7-manifolds with $G_2$ holonomy

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**What is $G_2$? $G_2$ holonomy and Ricci-flat metrics**

i. the automorphism group of the octonions $\mathcal{O}$

ii. the stabilizer of a generic 3-form in $\mathbb{R}^7$

Define a vector cross-product on $\mathbb{R}^7 = \text{Im}(\mathcal{O})$

$$ u \times v = \text{Im}(uv) $$

where $uv$ denotes octonionic multiplication. Cross-product has an associated 3-form

$$ \varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle $$

$\varphi_0$ is a generic 3-form so in fact

$$ G_2 = \{ A \in \text{GL}(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi \subset \text{SO}(7) \}. $$

$G_2$ can arise as the holonomy group of an irreducible non-locally-symmetric Riemannian 7-manifold (Berger 1955, Bryant 1987, Bryant-Salamon 1989, Joyce 1995). Any such manifold is automatically *Ricci-flat*. 
6 + 1 = 2 \times 3 + 1 = 7 \quad \& \quad \text{SU}(2) \subset \text{SU}(3) \subset G_2

∃ close relations between \(G_2\) holonomy and Calabi-Yau geometries in 2 and 3 dimensions.

- Write \(\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3\) with \((\mathbb{C}^3, \omega, \Omega)\) the standard \(\text{SU}(3)\) structure then

\[
\varphi_0 = dt \wedge \omega + \text{Re} \Omega
\]

Hence stabilizer of \(\mathbb{R}\) factor in \(G_2\) is \(\text{SU}(3) \subset G_2\). More generally if \((X, g)\) is a Calabi-Yau 3-fold then product metric on \(S^1 \times X\) has holonomy \(\text{SU}(3) \subset G_2\).

- Write \(\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{C}^2\) with coords \((x_1, x_2, x_3)\) on \(\mathbb{R}^3\), with standard \(\text{SU}(2)\) structure \((\mathbb{C}^2, \omega_I, \Omega = \omega_J + i\omega_K)\) then

\[
\varphi_0 = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_I + dx_2 \wedge \omega_J + dx_3 \wedge \omega_K,
\]

where \(\omega_I\) and \(\Omega = \omega_J + i\omega_K\) are the standard Kahler and holomorphic \((2, 0)\) forms on \(\mathbb{C}^2\). Hence subgroup of \(G_2\) fixing \(\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{C}^2\) is \(\text{SU}(2) \subset G_2\).
**G₂ structures and G₂ holonomy metrics**

- A **G₂ structure** is a 3-form \( \phi \) on an oriented 7-manifold \( M \) such that at every point \( p \in M \), \( \exists \) an oriented isomorphism

  \[
i : T_p M \rightarrow \mathbb{R}^7, \text{ such that } i^* \phi_0 = \phi.\]

- **G₂-structures on** \( \mathbb{R}^7 \leftrightarrow \text{GL}_+(7, \mathbb{R})/G_2 \).
- \( \text{dim}(\text{GL}_+(7, \mathbb{R})/G_2) = 35 = \text{dim} \Lambda^3 \mathbb{R}^7 \).
  \( \Rightarrow \text{ implies small perturbations of a } G_2\text{-structure are still } G_2\text{-structures.} \)

How to get a **G₂-holonomy metric from a G₂ structure?**

**Theorem**

Let \( (M, \phi, g) \) be a **G₂ structure** on a compact 7-manifold; the following are equivalent

1. \( \text{Hol}(g) \subset G_2 \) and \( \phi \) is the induced 3-form
2. \( \nabla \phi = 0 \) where \( \nabla \) is Levi-Civita w.r.t \( g \)
3. \( d\phi = d^*\phi = 0. \)

Call such a **G₂ structure** a **torsion-free G₂ structure**.

NB (3) is nonlinear in \( \phi \) because metric \( g \) depends nonlinearly on \( \phi \).
**Lemma**

Let $M$ be a compact 7-manifold.

