Knot contact homology and topological strings

Tobias Ekholm

Uppsala University and Institut Mittag-Leffler

Overview

These lectures will explain how certain properties of topological open string theories in non compact CY 3-folds can sometimes be captured from comparatively simple holomorphic curve theories at infinity.

We will start by introducing the relevant theories and then show how to relate them.

Knot contact homology and topological string

- Symplectic and contact geometry
- CE-algebras
- Conormals of knots and links
- The augmentation variety
- Chern-Simons, topological string, and large N duality
- Augmentations and the GW-disk potential
- Legendrian SFT and quantization of the augmentation variety
- Recursion relation for the colored HOMFLY
- A-model topological recursion
- The skein relation via holomorphic curves

Symplectic manifolds

(X, ω) $\quad \dim X = 2n$

ω non-deg 2-form, $d\omega = 0$

Example

$X = T^* M$, $\quad \omega = -d\theta$

where θ is the Liouville form

$$\theta = p \cdot dq = \sum_{j=1}^{n} p_j \, dq_j .$$

Lagrangian submanifolds

$L \subset X, \ \dim(L) = n, \ \omega|_L = 0$

Example

$M \subset T^*M \ , \quad T^*_m M \subset T^*M$

X is exact if $\omega = d\theta$

$L \subset X$ exact if $\theta|_L = df$.

Contact manifolds

Y, $\dim(Y) = 2n-1$

α 1-form, $\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n-1} \neq 0$.

$\xi = \ker(\alpha)$ – contact structure

Legendrian submanifolds

$\Lambda \subset Y$, $\dim(\Lambda) = n-1$, $\alpha|_{\Lambda} = 0$

Example

$Y = ST^*M$, $\alpha = \theta = p \cdot dq$

$\Lambda = ST^*_m M$

Symplectization and contact boundary

$\mathbb{R}\times Y$, $\omega = d(e^t\alpha) =$
$= e^t(dt\wedge\alpha + d\alpha)$

$\mathbb{R}\times\Lambda \subset \mathbb{R}\times Y$ Lag.

If $(X,L) \approx (T,\infty)\times(Y,\Lambda)$

outside compact then

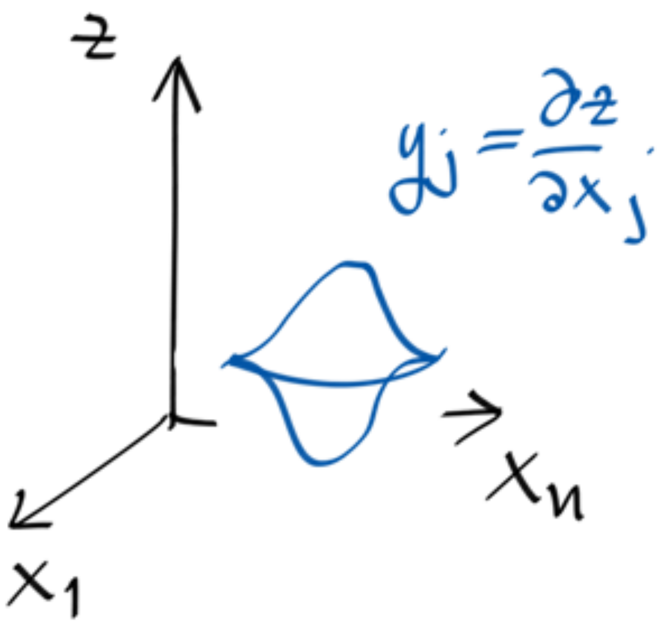
(Y,Λ) ideal contact boundary.

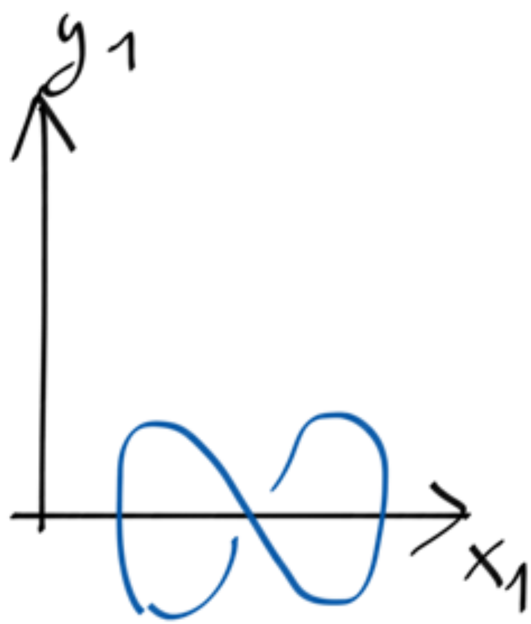
Darboux coordinate models

$$\mathbb{R}^{2n}_{st}, \quad \omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

$$\mathbb{R}^{2n-1}_{st}, \quad \alpha = dz - \sum_{j=1}^{n-1} y_j \, dx_j$$

Fronts





Contact homology dg-algebras

(Y, α) contact mfd

R - Reeb vector field

$d\alpha(\cdot, R) = 0$; $\alpha(R) = 1$

Reeb orbit — critical loop of

$$A : \mathrm{Loop}(Y) \to \mathbb{R}$$

$$A(\gamma) := \int_\gamma \alpha$$

Holomorphic curves in R x Y

Pick almost complex structure J on $\mathbb{R} \times Y$.

$J : \xi \to \xi$, $d\alpha(J -, \cdot) > 0$

$J \partial_t = R$.

$$\frac{\delta A}{\delta \eta} = \int_\gamma d\alpha(\dot{\gamma}, \eta)$$

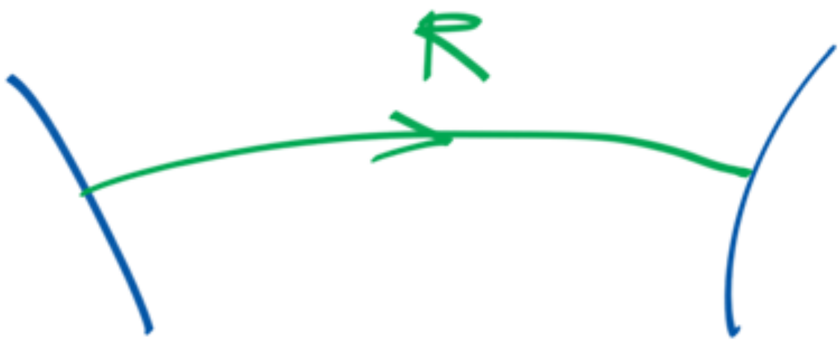
$$u : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times Y$$

$$\frac{\partial u}{\partial \sigma} + J \frac{\partial u}{\partial \tau} = 0 \approx \text{gradient.}$$

