Numerical Ricci flows and black holes

Pau Figueras

School of Mathematical Sciences, Queen Mary University of London
and
DAMTP, University of Cambridge

"Summer School on Geometric Flows and the Geometry of Spacetime",
University of Hamburg
Thursday 22nd of September 2016

European Research Council

THE ROYAL SOCIETY

General relativity

- Provides the “correct” classical description of the gravitational force at sufficiently large scales

- Einstein equations determine the spacetime metric g:

$$R_{ab}[g] - \frac{1}{2}\,R[g]\,g_{ab} = 8\pi\,G_N\,T_{ab}$$

- Non-linear PDEs $\Rightarrow$ difficult to solve in general

- Black holes are amongst the most important solutions of Einstein’s equations

4D black holes

4D asymptotically flat stationary vacuum black holes are well-understood in classical general relativity:

- Spherical topology [Hawking]
- Rotate rigidly [Hawking]
- Uniqueness [Carter; Robinson; Bunting; Mazur]
- Methods to construct all 4D black holes explicitly are known [Kerr; Belinskii and Zakharov]
- They exist in Nature: GW150914 [LIGO]
- Stability?

Why general relativity beyond 4D?

- Address fundamental questions about the nature of gravity in different settings

- Practical applications: general relativity with certain boundary conditions gives access to certain strongly interacting (non-gravitational) physics

- Mathematical interest: solutions have new properties

Outline of the talk

1. General relativity and black holes beyond 4D

2. Solving the Einstein equations

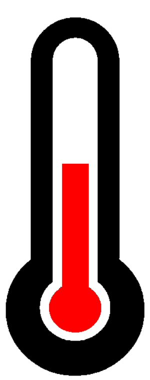
3. Application: gravitational dual of a CFT in a black hole background

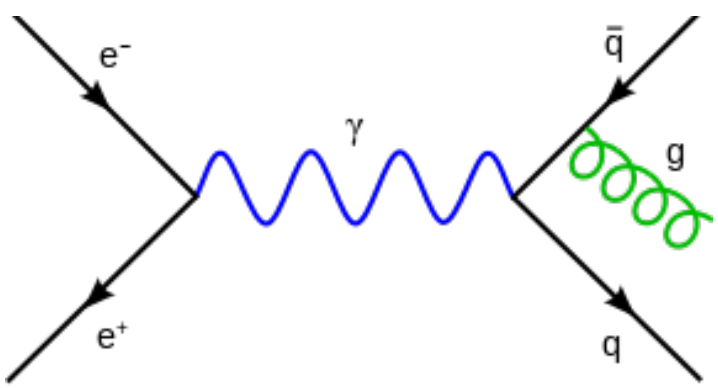
4. Summary and conclusions

General relativity and black holes beyond 4D

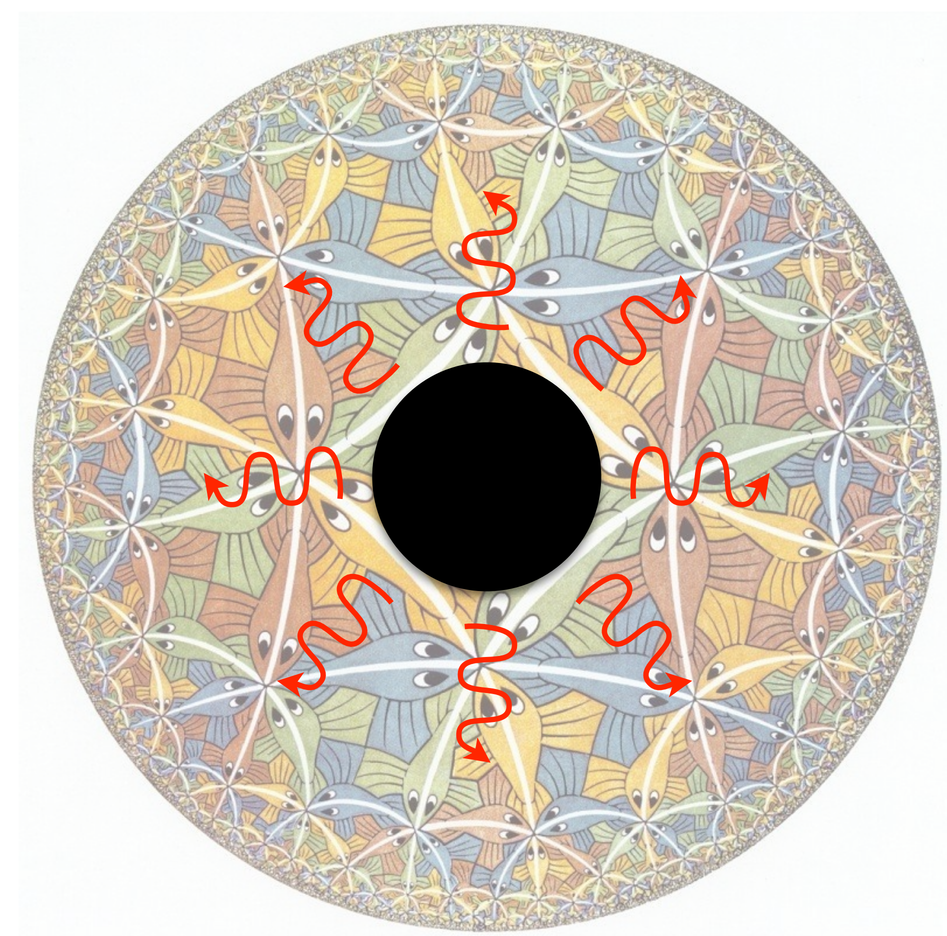
gauge/gravity correspondence [Maldacena]