1. $M$ admits a $G_2$ structure iff it is orientable and spinnable.
2. A torsion-free $G_2$ structure $(\phi, g)$ on $M$ has $\text{Hol}(g) = G_2$ iff $\pi_1 M$ is finite.
3. If $\text{Hol}(g) = G_2$ then $M$ has nonzero first Pontrjagin class $p_1(M)$.

**Ingredients of proof for 2 and 3.**

2. $M$ has holonomy contained in $G_2$, implies $g$ is Ricci-flat. Now combine structure results for non-simply connected compact Ricci-flat manifolds (application of Cheeger-Gromoll splitting theorem) with the classification of connected subgroups of $G_2$ that could appear as (restricted) holonomy groups of $g$.

3. Apply Chern-Weil theory for $p_1(M)$ and use $G_2$ representation theory to analyse refinement of de Rham cohomology on a $G_2$ manifold; full holonomy $G_2$ forces vanishing of certain refined Betti numbers and this leads to a sign for $\langle p_1(M) \cup [\phi], [M] \rangle$. 
Exceptional holonomy milestones

1984: (Bryant) locally \( \exists \) many metrics with holonomy \( G_2 \) and \( Spin(7) \). Proof uses Exterior Differential Systems.

1989: (Bryant-Salamon) explicit complete metrics with holonomy \( G_2 \) and \( Spin(7) \) on noncompact manifolds.
- total space of bundles over 3 & 4 mfds
- metrics admit large symmetry groups and are asymptotically conical

1994: (Joyce) Gluing methods used to construct compact 7-manifolds with holonomy \( G_2 \) and 8-manifolds with holonomy \( Spin_7 \). Uses a modified Kummer-type construction.

String/M-theorists become interested in using compact manifolds with exceptional holonomy for supersymmetric compactifications.

2000: Joyce’s book *Compact Manifolds with Special Holonomy*.

2003: Kovalev uses Donaldson’s idea of a *twisted connect sum* construction to find alternative constructions of compact \( G_2 \) manifolds.
The moduli space of holonomy $G_2$ metrics

Let $M$ be a compact oriented 7-manifold and let $\mathcal{X}$ be the set of torsion-free $G_2$ structures on $M$. Let $\mathcal{D}$ be the group of all diffeomorphisms of $M$ isotopic to the identity. Then $\mathcal{D}$ acts naturally on the set of $G_2$ structures on $M$ and on $\mathcal{X}$ by $\phi \mapsto \Psi_*(\phi)$.

Define the moduli space of torsion-free $G_2$ structures on $M$ to be $\mathcal{M} = \mathcal{X}/\mathcal{D}$.

**Theorem (Joyce)**

$\mathcal{M}$ the moduli space of torsion-free $G_2$ structures on $M$ is a smooth manifold of dimension $b^3(M)$, and the natural projection $\pi: \mathcal{M} \to H^3(M, \mathbb{R})$ given by $\pi(\phi \mathcal{D}) = [\phi]$ is a local diffeomorphism.

Main ingredients of the proof: (a) a good choice of ‘slice’ for the action of $\mathcal{D}$ on $\mathcal{X}$, i.e. a submanifold $S$ of $\mathcal{X}$ which is (locally) transverse to the orbits of $\mathcal{D}$, so that each nearby orbit of $\mathcal{D}$ meets $S$ in a single point. (b) Some fundamental technical results about (small) perturbations of $G_2$ structures to yield appropriate nonlinear elliptic PDE. (c) Linearise the PDE and apply standard Hodge theory and Implicit Function Theory.
Two fundamental technical results:
Denote by $\Theta$ the (nonlinear) map sending $\phi \mapsto *\phi$.