Lagrangian boundary conditions

$\Lambda \subset Y$ Legendrian

$\mathbb{R} \times \Lambda \subset \mathbb{R} \times Y$ Lagrangian



$$A : \mathrm{Path}(\Lambda, \Lambda) \longrightarrow \mathbb{R}$$

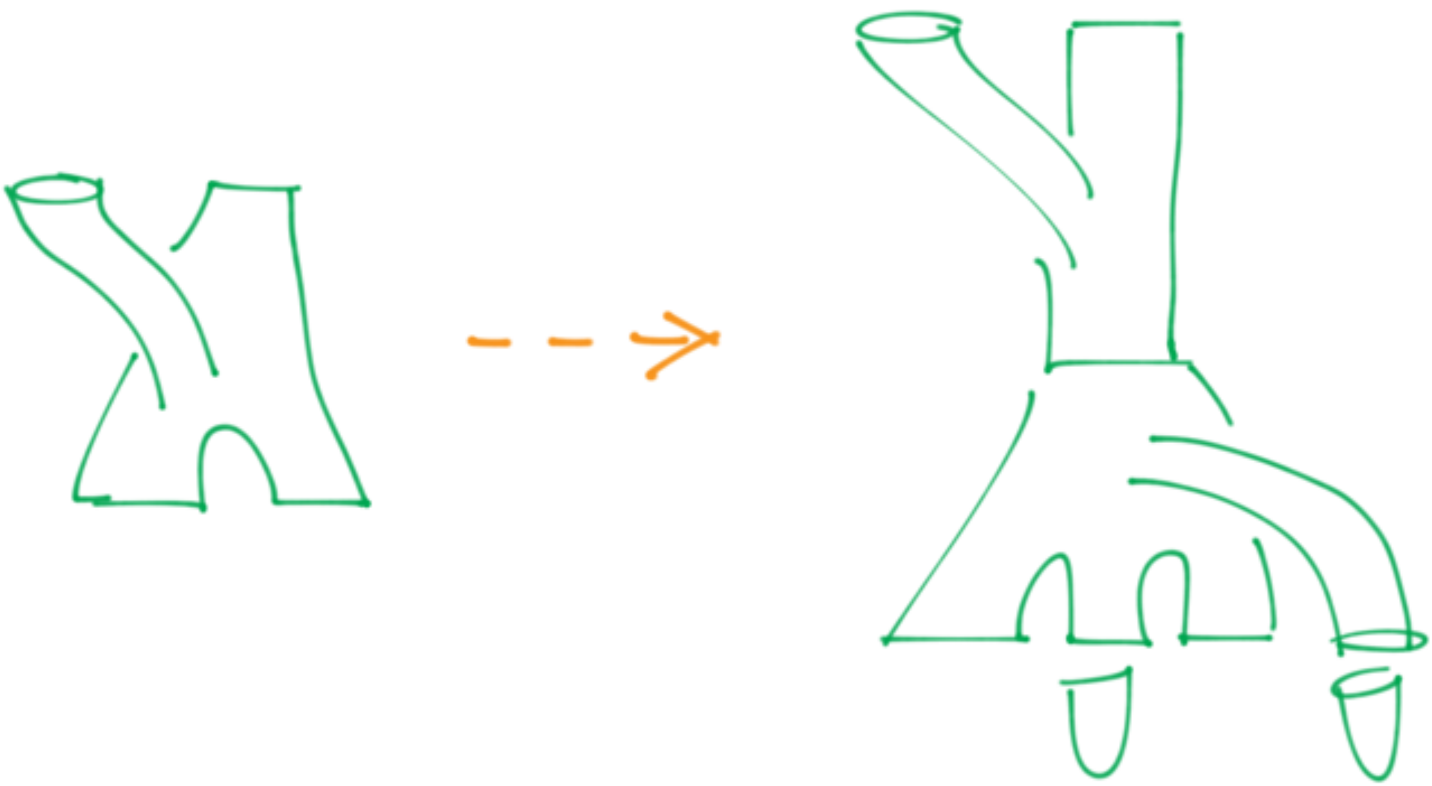
$$A(c) = \int_c \alpha$$

$$u : (\mathbb{R} \times [0,1], \mathbb{R} \times 0 \cup \mathbb{R} \times 1) \longrightarrow (\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$$

$$\frac{\partial u}{\partial \sigma} + J \frac{\partial u}{\partial \tau} = 0$$

SFT-compactness and dg-algebras

A sequence of finite energy punctured holomorphic curves in $(\mathbb{R} \times Y, \mathbb{R} \times \Lambda)$ has a subsequence that converges to a several level holomorphic building.



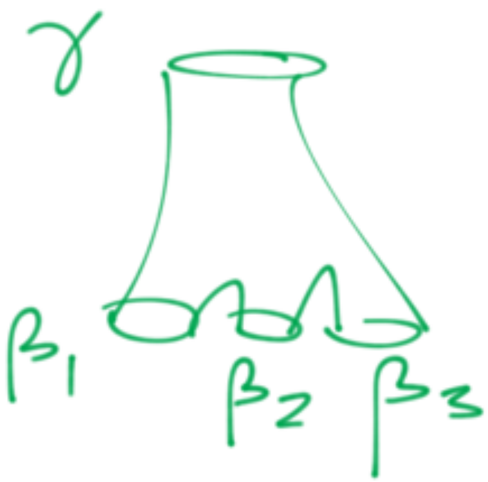
Orbit algebra

$$c_1(\xi) = 0 .$$

$$Q(Y) = \mathbb{C}\langle \text{good Reeb orbits} \rangle$$

$$|\gamma| = CZ(\gamma) + (n-3)$$

$d : Q(Y) \circlearrowleft$ counts



$$\dim \mathcal{M}(\gamma, \beta_1\beta_2\beta_3) = |\gamma| - \sum |\beta_j| = 1$$

$$d\gamma = \beta_1\beta_2\beta_3 + \dots$$

$d^2 = 0$:

1-dim orient mfd.

$= \partial\left(\mathcal{M}/\mathbb{R}\right)$

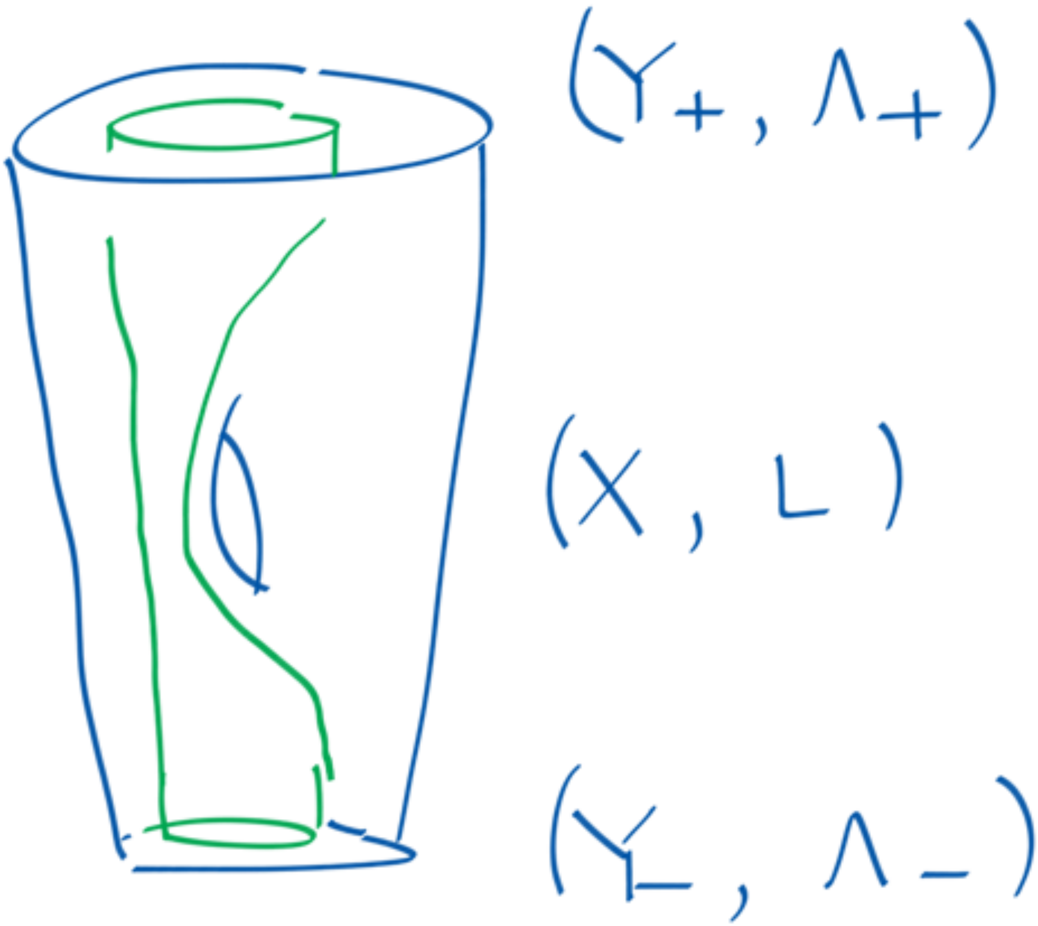
$CE(\Lambda) = \mathbb{Q}(Y)\langle \text{Reeb chords} \rangle$

