Exotic Einstein metrics on S^6 and $S^3 \times S^3$, nearly Kähler 6-manifolds and G_2 holonomy cones

Mark Haskins

Imperial College London

joint with Lorenzo Foscolo,
Stony Brook

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G_2 cones and nearly Kähler 6–manifolds

Riemannian cone over smooth compact Riemannian manifold M:

$C(M) = \mathbb{R}^+ \times M$ endowed with the Riemannian metric $g_c = dr^2 + r^2 g$

$\mathrm{Hol}(C) \subset \mathrm{G}_2 \iff$ parallel (and hence closed) 3-form φ and 4-form $*\varphi$

$$\varphi = r^2 dr \wedge \omega + r^3 \mathrm{Re}\,\Omega, \qquad *\varphi = -r^3 dr \wedge \mathrm{Im}\,\Omega + \tfrac{1}{2} r^4 \omega^2$$

$d\varphi = 0 = d * \varphi \iff$ the SU(3)–structure (ω, Ω) on M satisfies

$$\begin{cases} d\omega = 3\,\mathrm{Re}\,\Omega \\ d\mathrm{Im}\,\Omega = -2\,\omega^2 \end{cases} \qquad \text{(NK)}$$

A 6–manifold M endowed with an SU(3)–structure satisfying (NK) is called a (strict) **nearly Kähler** (nK) 6–manifold.

- every nK 6–manifold M is Einstein with $\mathrm{Scal} = 30 \Longrightarrow$ if M is complete, then it is compact with $|\pi_1(M)| < \infty \Longrightarrow$ wlog can assume $\pi_1(M) = 0$.
- nK 6-manifolds and real Killing spinors
- nK 2n-manifolds and Gray–Hervella classes of almost Hermitian manifolds

The 4 examples known!

- $S^6 \subset \mathrm{Im}\,\mathbb{O}$: dates back to at least **1947** (e.g. C. Ehresmann, A. Kirchoff)
- **1968**, in Gray–Wolf's classification of 3–symmetric spaces in 6d have

  $S^3 \times S^3 = \mathrm{SU}(2)^3/\triangle\mathrm{SU}(2) \quad CP^3 = \mathrm{Sp}(2)/\mathrm{U}(1)\times\mathrm{Sp}(1) \quad F_3 = \mathrm{SU}(3)/T^2$

  A 3-*symmetric space* has an automorphism σ with $\sigma^3 = 1$: define a homogeneous almost complex structure on $\ker(\sigma^2 + \sigma + \mathrm{Id})$ by

$$J = \frac{1}{\sqrt{3}}(2\sigma + \mathrm{Id})$$

- Connection with G_2–holonomy noted only in the 1980's, e.g. Bryant's **1987** first explicit example of a full G_2–holonomy metric is $C(F_3)$
- G_2–cones give local models for isolated singularities of G_2–spaces
- Infinitely many Calabi–Yau, hyperkähler and Spin(7)–cones. Why not G_2?
- **2005**, Butruille: the four known examples are the only homogeneous nK 6–manifolds
- **2006**, Bryant: local generality (via Cartan-Kähler theory) of 6d nK structures same as for 6d Calabi–Yaus (also Reyes Carrion thesis 1993)

Main Theorem and possible proof strategies

Main Theorem (Foscolo–Haskins, to appear **Annals of Mathematics**)
There exists a complete inhomogeneous nearly Kähler structure on S^6 *and on* $S^3 \times S^3$.

Two natural strategies to find nK 6–manifolds:

- Symmetries: cohomogeneity one nK 6–manifolds.
- Desingularisation of singular nK spaces.

Our proof uses elements from **both** viewpoints.

