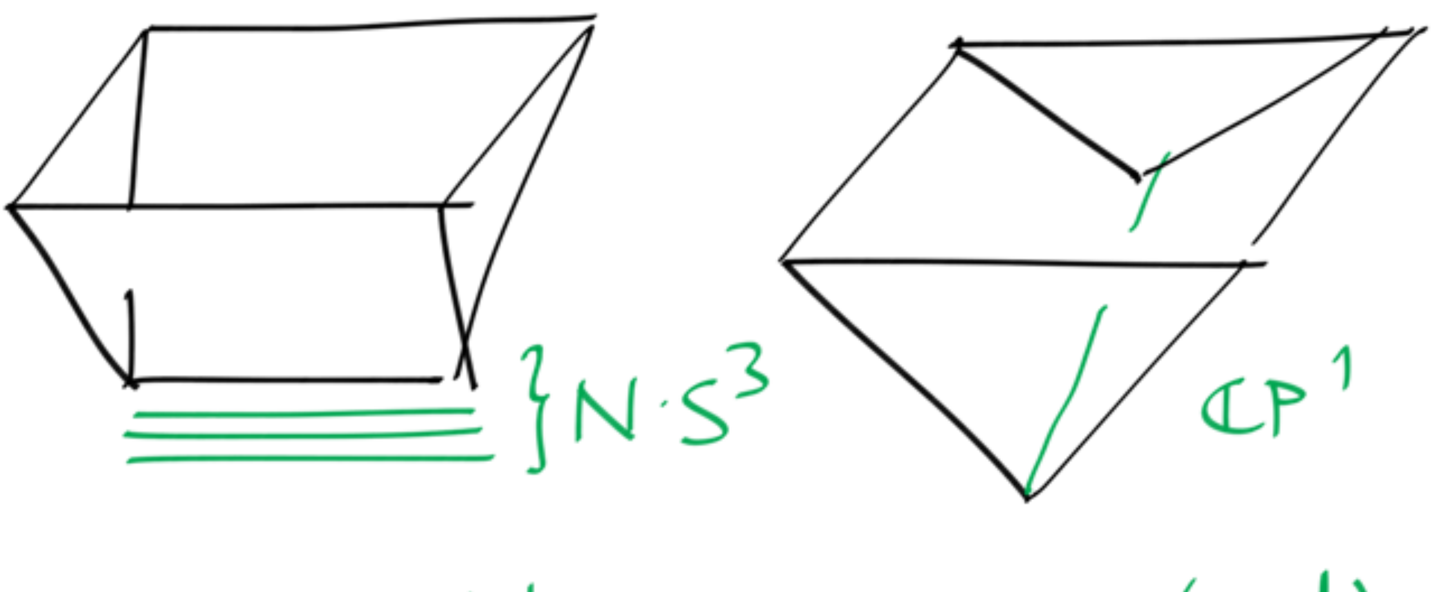
Gopakumar-Vafa large N duality

if $\text{area}(\mathbb{CP}^1) = t = N g_s$,

$Q = e^t = e^{N g_s}$ then

$$Z_{GW}(T^*S^3, N \cdot S^3) = Z_{GW}(X, Q)$$

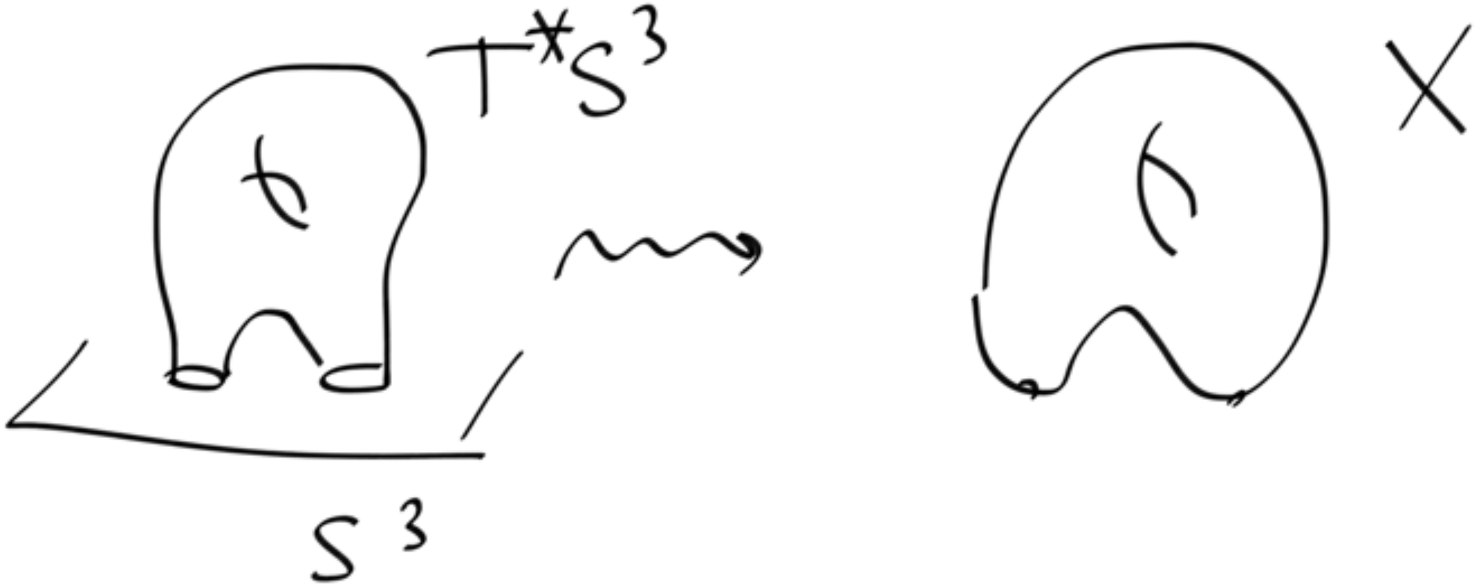
open string $\longleftrightarrow$ closed string



$$N \cdot g_s = \text{area}(\mathbb{CP}^1)$$

Ooguri-Vafa gives a world sheet proof of this result via a theory that interpolates between the two theories.

Intuitively boundaries on $N \cdot S^3$ shrinks and closes off smoothly



$K \subset S^3$

Shift L_K off S^3 along $d\theta$

Then we get $L_K \subset X$
and

$$\Psi_K(X) = Z_{GW}(X, L_K) / Z_{GW}(X)$$

Witten's argument relates constant curves on L_K to $GL(1)$ gauge theory on a solid torus.

The path integral becomes the path integral of 1-dim QM in periods and by usual quantization:

$$p\Psi_K(x) = g_s \frac{\partial}{\partial x} \Psi_K(x)$$

(We will give a purely geometric explanation of this later)

Then by short wave asymp

$$\Psi_K(x) = \exp\left(\frac{1}{g_s}\int p\,dx + \dots\right)$$

From GW-perspective

$$\Psi_K(x) = \exp\left(\frac{1}{g_s} W_K(x) + \overset{[\chi<1]}{\dots}\right)$$

$$W_K(x) = \sum_{k,m} C_{k,m}\, Q^k e^{mx}$$

the GW disk-potential