**Lemma (A)**

If $\phi$ is a closed $G_2$-structure on $M$ and $\chi$ a sufficiently small 3-form then $\phi + \chi$ is also a $G_2$-structure with $\Theta$ given by

$$\Theta(\phi + \chi) = *\phi + *(\text{explicit terms linear in } \chi) - F(\chi)$$

where $F$ is a smooth function from a closed ball of small radius in $\Lambda^3 T^*M$ to $\Lambda^4 T^*M$ with $F(0) = 0$ satisfying some additional controlled growth properties.

**Lemma (B)**

If $(M, \phi, g)$ is a compact $G_2$ manifold and $\tilde{\phi}$ is a closed 3-form $C^0$-close to $\phi$, then $\tilde{\phi}$ can be written uniquely as $\tilde{\phi} = \phi + \xi + d\eta$ where $\xi$ is a harmonic 3-form and $\eta$ is a $d^*$-exact 2-form. Moreover, $\tilde{\phi}$ is a torsion-free $G_2$ structure also satisfying the “gauge fixing”/slice condition if and only if

$$(*) \quad \Delta \eta = *dF(\xi + d\eta).$$

The latter gives us the nonlinear elliptic PDE (for the coexact 2-form $\eta$) we seek.
How to construct compact $G_2$ manifolds

Meta-strategy to construct compact $G_2$ manifolds

I. Find a closed $G_2$ structure $\phi$ with sufficiently small torsion on a 7-manifold with $|\pi_1| < \infty$

II. Perturb to a torsion-free $G_2$ structure $\phi'$ close to $\phi$.

- II was understood in great generality by Dominic Joyce using an extension of Lemma (B) to $G_2$ structures $\phi$ that are closed and sufficiently close to being torsion-free.

- Condition that the perturbed $G_2$ structure $\phi + d\eta$ be torsion-free still becomes a nonlinear elliptic PDE (*)' for the 2-form $\eta$; get extra terms on RHS of (*) coming from failure of background $G_2$-structure $\phi$ to be torsion-free.

- Joyce solves (*)' by iteratively solving a sequence of linear elliptic PDEs together with a priori estimates (of appropriate norms) on the iterates to establish their convergence to a limit satisfying (*)'.

Q: How to construct closed almost torsion-free $G_2$ structures?!
Degenerations of compact $G_2$-manifolds I

Q: How to construct closed almost torsion-free $G_2$ structures?!

- Key idea: Think about possible ways a family of $G_2$ holonomy metrics on a given compact 7-manifold might degenerate.
- Find instances in which the singular “limit” $G_2$ holonomy space $X$ is simple to understand.
- Try to construct a smooth compact 7-manifold $M$ which resolves the singularities of $X$; use the geometry of the resolution to build by hand a closed $G_2$-structure on $M$ that is close enough to torsion-free.

$M$ has holonomy $G_2 \Rightarrow M$ is Ricci-flat; so think about how families of compact Ricci-flat manifolds (more generally Einstein manifolds or just spaces with lower Ricci curvature bounds) can degenerate.
Degenerations of compact $G_2$-manifolds II

Case 1. Neck stretching degeneration.

- A degeneration in which $(M, g_i)$ develops a long “almost cylindrical neck” that gets stretched longer and longer.
- In the limit we decompose $M$ into a pair of noncompact 7-manifolds $M_+$ and $M_-; M_\pm$ should each be asymptotically cylindrical $G_2$ manifolds.

Given such a pair $M_\pm$ with appropriately compatible cylindrical ends we could try to reverse this construction, i.e. to build a compact $G_2$ manifold $M$ by truncating the infinite cylindrical end sufficiently far down to get a $G_2$-structure with small torsion and a long “almost cylindrical” neck region.

**Big disadvantage:** doesn’t seem any easier to construct asymptotically cylindrical $G_2$ manifolds than to construct compact $G_2$ manifolds.

**Advantage:** maintain good geometric control throughout, e.g. lower bounds on injectivity radius, upper bounds on curvature etc.

$\Rightarrow$ perturbation analysis remains relatively simple technically.