Simplest singular nK spaces: sine-cones (reduced holonomy $SU(3) \subset G_2$)
cross-section of a "split" G_2 *cone, i.e.* $\mathbb{R} \times C$ *for* C *a Calabi–Yau cone*

(N^5, g_N) smooth Sasaki–Einstein $\Leftrightarrow$ $C(N)$ is a Calabi–Yau (CY) cone

The **sine-cone** over N: $SC(N) = [0, \pi] \times N$ endowed with the Riemannian metric $dr^2 + \sin^2 r\, g_N$ (aka **metric suspension** of N)

$SC(N)$ is nK but has 2 isolated singularities each modelled on CY cone $C(N)$

Idea: *Try to desingularise* $SC(N)$ *by replacing conical singularities with smooth asymptotically conical CY 3–folds.*

A simple nK sine-cone and desingularisations

A simple example comes from the so-called **conifold**:

- $C(N)$ is the **conifold** $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$
- $N = \mathrm{SU}(2) \times \mathrm{SU}(2)/\triangle \mathrm{U}(1)$ which is diffeomorphic to $S^2 \times S^3$

$C(N)$ has 2 Calabi–Yau desingularisations (*Candelas–de la Ossa, Stenzel*)

- $Y =$ the small resolution $\simeq$ total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$
  vertex of cone replaced with a totally geodesic holomorphic $\mathbb{P}^1$
- $Y' =$ the smoothing $\simeq T^*S^3$
  vertex of cone replaced with a totally geodesic special Lagrangian S^3

The **conifold** itself and its asymptotically conical CY desingularisations are **cohomogeneity one**, i.e. $\exists$ some Lie group G acting isometrically with generic orbit of codimension one

Two examples above have only 1 singular orbit: $\mathbb{P}^1$ or S^3

Sine-cone $C(N)$, conifold and its desingularisations are cohomogeneity one.

So obvious question is: **Can we desingularise this sine-cone as a cohomogeneity one space?**

Cohomogeneity one nK 6-manifolds

2010, **Podestà–Spiro**: *potential complete cohomogeneity one nK* 6*–mfds* M. Compact Lie group G acts with K, K_1, K_2 as its principal and singular isotropy groups. Principal orbit is G/K; 2 singular orbits G/K_i.

G	K	K_1	K_2	M
$SU(2)\times SU(2)$	$\triangle U(1)$	$\triangle SU(2)$	$\triangle SU(2)$	$S^3\times S^3$
$SU(2)\times SU(2)$	$\triangle U(1)$	$\triangle SU(2)$	$U(1)\times SU(2)$	S^6
$SU(2)\times SU(2)$	$\triangle U(1)$	$U(1)\times SU(2)$	$SU(2)\times U(1)$	CP^3
$SU(2)\times SU(2)$	$\triangle U(1)$	$U(1)\times SU(2)$	$U(1)\times SU(2)$	$S^2\times S^4$
$SU(3)$	$SU(2)$	$SU(3)$	$SU(3)$	S^6

$\Rightarrow N_{1,1} = SU(2)\times SU(2)/\Delta U(1)$ is only possible interesting principal orbit!

Rough outline of proof

1. Understand the local theory for cohomogeneity one nK 6–mfds in neighbourhood of principal orbit $N_{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\Delta\mathrm{U}(1)$.
   - Our approach: study the geometry induced on (invariant) hypersurfaces and how it varies. Decomposes into a "static" and "dynamic" part.
   - Static = understand exactly what geometric structures can appear on an (invariant) hypersurface.

     Answer = (invariant) *nearly hypo* $SU(2)$ structures (Fernandez et al);

     Space of invariant nearly hypo structures can be identified with a connected open subset of $SO_0(1,2) \times S^1$. S^1 factor corresponds to obvious continuous symmetries of the equations.

     So up to symmetry *there exists a 3-dimensional family of invariant nearly hypo structures.*
   - Dynamic = (cohom 1) nK metrics correspond to differential equations for evolving a 1–parameter family of (invariant) nearly hypo structures.

     Answer in cohom 1 case = explicit *1st order* ODEs for a curve in the space of invariant nearly hypo structures.
   - **Upshot:** $\exists$ **2–parameter family of cohomogeneity 1 local nK metrics.**

Rough outline of proof II

• Don't know how to find explicit form for general solution to the ODEs. Special explicit solutions do exist, have geometric significance and play important role in our proof.

• Generic solution in 2–parameter family does NOT extend to a complete metric.