$d : CE(\Lambda) \circlearrowleft$ counts

$\dim = |a| - \sum |b_i| = 1$

$da = b_1 b_2 b_3 + \dots$

Functoriality



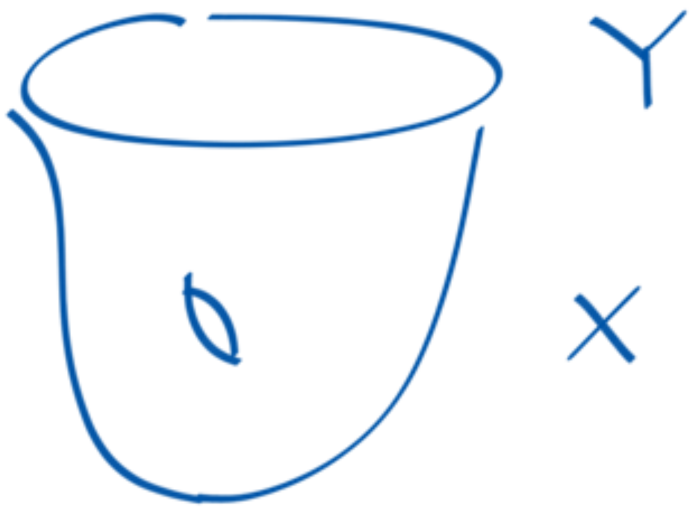
$$\Phi_{(X,L)} : CE(\Lambda_+) \longrightarrow CE(\Lambda_-)$$

counts

dim = 0 .

Orbit augmentations

If Y bounds

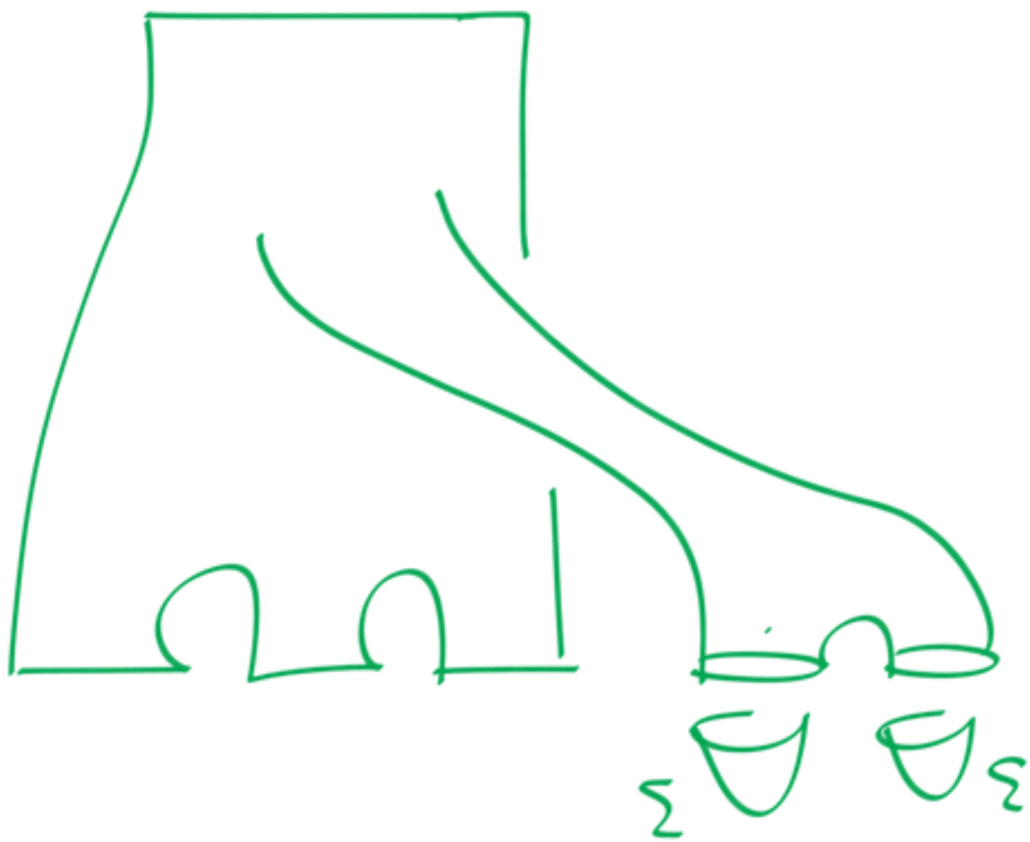


$$\varepsilon = \Phi_X : Q(Y) \longrightarrow \mathbb{C}$$

chain map gives augmented

$$d : CE(\Lambda) \longrightarrow CE(\Lambda)$$

which counts

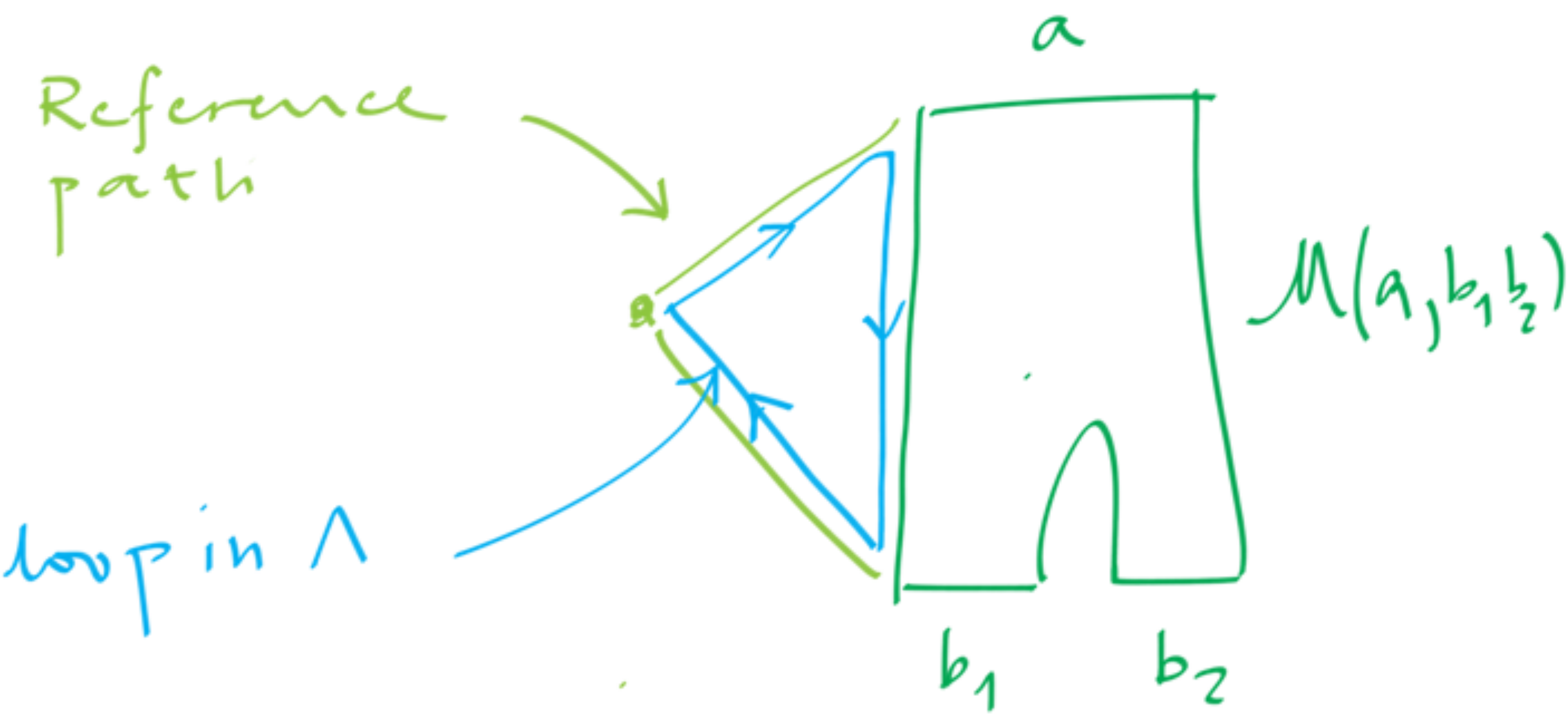


and gives $CE(\Lambda)$ as algebra over $\mathbb{C}$.