Donaldson suggested a way to circumvent the problem above.
Degenerations of compact $G_2$-manifolds III

Case 2. Diameter bounded with lower volume control

How can sequences of compact Ricci-flat spaces degenerate with bounded diameter and lower volume bounds?

Simplest answer: they could develop orbifold singularities in codimension 4. Simplest model is a metric version of the *Kummer construction* for K3 surfaces.

Choose a lattice $\Lambda \simeq \mathbb{Z}^4$ in $\mathbb{C}^2$ and form 4-torus $T^4 = \mathbb{C}^2/\Lambda$. Look at involution $\sigma : T^4 \to T^4$ induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$.

- $\sigma$ fixes $2^4 = 16$ points $\{[z_1, z_2] : (z_1, z_2) \in \frac{1}{2} \Lambda\}$.
- $T^4/\langle \sigma \rangle$ is a flat hyperkähler orbifold with 16 singular points modelled on $\mathbb{C}^2/\{\pm 1\}$.
- $S$ the blow-up of $T^4/\langle \sigma \rangle$ is a smooth K3 surface: a *Kummer surface*.
- Pulling back flat orbifold metric $g_0$ from $T^4$ to $S$ gives a singular Kähler metric on $S$, degenerate at the 16 $\mathbb{P}^1$ introduced by blowing-up.
The metric Kummer construction

Want to build a family of smooth metrics $g_t$ on $S$ which converges as $t \to 0$ to this singular flat orbifold metric.

Key is the *Eguchi-Hanson metric*, which gives a hyperkähler metric on the blowup of $\mathbb{C}^2/\langle \pm 1 \rangle$ (which is biholomorphic to $T^*\mathbb{P}^1$).

To get a nonsingular Kahler metric on $S$ near each $\mathbb{P}^1$ we replace the degenerate metric with a suitably scaled copy of Eguchi-Hanson metric and interpolate to get $\omega'_t$ on $S$, where parameter $t$ controls the diameter of the $16 \mathbb{P}^1$.

- Page observed that $\omega'_t$ is close to Ricci-flat.
- Topiwala, LeBrun-Singer then proved that it can be perturbed to a Ricci-flat Kahler metric $\omega_t$.
- $\omega_t$ converges to the flat orbifold metric as $t \to 0$ and the size of each $\mathbb{P}^1$ goes to 0.

Could try similar thing using other ALE hyperkähler 4-manifolds constructed by Gibbons-Hawking, Hitchin, Kronheimer for all the ADE singularities $\mathbb{C}^2/\Gamma$, i.e. where $\Gamma$ is a finite subgroup of $SU(2)$. 
Joyce’s orbifold resolution construction of compact $G_2$ manifolds

Basic idea: seek a $G_2$ analogue of the metric Kummer construction above.

- look at finite subgroups $\Gamma \subset G_2$ and consider singular flat orbifold metrics $X = T^7/\Gamma$.
- analyse the singular set of $T^7/\Gamma$; this is never an isolated set of points and often can be very complicated with various strata.
- look for $\Gamma$ for which the singular set is particularly simple, e.g. a disjoint union of smooth manifolds.
- find appropriate $G_2$ analogues of Eguchi-Hanson spaces, i.e. understand how to find resolutions of $\mathbb{R}^7/G$ and put (Q)ALE $G_2$ holonomy metrics on them.
- Use these ingredients to find a smooth 7-manifold $M$ resolving the singularities of $X$, admitting a 1-parameter family of closed $G_2$ structures $\phi_t$ with torsion sufficiently small compared to lower bounds for injectivity radius and upper bound for curvature; apply the general perturbation theory for closed $G_2$ structures with small torsion; analysis is delicate because induced metric is nearly singular.
Simplest generalised Kummer construction

If $G \subset SU(2)$ is a finite group and $Y$ an ALE hyperkahler manifold then $\mathbb{R}^3 \times Y$ is naturally a (Q)ALE $G_2$-manifold, e.g. $Y$ could be Eguchi-Hanson space for $G = \mathbb{Z}_2$.