Fundamental difficulty: *recognise which local solutions extend to complete metrics.*

Proceed in two steps; separate the two singular orbits that appear and study separately.

1. *Understand the possible singular orbits (uses Lie group theory) and which solutions extend over a given singular orbit (need to solve singular IVP).*
2. *Understand how to "match" a pair of solutions from the previous step.*

Step 1 fits into a general framework for cohomogeneity 1 Einstein metrics (**Eschenburg–Wang** 2000); extra care needed because of isotropy repn.
Step 2 is the most subtle part of argument. Closest to previous work of **Böhm** on Einstein metrics on spheres (**Inventiones** 1998).

Local solutions extending over a singular orbit

Neighbourhood of singular orbit is a G-equivariant disc bundle over singular orbit. Use representation theory to express conditions that a G-invariant section extend smoothly over the zero section. Get a *singular initial value problem* for 1st order nonlinear ODE system. Smoothness gives constraints on the initial values permitted.

Podestà–Spiro: up to symmetries *possible singular orbits are*

$$\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)\times\mathrm{SU}(2)\simeq S^2 \qquad \mathrm{SU}(2)\times\mathrm{SU}(2)/\triangle\mathrm{SU}(2)\simeq S^3$$

Proposition (Nearly Kähler deformations of small resolution & smoothing)

- *There exist two* 1*–parameter families* $\{\Psi_a\}_{a>0}$ *and* $\{\Psi_b\}_{b>0}$ *of solutions to the fundamental ODE system which extend smoothly over a singular orbit* S^2 *and* S^3*, respectively.* a *and* b *measure size of singular orbits.*
- *As* $a, b \to 0$*, appropriately rescaled, the local nK structures* Ψ_a *and* Ψ_b *converge to the CY structures on the small resolution and the smoothing.*

Think of the two 1–parameter families as *local* **nearly Kähler deformations** of CY metrics on small resolution and smoothing.

Now the parameter a or b is NOT just a global rescaling (as in CY case).

Matching pairs of solns: maximal volume orbits

M complete cohom 1 nK $\Longrightarrow$ orbital volume $V(t)$ has a unique maximum.
But generic member of our 1-parameter families of solutions is **not** complete.

Key properties of space of invariant maximal volume orbits $\mathcal{V}$:

- $\mathcal{V} \simeq \mathbb{R}^2 \times S^1 \subset \mathbb{R}^3 \times S^1$
- $V \geq 1$ on $\mathcal{V}$ and $V = 1$ precisely for the Sasaki–Einstein structure on $\mathrm{N}_{1,1}$
- $\mathcal{V} \cap \{V \leq C\}$ is *compact*

Key Proposition *Every member of the families $\{\Psi_a\}_{a>0}$ and $\{\Psi_b\}_{b>0}$ has a unique maximal volume orbit.*

Idea of proof: a continuity argument in the parameter a or b.

Nonempty; open; closed.

Nonempty: 3 of 4 known homogeneous examples appear in these families; these clearly have max vol orbits.
Openness: easy using nondegeneracy conditions that are satisfied.
Closedness is main point: uses compactness of $\mathcal{V} \cap \{V \leq C\}$ plus standard ODE theory and basic comparison theory.

Strategy for finding complete nK metrics: *Match pairs of solutions in the two families across their maximal volume orbits using discrete symmetries.*

- α, β continuous curves in $\mathbb{R}^2 \simeq \mathcal{V}/S^1$ parametrising the maximal volume orbits of $\{\Psi_a\}_{a>0}$ and $\{\Psi_b\}_{b>0}$
- Discrete symmetries = reflections along the axes
- **Matching** means:
  (i) curves α, β must intersect (up to a discrete symmetry), or
  (ii) self-intersect, or
  (iii) intersect either axis.
- Intersection points with axes correspond to solutions with a special *"doubling symmetry"*, i.e. $\exists$ *a reflection that exchanges the two singular orbits* (therefore are of same type and size).
- Intersection points of α curve with axes give $S^2 \times S^4$ or CP^3.
- Intersection points of β curve with either axis gives $S^3 \times S^3$.
- Intersection points of α and β curves (up to action of reflection) gives S^6.