We conclude that as L_K varies

$$p = \frac{\partial W_K}{\partial x} \qquad (*)$$

Since colored HOMFLY is q-holonomic $(*)$ parametrizes an algebraic curve $A_K = 0$

In all examples A_K is the augmentation polyn.

Example - the disk potential for the unknot

$$A_u = 1 - e^x - e^p + Qe^x e^p$$

$$e^p = \frac{1-e^x}{1-Qe^x}$$

$$p = \log(1-e^x) - \log(1-Qe^x)$$

$$= \sum \frac{1}{k}(1-Q^k)\,e^{kx}$$

$$W(x) = \sum \frac{1}{k^2}(1-Q^k)\,e^{kx}$$

Augmentations and exact Lagrangian fillings

If L is an exact Lagrangian filling of Λ_K in T^*S^3

then $\varepsilon : \mathcal{A}_K \longrightarrow \mathbb{C}[H_1(L)]$

$$\varepsilon(a) = \sum_{|a|=0} |\mathcal{M}_A(a)|\, e^A$$

chain map

$\varepsilon \circ d =$ 

$= \partial\Big($  dim $= 1 \Big)$

Λ_K has two natural fillings

L_K and $M_K \approx S^3 - K$

$e^P = 1$, $e^x = 1$ belong

to $V_K|_{Q=1}$

$$A_u(Q=1) = (1-e^x)(1-e^P)$$

$$A_T(Q=1) = (1-e^x)(1-e^P)(e^{3P}-1)$$

String topological arguments show that the Q=1 degree 0 knot contact homology can be described in terms of the knot group $\pi = \pi_1(S^3 - K)$ as an algebra over $\mathbb{C}[e^{\pm x}, e^{\pm p}]$ freely generated by the elements of π subject to the following relations:

- $[e] = 1 - e^p$
- If l is the longitude and m the meridian then $[l\gamma] = [\gamma l] = e^x[\gamma]$, $[m\gamma] = [\gamma m] = e^p[\gamma]$
- $[\gamma_1 \gamma_2] - [\gamma_1 m \gamma_2] - [\gamma_1][\gamma_2] = 0$

Then if $\rho : \pi \longrightarrow GL(n)$ is a rep with

$$\rho(m) = \mathrm{diag}(e^{p_0}, 1, \dots, 1), \quad \rho(l) = \mathrm{diag}(e^{x_0}, *, *, \dots *)$$

then (e^{x_0}, e^{p_0}) lies in the augmentation variety.

In fact all Q=1 augmentations arises this way by a result of Cornwell.

For $SU(2)$ $\quad \rho'(m) = \mathrm{diag}(e^{p}, e^{-p})$

$\rho'(\ell) = \mathrm{diag}(e^{x}, e^{-x})$

$\rho(\gamma) = e^{\ell k(\gamma, K)} \rho'(\gamma)$ in

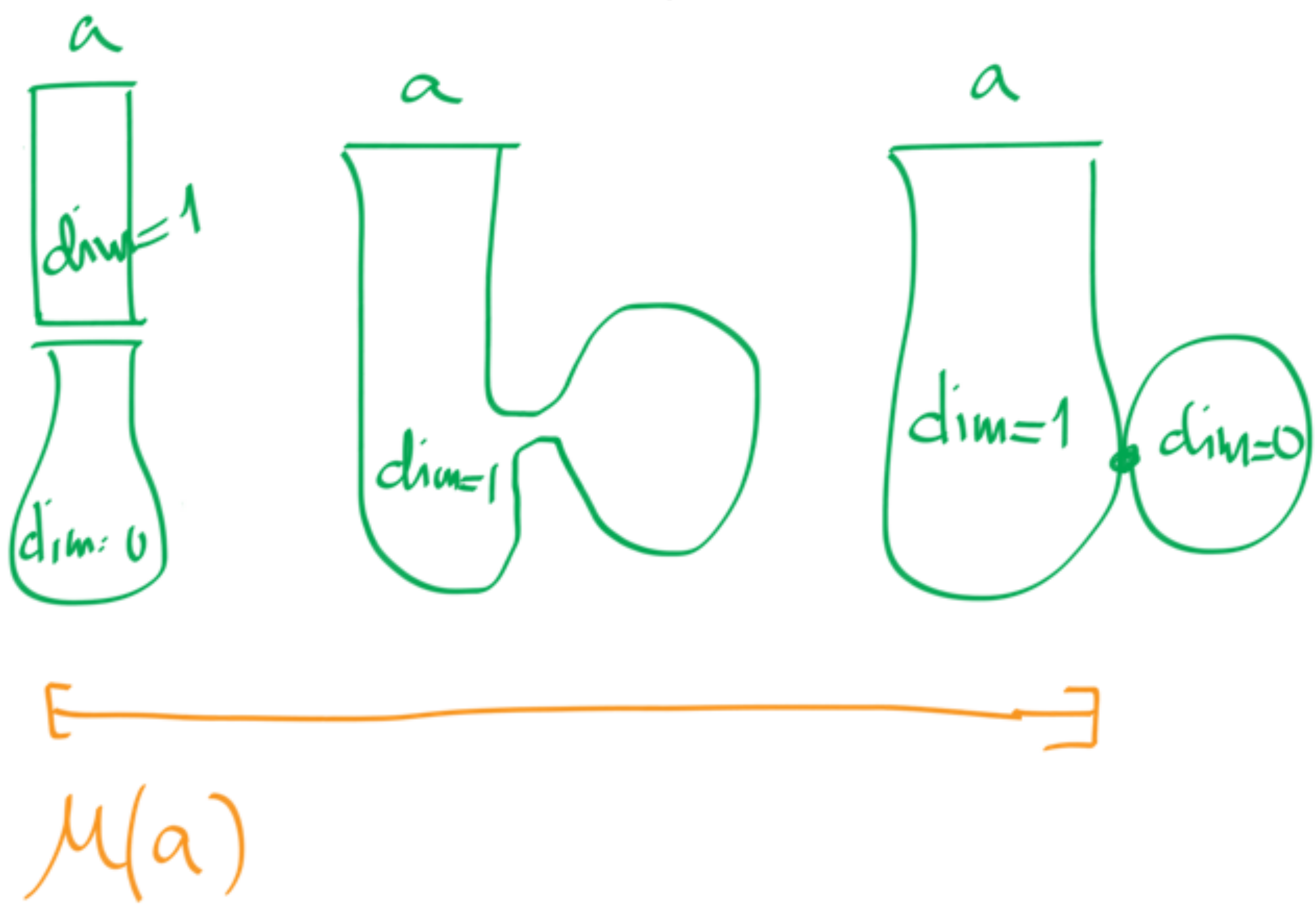
$GL(2)$ satisfies the above w/ $e^{p_0} = e^{2p}$, $e^{x_0} = e^{x}$