Coefficients for CE

Chains on based loop space

$C_*(\Omega\Lambda)$



$$da = [\mathcal{M}(a; b_1, b_2)] + \cdots =$$

$$= \sum \sigma_1 b_1 \sigma_2 b_2 \sigma_3 \quad + \ldots$$

Group algebra of 2nd homology

$$\mathbb{C}\left[H_2(Y,\Lambda)\right]$$

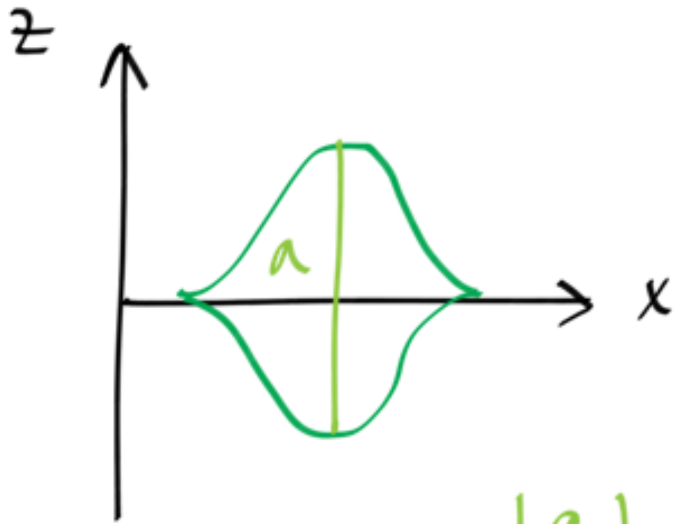
$\tau \in H_2(Y,\Lambda)$

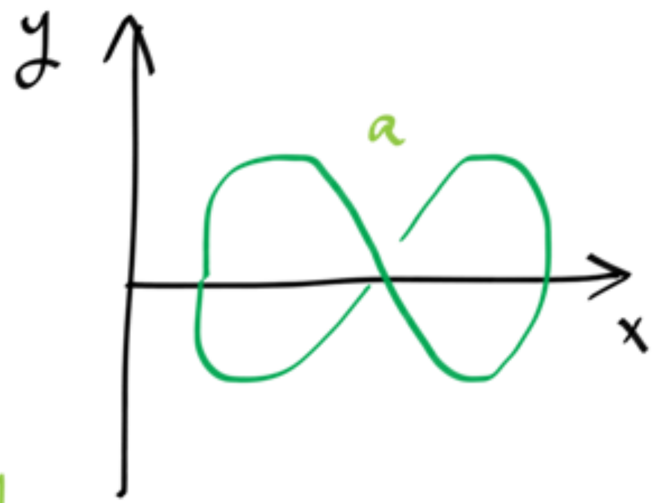
$\dim \mathcal{M}(a, b_1 b_2) = 0$

$$da = e^{\tau} b_1 b_2 + \cdots$$

Examples

Unknot in $\mathbb{R}^3_{st}$ (or S^3_{st})

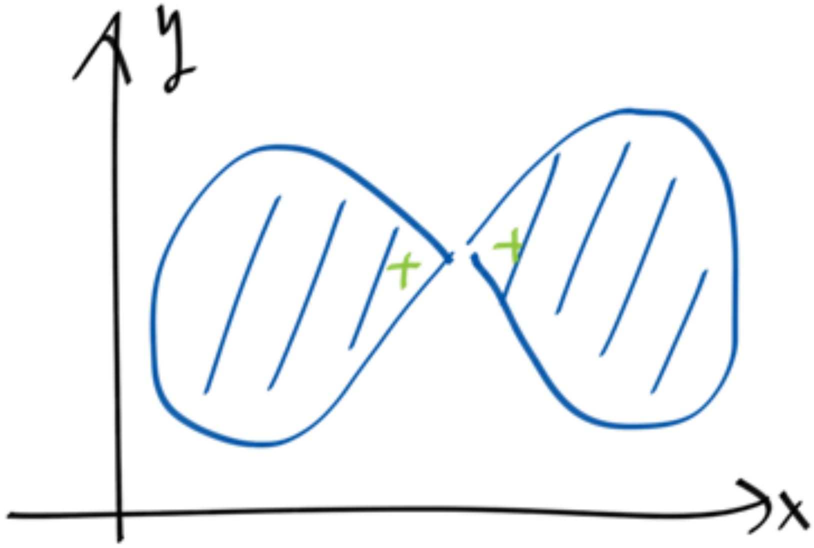




$|a| = 1$

$C_*(\Omega S^1) = \mathbb{C}[t, t^{-1}]$

$da = 1 - t$



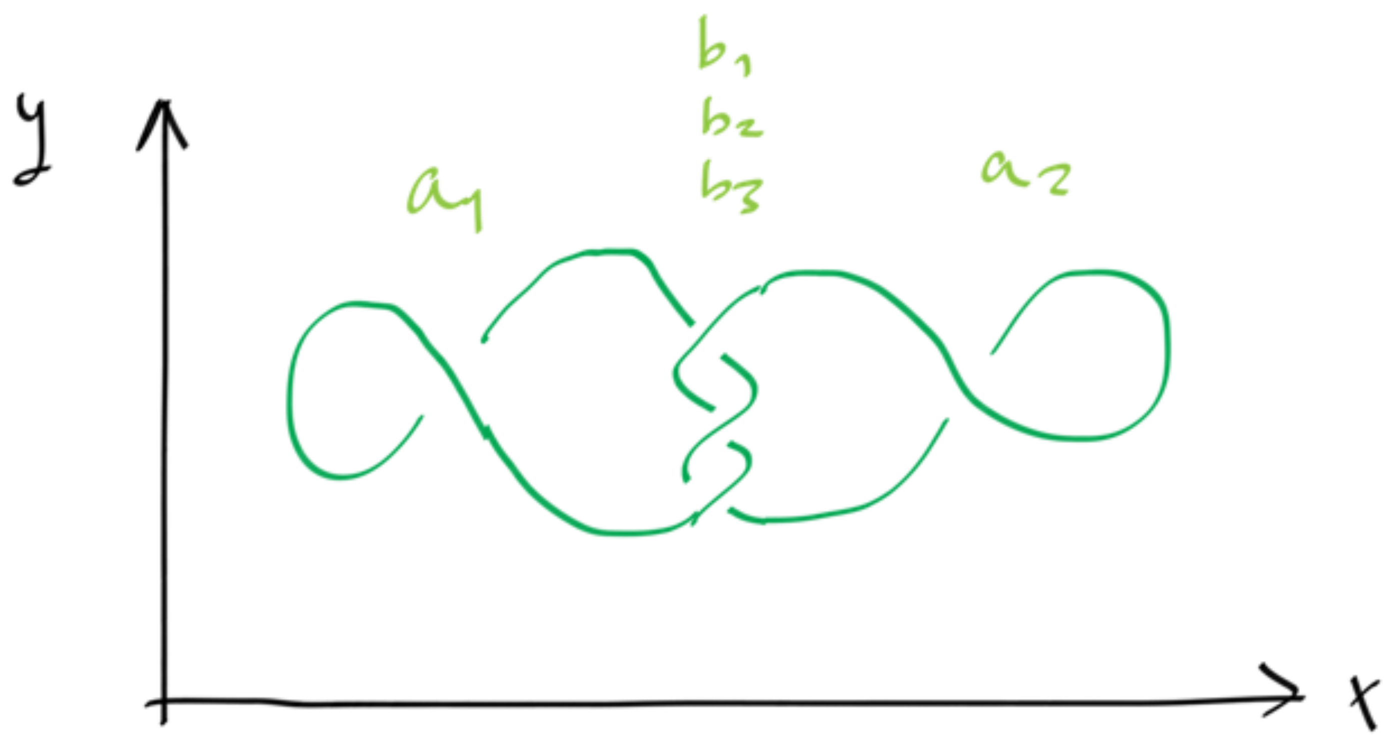
$\mathbb{C}$-coeff, set $t=1$

$da=0$, $CE(\Lambda) \approx \mathbb{C}[a]$.