To understand if there exist new complete cohomogeneity 1 nK metrics is equivalent to:

How many axis crossings/intersection/self-intersection points do the curves α and β (and their images under the reflections) have?

Geometry of the α and β curves

Two obvious ways to get some information about the α and β curves.

1. Standard nK metrics on CP^3, S^6 and $S^3 \times S^3$ give points on these curves.
   CP^3 and $S^3 \times S^3$ give intersection points of α and β curves with the axes; S^6 gives an intersection point between α and (reflection of) β curves.
   What about the sine-cone?
2. Study limits of α and β curves as the parameters a and $b \to 0$.
   Desingularisation philosophy suggests: Ψ_a and Ψ_b should both converge to the *sine-cone* away from the two singular orbits. Max vol orbit in sine-cone is the origin in the plane ("rotated" SE structure). So expect that the α and β curves both limit to the origin. Need to prove:

Proposition. *As $a, b \to 0$ Ψ_a and Ψ_b converge to the sine-cone over the standard Sasaki–Einstein structure on* $\mathrm{N}_{1,1}$.

Proof ingredients: use convergence of "bubbles" to asymptotically conical CY structures; the *Böhm functional* $\mathcal{B}$ for cohom 1 Einstein metrics; invariance of $\mathcal{B}$ under rescaling and fact that it gives a power of *Vol* on a max vol orbit; rotated SE metric is the absolute min of *Vol* on all max vol orbits.

Existence of the new metric on $S^3 \times S^3$

First look only for solutions obtained by "doubling" some Ψ_b.

Idea: *exploit the convergence of Ψ_b as $b \to 0$ to the sine-cone and the existence of the homogeneous nK metric on $S^3 \times S^3$ (this has $b = 1$). Find a new nK metric "between" these two metrics, i.e. with $0 < b < 1$.*

Observation: Can detect a doubled metric on $S^3 \times S^3$ via condition that

$$v_0 = 0$$

on a *max volume orbit* where v_0 is one component of nearly hypo structure. *ODE system $\Rightarrow$ Zeros of v_0 are nondegenerate; count the number of them that occur before a max vol orbit: we call this $\mathcal{C}(b)$.*

Key fact: *$\mathcal{C}(b)$ is locally constant in b unless we hit a* **doubled metric**.

Idea of proof: $b = 1$ is standard $S^3 \times S^3$ and we check $C(b) = 1$. To get a new cohomogeneity 1 metric on $S^3 \times S^3$ it's enough to show $C(b) \geq 2$ for $b > 0$ sufficiently small. This implies there is another doubled metric for some $b \in (0, 1)$.

Need to prove: *$C(b) \geq 2$ for $b > 0$ sufficiently small.*

$C(b) \geq 2$ for $b > 0$ sufficiently small.

Want to count zeroes of v_0 before max vol orbit; by ODE system this is equivalent to counting *critical points of* u_0 (another component of the nearly hypo structure)
u_0 is a solution of the second order IVP

$$(*) \quad (\lambda u_0')' + 12\lambda u_0 = 0, \quad u_0(0) = 0, \quad u_0'(0) = 2b^2 > 0.$$

Convergence of Ψ_b to sine-cone implies $\lambda(t) \to \sin t$ as $b \to 0$.
Idea: *Compare* u_0 *to a solution of the limiting equation*

$$(\sin t\, \xi')' + 12 \sin t\, \xi = 0.$$

This is Legendre's equation with $k = 3$. There are explicit solutions:

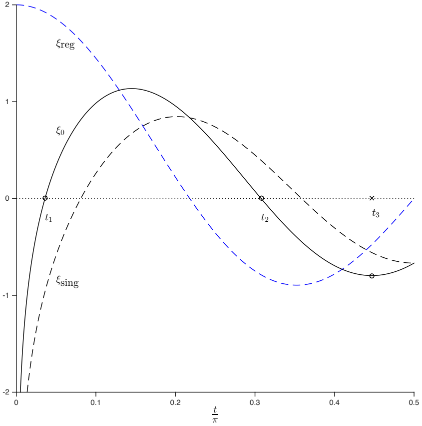
$$\xi_0(t) = C_1(5\cos^3 t - 3\cos t) +$$
$$C_2 \left(\frac{5}{2}\cos^2 t + \frac{1}{8}\cos t(4\cos^2 t - 6\sin^2 t) \log \frac{1 - \cos t}{1 + \cos t} - \frac{2}{3} \right)$$

1st solution is **regular** at endpoints 0 and π while 2nd is **singular**.