Simplest Kummer construction:

- find finite $\Gamma \simeq \mathbb{Z}_2^3 \subset G_2$ so that singular set $S$ of $T^7/\Gamma$ is a disjoint union of 3-tori for which some open neighbourhood of each torus is isometric to $T^3 \times B^4/\langle \pm 1 \rangle$.
- Replace $B^4/\langle \pm 1 \rangle$ by its blowup $U$ and (using explicit form of Kähler potential) put a 1-parameter family of triples of 2-forms $\omega_i(t)$ on $U$ that interpolates between the hyperkähler structures of Eguchi-Hanson and of $\mathbb{C}^2/\langle \pm I \rangle$.
- Obtain a compact smooth 7-manifold $M$ by replacing a neighbourhood of each component of singular set $S$ by $T^3 \times U$.
- The triple of 2-forms $\omega_i(t)$ on $U$ gives rise to a closed $G_2$ structure on $T^3 \times U$ for $t$ sufficiently small and which is flat far enough away from $T^3$; so $M$ has a 1-parameter family of closed $G_2$-structures $\phi'_t$ with small torsion supported in some “annulus” around the $T^3$.

Now apply the perturbation theory to get a 1-parameter family of torsion-free $G_2$ structures $\phi_t$ and verify that $M$ has finite (actually trivial) fundamental group so that $g_t$ all have full holonomy $G_2$. Can also compute Betti numbers of $M$: $b^2 = 12$, $b^3 = 43$. 
Donaldson suggested constructing compact $G_2$ manifolds from a pair of asymptotically cylindrical Calabi-Yau 3-folds via a neck-stretching method.

i. Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Calabi-Yau 3-folds $V$ with one end $\sim \mathbb{C}^* \times D \sim \mathbb{R}^+ \times S^1 \times D$, with $D$ a smooth $K3$.

ii. $M = S^1 \times V$ is a 7-manifold with $\text{Hol } g = SU(3) \subset G_2$ with end $\sim \mathbb{R}^+ \times T^2 \times K3$.

iii. Take a twisted connected sum of a pair of $M_\pm = S^1 \times V_\pm$

iv. For $T \gg 1$ construct a $G_2$-structure w/ small torsion (exponentially small in $T$) and prove it can be corrected to torsion-free.

Kovalev (2003) carried out Donaldson’s proposal for AC CY 3-folds arising from Fano 3-folds. However the paper contains two serious mistakes.
Twisted connected sums & hyperkähler rotation

Product $G_2$ structure on $M_{\pm} = S^1 \times V_{\pm}$ asymptotic to

$$d\theta_1 \wedge d\theta_2 \wedge dt + d\theta_1 \wedge \omega_i^{\pm} + d\theta_2 \wedge \omega_j^{\pm} + dt \wedge \omega_K^{\pm}$$

$\omega_i^{\pm}, \omega_j^{\pm} + i \omega_K^{\pm}$ denote Ricci-flat Kähler metric & parallel $(2, 0)$-form on $D_{\pm}$.

To get a well-defined $G_2$ structure using

$$F : [T - 1, T] \times S^1 \times S^1 \times D_- \to [T - 1, T] \times S^1 \times S^1 \times D_+$$

given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T - 1 - t, \theta_2, \theta_1, f(y))$$

to identify end of $M_-$ with $M_+$ we need $f : D_- \to D_+$ to satisfy

$$f^*\omega_i^{+} = \omega_j^{-}, \quad f^*\omega_j^{+} = \omega_i^{-}, \quad f^*\omega_K^{+} = -\omega_K^{-}.$$

- Constructing such hyperkähler rotations is nontrivial and a major part of the construction.
- Some problems in Kovalev’s original paper here.
Twisted connected sum $G_2$-manifolds