$t=1$ induced from Lagr. disk

Orientations uses spin str on Λ, $t=1$ corresponds to the spin str that extends to L.

Trefoil in $\mathbb{R}^3_{st}$



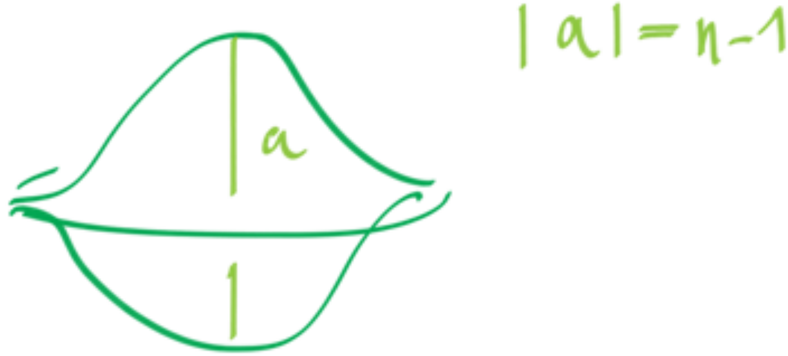
$|a_j| = 1, \quad |b_j| = 0$

$$da_1 = 1 - b_1 - b_3 - b_3 b_2 b_1$$
$$da_2 = 1 + b_1 + b_3 + b_1 b_2 b_3$$

5 augmentations from torus filling

$$CE(\Lambda) \simeq \mathbb{C}[b_1, b_2, b_3] / (1 - b_1 - b_3 - b_1 b_2 b_3)$$

Unknot in dim $n-1$



$$C_*(\Omega\Lambda) \approx \mathbb{C}[y] \qquad |y|=n-2$$

$$da = y$$

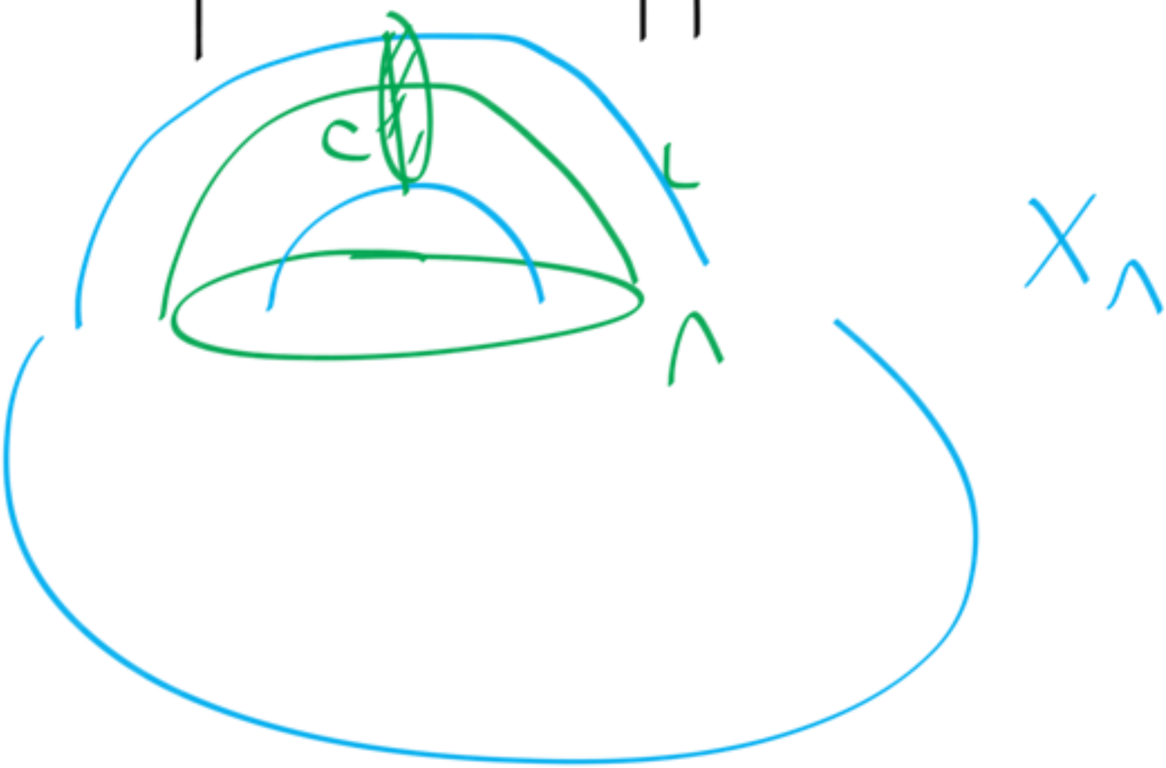
with $\mathbb{C}$-coeff $\quad da = 0$

$$CE(\Lambda) \approx \mathbb{C}[a].$$

Legendrian surgery and Floer cohomology

$CE(\Lambda)$ with $\mathbb{C}$-coeffs

computes wrapped Floer cohomology



$$CE(\Lambda) \approx CW(c)$$

$$\Lambda = \text{unknot} \Rightarrow X_\Lambda = T^*S^n$$

$$CW(c) \approx C_*(\Omega S^n) \approx \mathbb{C}[a]$$

$\Lambda =$ trefoil $\Rightarrow X_\Lambda = \{1+x+y+xyz=0\}$

$CW(\Lambda) = CE(\Lambda)$ proves mirror symmetry for X_Λ.

$CE(\Lambda)$ with $C_*(\Omega\Lambda)$-coeff computes $CW(c)$ in X'_Λ

$\Lambda =$ Leg unknot $\quad X'_\Lambda = T^*\mathbb{R}^n$

$CW(\Lambda) = \mathbb{C}$.

Knot contact homology - definition

$K \subset S^3$ link

$L_K \subset T^*S^3$

$L_K = \{p \in T^*_K S^3 : p|_{TK} = 0\} \approx S^1 \times \mathbb{R}^3$

$\Lambda_K = L_K \cap ST^*S^3 \approx S^1 \times S^1$

$$0 \to H_2(ST^*S^3) \to H_2(ST^*S^3, \Lambda_K) \to H_1(\Lambda_K) \to 0$$

$$t \longrightarrow t, x, p \longrightarrow x, p$$

$$\mathbb{C}\left[H_2(ST^*S^3, \Lambda_K)\right] \cong$$

$$\mathbb{C}\left[e^{\pm x_j}, e^{\pm p_j}, Q^{\pm 1}\right]_{j=1,\dots,k}$$

$$K = K_1 \cup \dots \cup K_k \, , \quad Q = e^t \, .$$

Knot contact homology

$$\mathcal{A}_K := CE(\Lambda_K)$$

algebra over $\mathbb{C}[e^{\pm x_i}, e^{\pm p_i}, Q]$

Knot contact homology - loop space coefficients

K knot. Fill Λ_K w/ $L_K \subset T^*S^3$
then $Q=1$.