Existence of the new metric on $S^3 \times S^3$

Lemma: *There exists a solution ξ_0 of this Legendre eqn with the following properties: there exists $0 < t_1 < t_2 < t_3 < \frac{\pi}{2}$ such that $\xi_0(t_1) = \xi_0(t_2) = 0$, $\xi_0 \geq 0$ on $[t_1, t_2]$ and ξ_0 has a negative minimum at t_3.*

Proof:



Existence of the new metric on $S^3 \times S^3$

Recall that u_0 solves

$$(*) \quad (\lambda u_0')' + 12\lambda u_0 = 0, \quad u_0(0) = 0, \quad u_0'(0) = 2b^2 > 0.$$

and $\lambda \to \sin t$ on $(0, \pi)$ as $b \to 0$.

Theorem: *There exists $\epsilon > 0$ such that for all $b < \epsilon$, u_0 the solution of (*) has a strict negative minimum before the maximal volume orbit.*

Proof sketch: Apply a (generalised) Sturm-Picone comparison argument to prove the same conclusion about the minimum holds for solution of $(*)$, using uniform convergence of $\lambda(t)$ to $\sin t$ on compact subsets of $(0, \pi)$.

Finally: initial conditions for u_0 also force a *maximum* before the minimum.

Existence of the new metric on S^6

Need to force planar curves α and β to intersect in another point (2 intersection points already exist: standard nK S^6 and sine-cone).

Idea: *use the new and old solutions on $S^3 \times S^3$ to find a closed bounded region D in the plane encircling the origin.* The α curve starts at the origin (as $a \to 0$); we want to show that eventually the α curve leaves D passing through its boundary; this point gives the new intersection point of the α and β curves.

Proposition. *The curve α exits any compact subset of $\mathbb{R}^2$ as $a \to \infty$.*

Idea of proof: based on explicit Taylor series for solutions Ψ_a and their dependence on the parameter a we consider a very particular (but non geometric) rescaling of the solution to the ODE system. Show that the rescaled solutions are well behaved as $a \to \infty$ and converge smoothly to some limiting object. Scaling used shows $V_{max} \sim ca^4$.

Heuristically: making the size of the singular orbit 2-sphere large ($a \to \infty$) forces the size of the maximal volume orbit to be large.

The intersection points of α and β

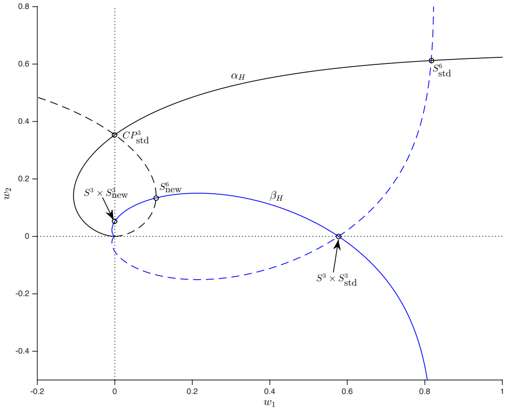


Figure: α and β curves and the locations of the 5 complete cohomogeneity one nK structures computed numerically

Nearly hypo structures

$N_{1,1} = SU(2) \times SU(2)/\triangle U(1)$
On $M^* = (a, b) \times N_{1,1}$ we write

$$\omega = \eta \wedge dt + \omega_1 \qquad \Omega = (\omega_2 + i\omega_3) \wedge (\eta + idt),$$

where $(\eta, \omega_1, \omega_2, \omega_3)$ defines an invariant SU(2)–structure on $N_{1,1}$.