so

$(e^{2p} - 1)\, A\text{-poly} \mid A_K$.

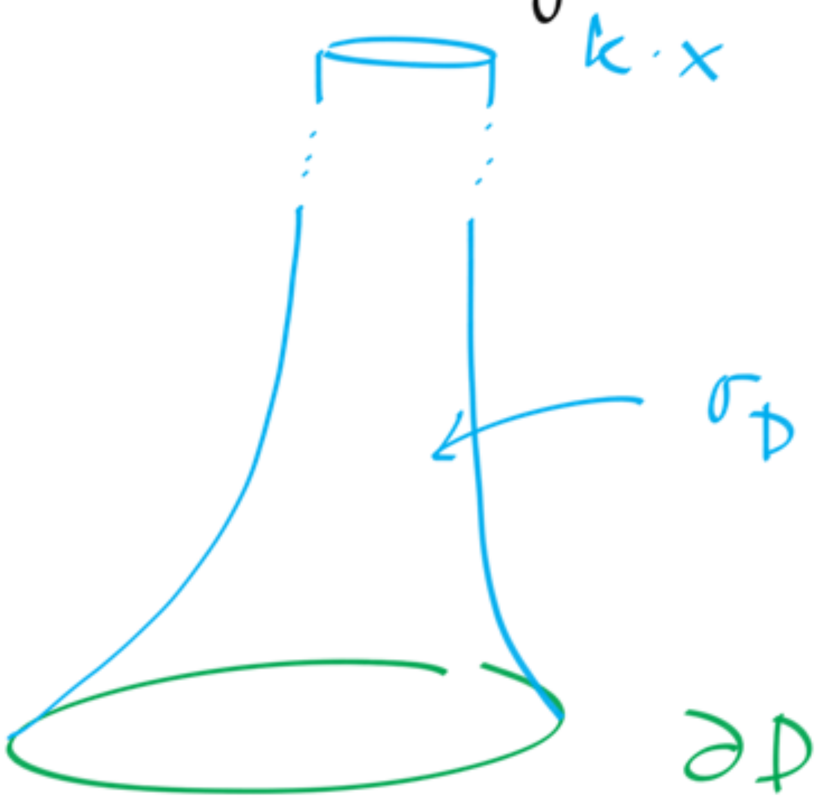
Augmentations and non-exact fillings

Consider $L_K \subset X$. The chain map argument from the exact case fails:

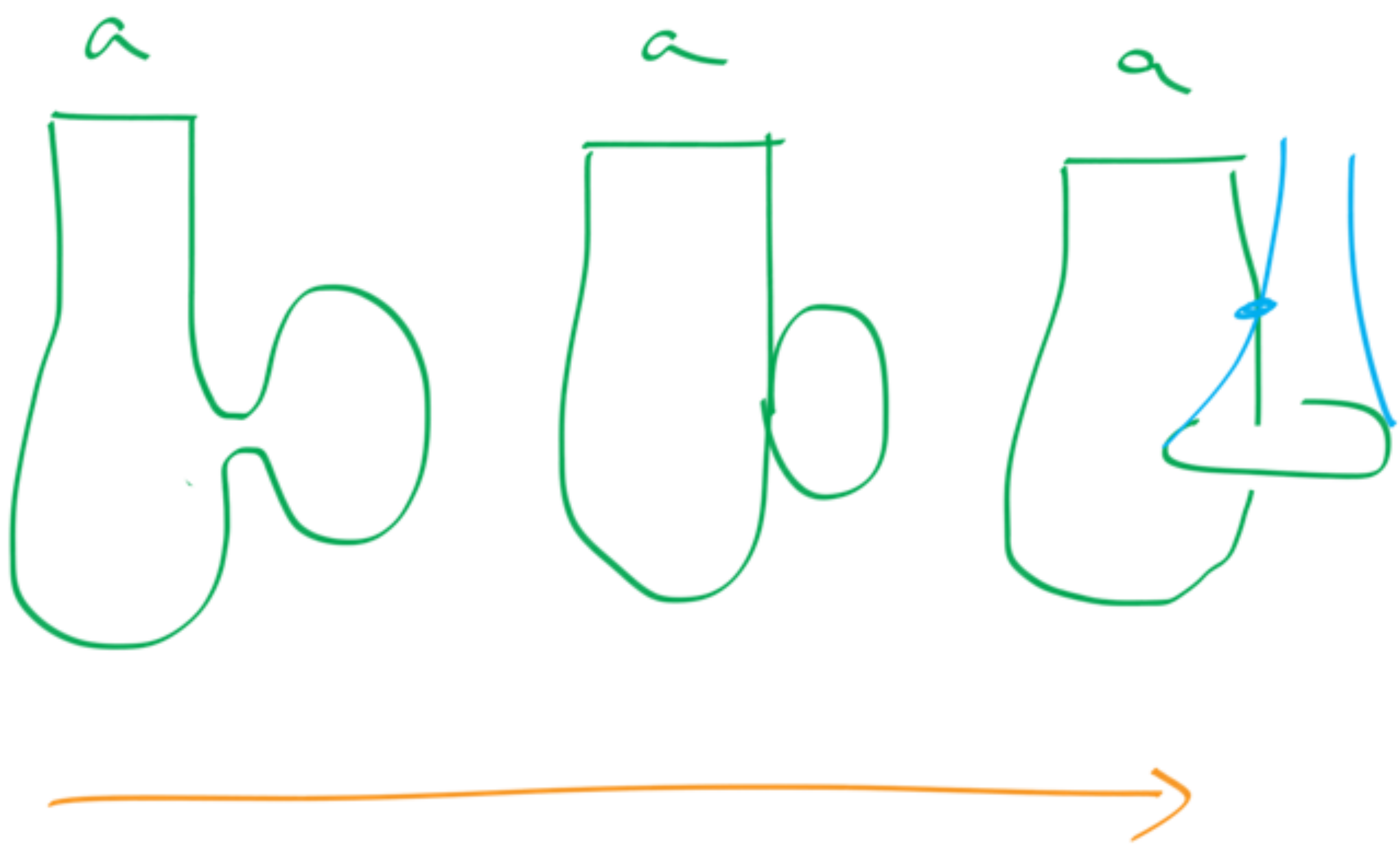


The bounding cochain techniques of Fukaya-Oh-Ohta-Ono allows us to overcome this problem:

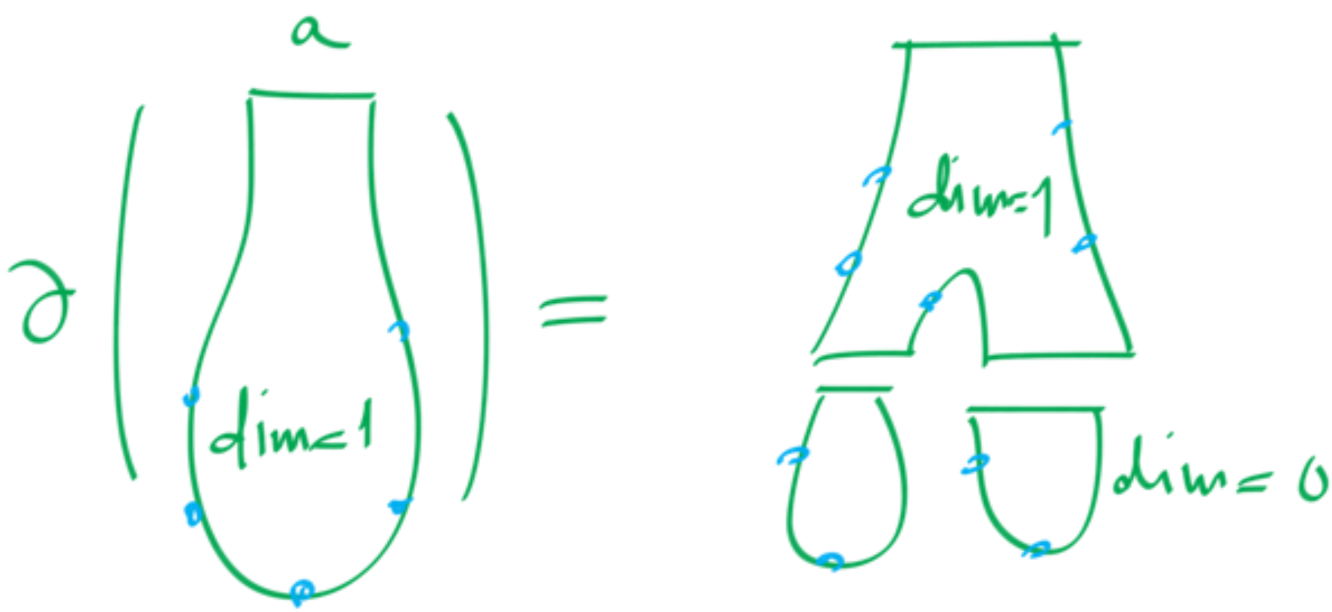
For each rigid disk D fix a chain σ_D in L_K that interpolates between ∂D and longitude x



Look at disks with insertions of σ_D at boundary



$\mu'(a)$



Thus if we define the differential counting disks w/ insertions then

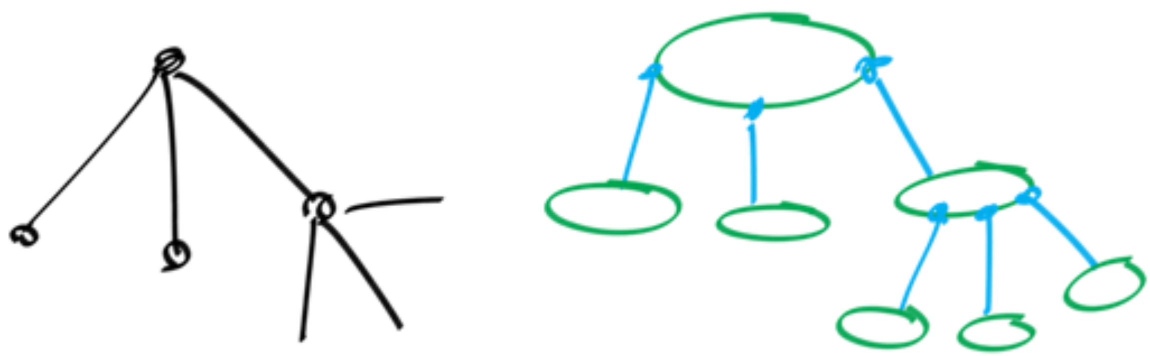
$$\varepsilon : \mathcal{A}_K \longrightarrow \mathbb{C}$$

augmentation

At ∞ $\sigma_D = k \cdot \xi$ hence

$p = \dfrac{\partial W_K}{\partial x}$ counts w/ insertion.

Note W_K counts trees



This corresponds to boundary term in path integral.

We conclude

$$p = \frac{\partial w_K}{\partial x}$$

parameterizes a branch of V_K.