Use coeff in

$$C_*(\Omega\Lambda_K) \approx \mathbb{C}[e^{\pm x}, e^{\pm p}]$$

but $e^{\pm x}, e^{\pm p}$ does <u>not</u>

commute w/ Reeb chords

Let $\Sigma_q = ST^*_q S^3$
$q \notin K$

<u>Thm</u> (E.-Ng-Shende, '16)

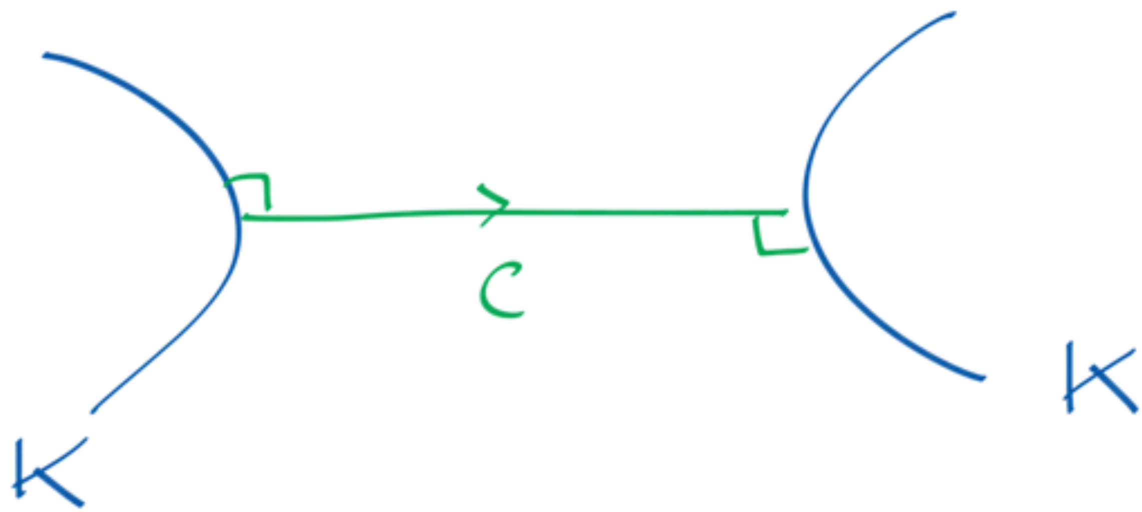
$CE(\Lambda_K \cup \Sigma_q)$ is a complete knot invariant (knows $\pi_1(S^3-K)$ & peripheral structure)

The proof uses a connection to string topology developed by Cieliebak-E-Latschev-Ng.

Knot contact homology - calculation

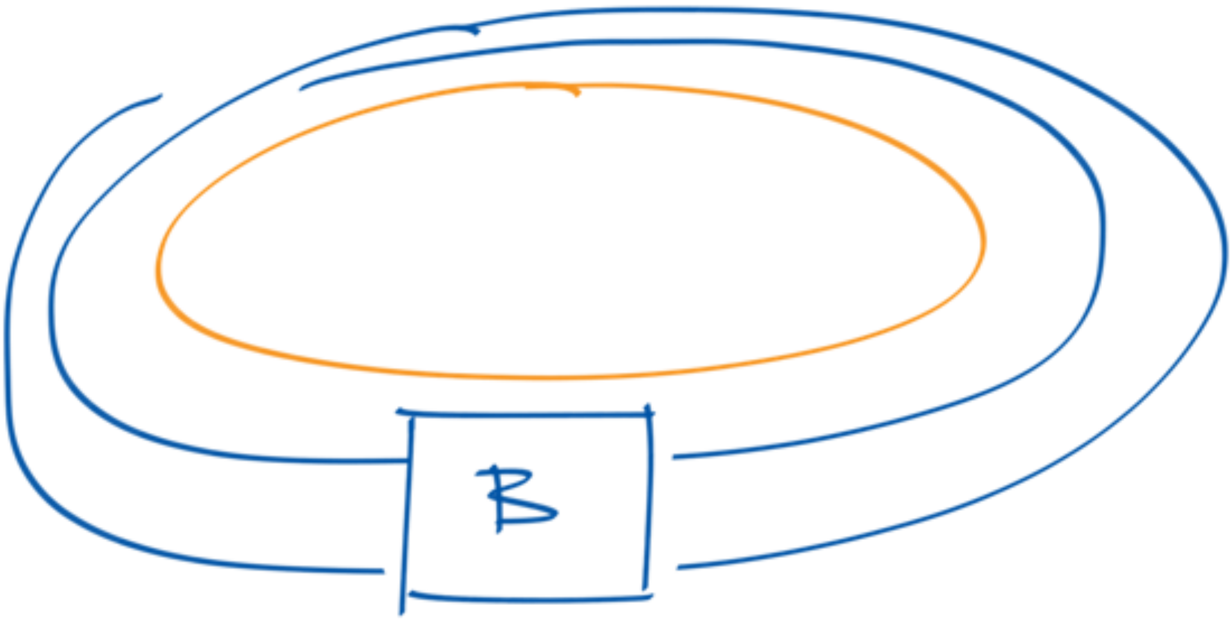
Reeb chords on $\Lambda_K \subset ST^*S^3$ $\sim$

Oriented binormal geodesics on K



$|c| =$ Morse index (c)
$(\in \{0,1,2\}$ in flat $\mathbb{R}^3)$

To compute $\mathcal{A}_K$, braid K around the unknot



then $\Lambda_K \subset N(\Lambda_u) \approx J^1(\Lambda_u)$

and holomorphic disks on Λ_K can be computed from disks on Λ_u and flow trees of $\Lambda_K \subset J^1(\Lambda_u)$.

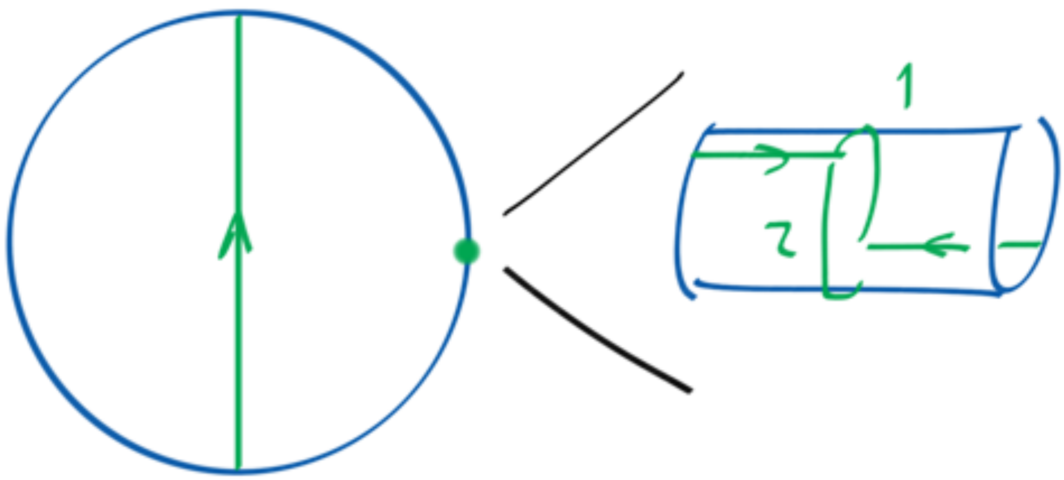
For braid on n strands:

$n(n-1)$ deg 0 chords

$n(2n-1)$ deg 1

n^2 deg 2

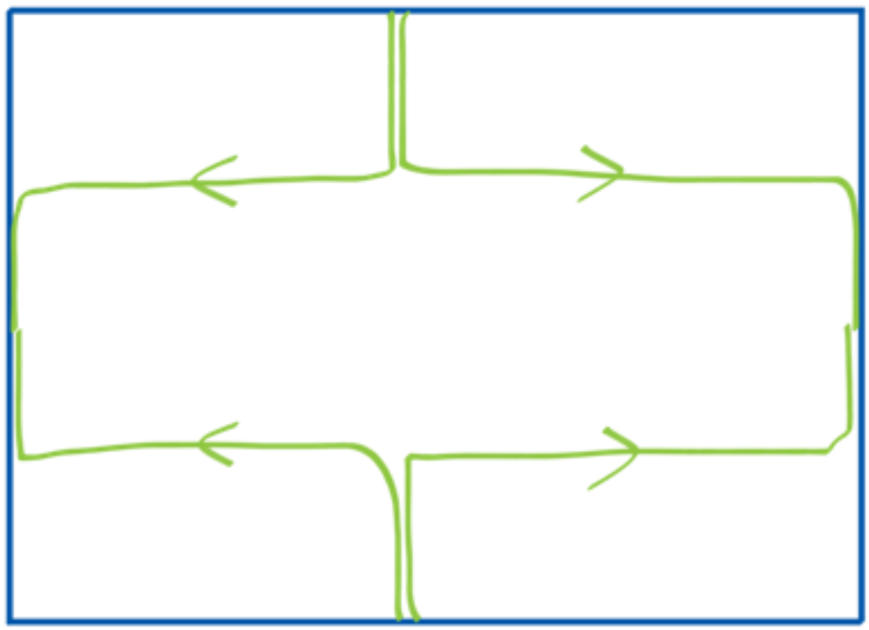
Knot contact homology of the unknot



$$\mathcal{A}_U = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle c, e\rangle \ ; |c|=1, |e|=2$$

$$\partial e = c - c = 0; \quad \partial c = 1 - e^x - e^p + Qe^x e^p.$$

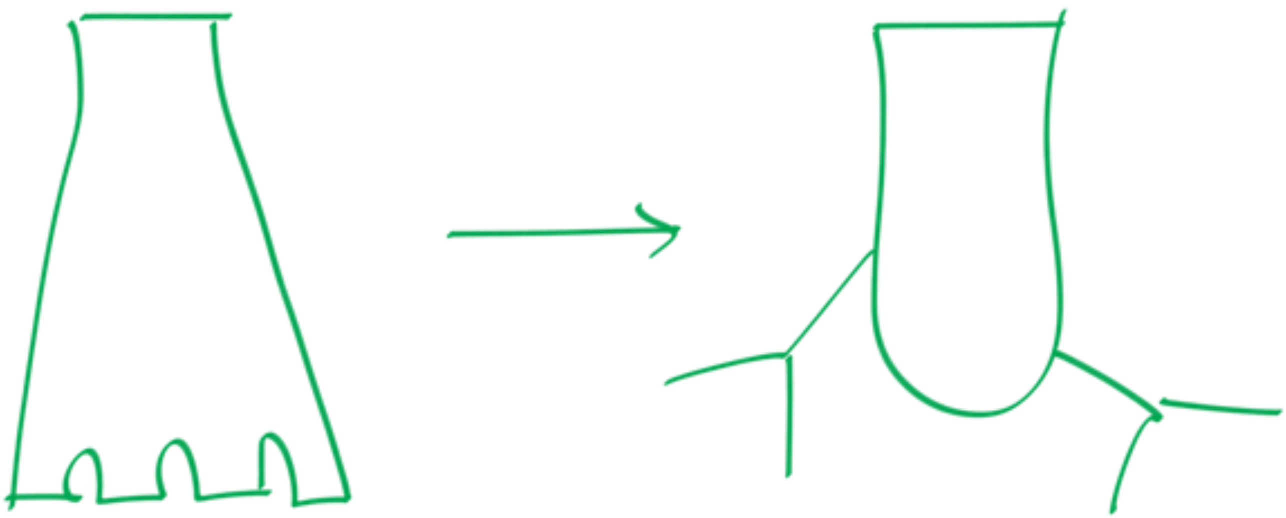
Boundaries of the four disks on the conormal



Knot contact homology of the trefoil

General: as $\Lambda_K \longrightarrow \Lambda_U$

disk on $\Lambda_K \longrightarrow$ disk on Λ_U w/ tree

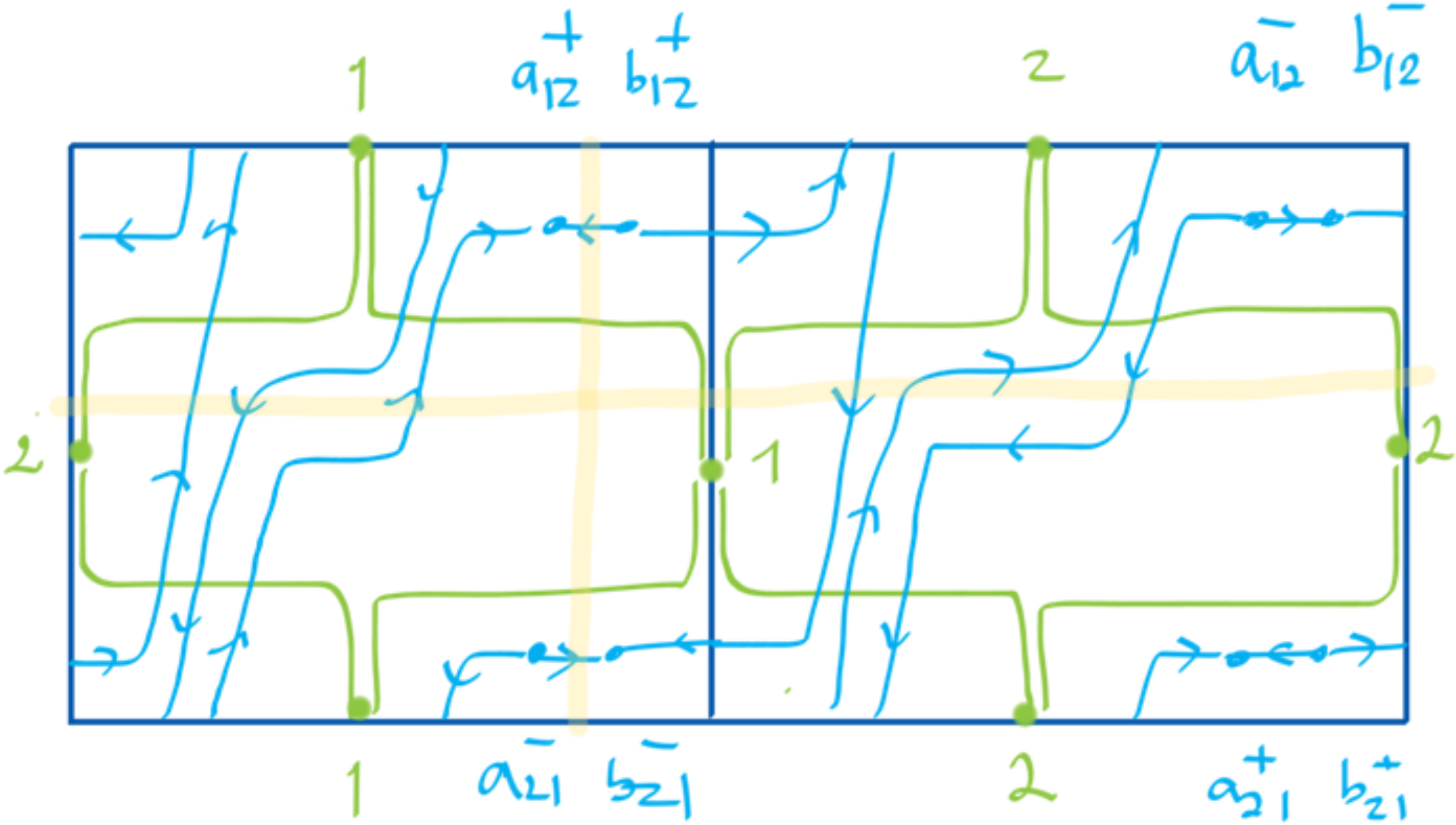


$$\mathcal{A}_T = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle a_{12}, a_{21}, b_{12}, b_{21}, c_{ij}, e_{ij} \rangle$$

$ij = 1,2$

$|a_{ij}| = 0, \quad |b_{ij}| = |c_{ij}| = 1$

$|e_{ij}| = 2$


1
a_{12}^{+} b_{12}^{+}
2
a_{12}^{-} b_{12}^{-}
2
1
2
1
a_{21}^{-} b_{21}^{-}
2
a_{21}^{+} b_{21}^{+}