The SU(2)–structure induced on a hypersurface in a nK 6–manifold is called a **nearly hypo** structure. It is defined by the following equations:

$$d\omega_1 = 3\eta \wedge \omega_2 \qquad d(\eta \wedge \omega_3) = -2\omega_1 \wedge \omega_1$$

The evolution equations to obtain a nK manifold by flowing a nearly hypo structure are:

$$\partial_t \omega_1 = -d\eta - 3\omega_3 \qquad \partial_t(\eta \wedge \omega_2) = -d\omega_3 \qquad \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1$$

Lemma. *The space of invariant nearly hypo structures on* $N_{1,1}$ *is a smooth manifold diffeomorphic to* $\mathbb{R}^3 \times S^1$*. (The* S^1*–factor is generated by the action of the Reeb vector field.)*

The fundamental ODE system

We parametrise invariant nearly hypo structures modulo the Reeb action by tuples $(\lambda, u, v) \in \mathbb{R}^+ \times \mathbb{R}^{1,2} \times \mathbb{R}^{1,2}$ subject to the constraints

$$\lambda^2|u|^2 = |v|^2 > 0 \qquad \langle u, v\rangle = 0 \qquad v_1 = |u|^2 \qquad u_2 = -\lambda|u|$$

The basic equations then are:

$$\begin{aligned} &\lambda\dot{u}_0 + 3v_0 = 0, && \dot{v}_0 - 4\lambda u_0 = 0, \\ &\lambda\dot{u}_1 + 3v_1 - 2\lambda^2 = 0, && \dot{v}_1 - 4\lambda u_1 = 0, \\ &\lambda\dot{u}_2 + 3v_2 = 0, && \lambda\dot{v}_2 - 4\lambda^2 u_2 + 3u_2 = 0, \end{aligned}$$

$$\lambda^2|u|^2\dot{\lambda} + 2\lambda^4 u_1 + 3u_2 v_2 = 0.$$

Proposition. *Up to symmetries, there exists a 2–parameter family of local cohomogeneity one nK stuctures on $(a, b) \times \mathrm{N}_{1,1}$.*

The homogeneous nK structure on $S^3 \times S^3$ is

$$\lambda = 1, \quad u_0 = u_1 = \frac{1}{\sqrt{3}} \sin(2\sqrt{3}t), \quad u_2 = -\frac{2}{\sqrt{3}} \sin(\sqrt{3}t),$$

$$v_0 = -\frac{2}{3} \cos(2\sqrt{3}t), \quad v_1 = \frac{2}{3}\left(1 - \cos(2\sqrt{3}t)\right), \quad v_2 = \frac{2}{3} \cos(\sqrt{3}t),$$

for $t \in [0, \frac{\pi}{\sqrt{3}}]$.

The sine-cone is:

$$\lambda = \sin t, \quad u_0 = 0, \quad u_1 = \sin^2 t \cos t, \quad u_2 = -\sin^3 t,$$

$$v_0 = 0, \quad v_1 = \mu^2 = \sin^4 t, \quad v_2 = \sin^3 t \cos t,$$

for $t \in [0, \pi]$.

The first few terms of the Taylor series of Ψ_a at $t = 0$ are:

$$\lambda(t) = \tfrac{3}{2}t - \frac{2a^2+3}{12a^2}t^3 + \frac{116a^4 - 381a^2 + 261}{1440a^4}t^5 + \cdots$$

$$u_0(t) = a^2 - 3a^2t^2 + \frac{52a^2-3}{24}t^4 + \cdots$$

$$u_1(t) = a^2 - \tfrac{3}{2}(2a^2-1)t^2 + \frac{52a^4 - 32a^2 - 3}{24a^2}t^4 + \cdots$$

$$u_2(t) = \tfrac{-3\sqrt{3}}{2}at^2 + \frac{\sqrt{3}(16a^2-3)}{12a}t^4 + \cdots$$

Conclusion

Conjecture The Main Theorem yields all (inhomogeneous) cohomogeneity one nK structures on simply connected 6–manifolds.