1. Construct suitable ACyl Calabi-Yau 3-folds $V$;
2. Find sufficient conditions for existence of a hyperkähler rotation between $D_-$ and $D_+$;
   - Use global Torelli theorems and lattice embedding results (e.g. Nikulin) to find hyperkähler rotations from suitable initial pairs of (deformation families of) ACyl CY 3-folds.
3. Given a pair of ACyl CY 3-folds $V_\pm$ and a HK-rotation $f : D_- \to D_+$ can always glue $M_-$ and $M_+$ to get a 1-parameter family of closed manifolds $M_T$ with holonomy $G_2$.
   - In general for the same pair of ACyl CY 3-folds different HK rotations can yield different 7-manifolds (e.g. different Betti numbers $b^2$ and $b^3$).

$\Rightarrow$ have reduced solving nonlinear PDEs for $G_2$-metric to two problems about complex projective 3-folds.
**ACyl Calabi-Yau 3-folds**

**Theorem (H-Hein-Nordström JDG 2015)**

Any simply connected ACyl Calabi-Yau 3-fold $X$ with split end $S^1 \times K3$ is quasiprojective, i.e. $X = \overline{X} \setminus \overline{D}$ for some smooth projective variety $\overline{X}$ and smooth anticanonical divisor $\overline{D}$. Moreover $\overline{X}$ fibres holomorphically over $\mathbb{P}^1$ with generic fibre a smooth anticanonical K3 surface. Conversely, the complement of any smooth fibre in any such $\overline{X}$ admits (exponentially) ACyl CY metrics with split end.

Builds on previous work of Tian-Yau and Kovalev; HHN proved more general compactification for ACyl CY manifolds (ends need not split; compactification can be singular).

3 main sources of examples of such K3 fibred 3-folds:

- Fano 3-folds, K3 surfaces with nonsymplectic involution (Kovalev); gives several hundred examples.
- weak or semi-Fano 3-folds (Corti-H-Nordström-Pacini); gives at least several hundred thousand examples!
Simple example of a semi-Fano 3-fold

Example 1: start with a (singular) quartic 3-fold $Y \subset \mathbb{P}^4$ containing a projective plane $\Pi$ and resolve. If $\Pi = (x_0 = x_1 = 0)$ then eqn of $Y$ is

$$Y = (x_0 a_3 + x_1 b_3 = 0) \subset \mathbb{P}^4$$

where $a_3$ and $b_3$ are homogeneous cubic forms in $(x_0, \ldots, x_4)$. Generically the plane cubics

$$(a_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi,$$

$$(b_3(0, 0, x_2, x_3, x_4) = 0) \subset \Pi$$

intersect in 9 distinct points, where $Y$ has 9 ordinary double points. Blowing-up $\Pi \subset Y$ gives a smooth 3-fold $X$ such that $f : X \to Y$ is a projective small resolution of all 9 nodes of $Y$.

$X$ is a smooth (projective) semi-Fano 3-fold; it contains 9 smooth rigid rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$; $X$ has genus 3 and Picard rank 2.
**$G_2$-manifolds and toric semi-Fano 3-folds**

**Theorem (Corti-Haskins-Nordström-Pacini (Duke 2015) + CHK)**

There exist over 900 million matching pairs of ACyl CY 3-folds of semi-Fano type for which the resulting $G_2$-manifold is 2-connected.

**Main ingredients of proof.**

- Use a pair of ACyl CY 3-folds with one of toric semi-Fano type and the other a semi-Fano (or Fano) of rank at most 2.
- Use further arithmetic information about polarising lattices (discriminant group information) to prove there are over 250,000 toric semi-Fanos that can be matched to any ACyl CY 3-fold of Fano/semi-Fano type of rank at most 2. Over 250,000 rigid toric semi-Fanos arise from only the 12 most “prolific” polytopes.
- There are over 200 deformation types of Fanos/semi-Fanos of rank at most 2.