Differential in degree 1

$$db_{12} = e^{-x}a_{12} - a_{21}$$

$$db_{21} = e^{x}a_{21} - a_{12}$$

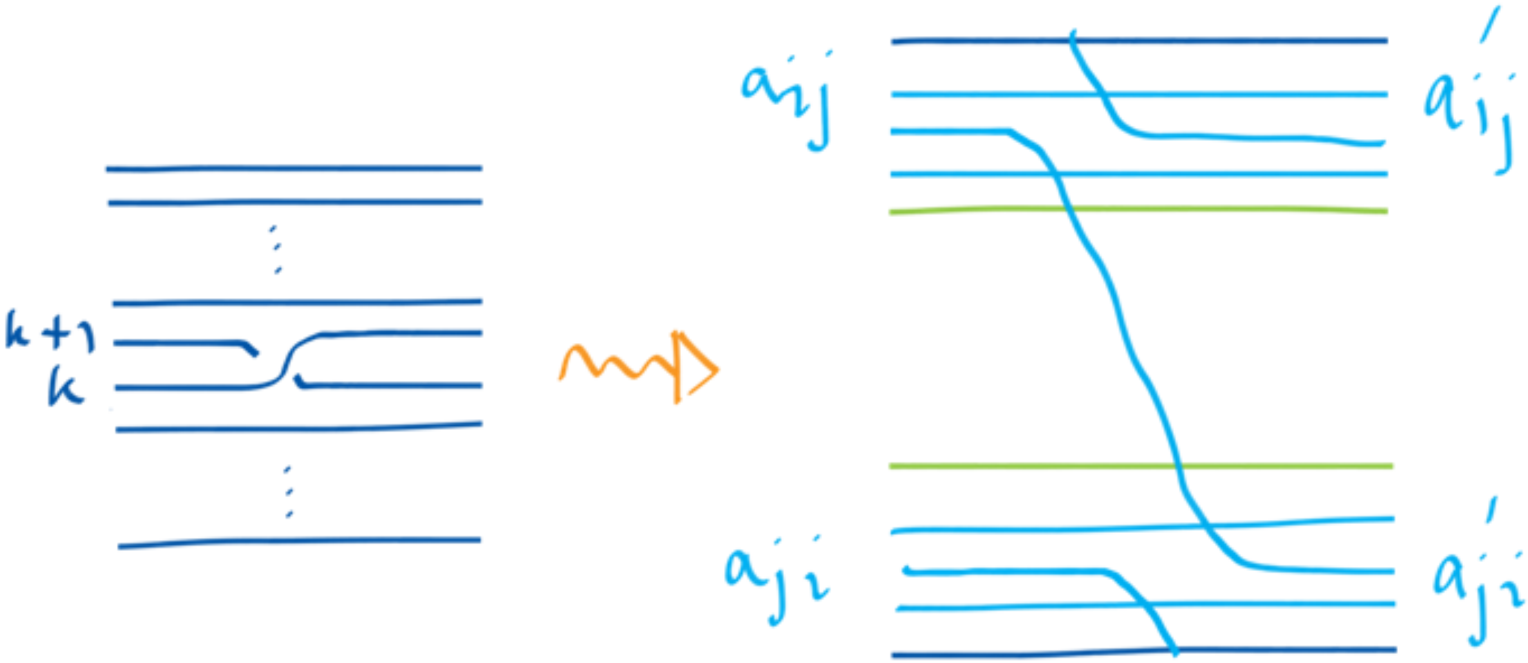
$$dc_{11} = e^{P}e^{x} - e^{x} - (2Q - e^{P})a_{12} - Qa_{12}^{2}a_{21}$$

$$dc_{12} = Q - e^{P} + e^{P}a_{12} + Qa_{12}a_{21}$$

$$dc_{21} = Q - e^{P} + e^{P}e^{x}a_{21} + Qa_{12}a_{21}$$

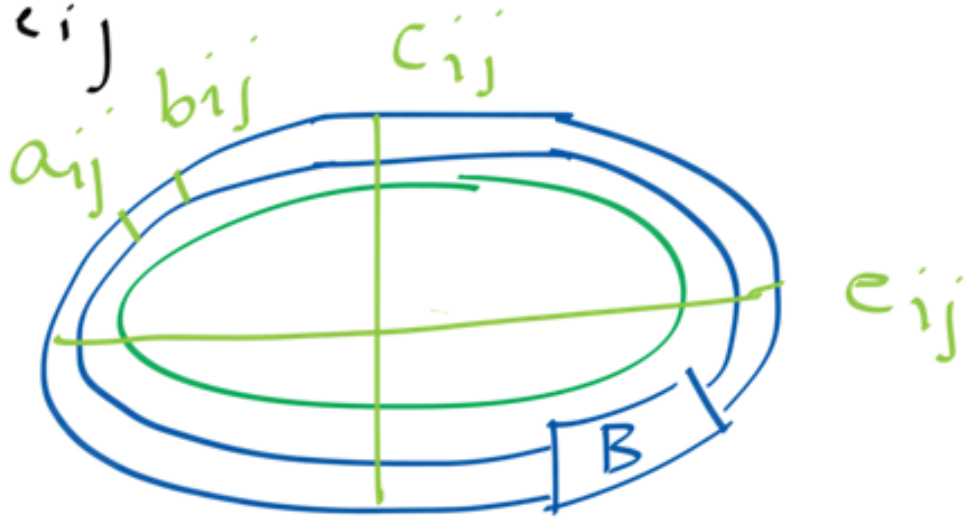
$$dc_{22} = e^{P} - 1 - Qa_{21} + e^{P}a_{12}a_{21}$$

General links



A twist in the braid gives rise to a homomorphism of the algebra of degree 0 chords that is a linear change of variables except for two quadratic terms. Composing these gives explicit matrices Φ^L_B and Φ^R_B that gives the effect of these isomorphism on an algebra corresponding to a braid with one extra trivial strand.

Write generators as a_{ij}, b_{ij}, c_{ij}, e_{ij}



Let $A = a_{ij}$ etc.

<u>Then</u> d on $\mathcal{A}_K$ is given by

$$dA = 0$$

$$dB = -\lambda^{-1}\cdot A\cdot\lambda + \phi_B^L \cdot A \cdot \phi_B^R$$

$$dC = A\cdot\lambda + A\cdot\phi_B^R$$

$$dE = B\cdot(\phi_B^R)^{-1} + B\cdot\lambda^{-1} - \phi_B^L\cdot C\cdot\lambda^{-1} + \lambda^{-1}\cdot C\cdot(\phi_B^R)^{-1}$$

Augmentations

Consider $\mathcal{A}_K$ as a family of algebras over

$$(\mathbb{C}^*)^{2k} \times \mathbb{C}^*$$

where pts correspond to values of $(e^{x_j}, e^{p_j}, Q)_{j=1}^{k}$

An <u>augmentation</u> is a chain map

$$\varepsilon : \mathcal{A}_K \longrightarrow \mathbb{C}$$

i.e. $\varepsilon \circ d = 0$.

The augmentation variety V_K is the alg. closure of

$$\left\{ (e^{x_j}, e^{p_j}, Q)_{j=1}^{k} : A_K \text{ has aug} \right\}.$$

If $k=1$ the augmentation polynomial A_K is the polynomial of V_K.

and can be computed by elimination theory.

Examples

$$A_u = 1 - e^x - e^p + Q e^x e^p$$

$$A_T = \left(e^{4p} - e^{3p}\right) e^{2x}$$
$$+ \left(e^{4p} - Q e^{3p} + 2Q^2 e^{2p} - 2Q e^{2p} - Q^2 e^p + Q^2\right) e^x$$
$$+ \left(-Q^3 e^p + Q^4\right)$$

Properties

The A-polynomial divides $A_K|_{Q=1}$. Other factors from flat $GL(n)$ conn.

We will eventually relate V_K to open string (GW-theory).