strongly interacting gauge theory $\longleftrightarrow$ classical GR in anti-de Sitter space



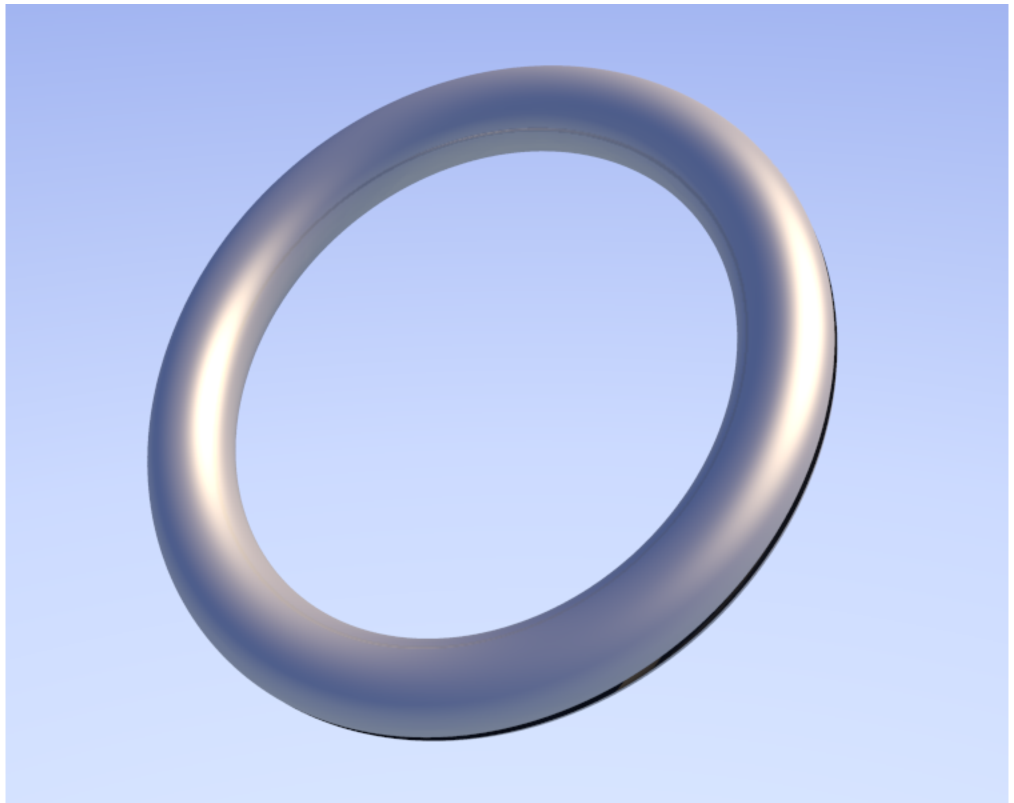


$\longleftrightarrow$



$\Rightarrow$ GR as a calculational tool

What's new?



- New topologies: black rings [Emparan and Reall], black saturns [Elvang and PF],. . .
- Non-uniqueness
- Symmetries
- Rotation is different:
  - Can have multiple spins
  - The angular momentum can be arbitrarily large

What's new?

- Phase transitions and *generic* instabilities

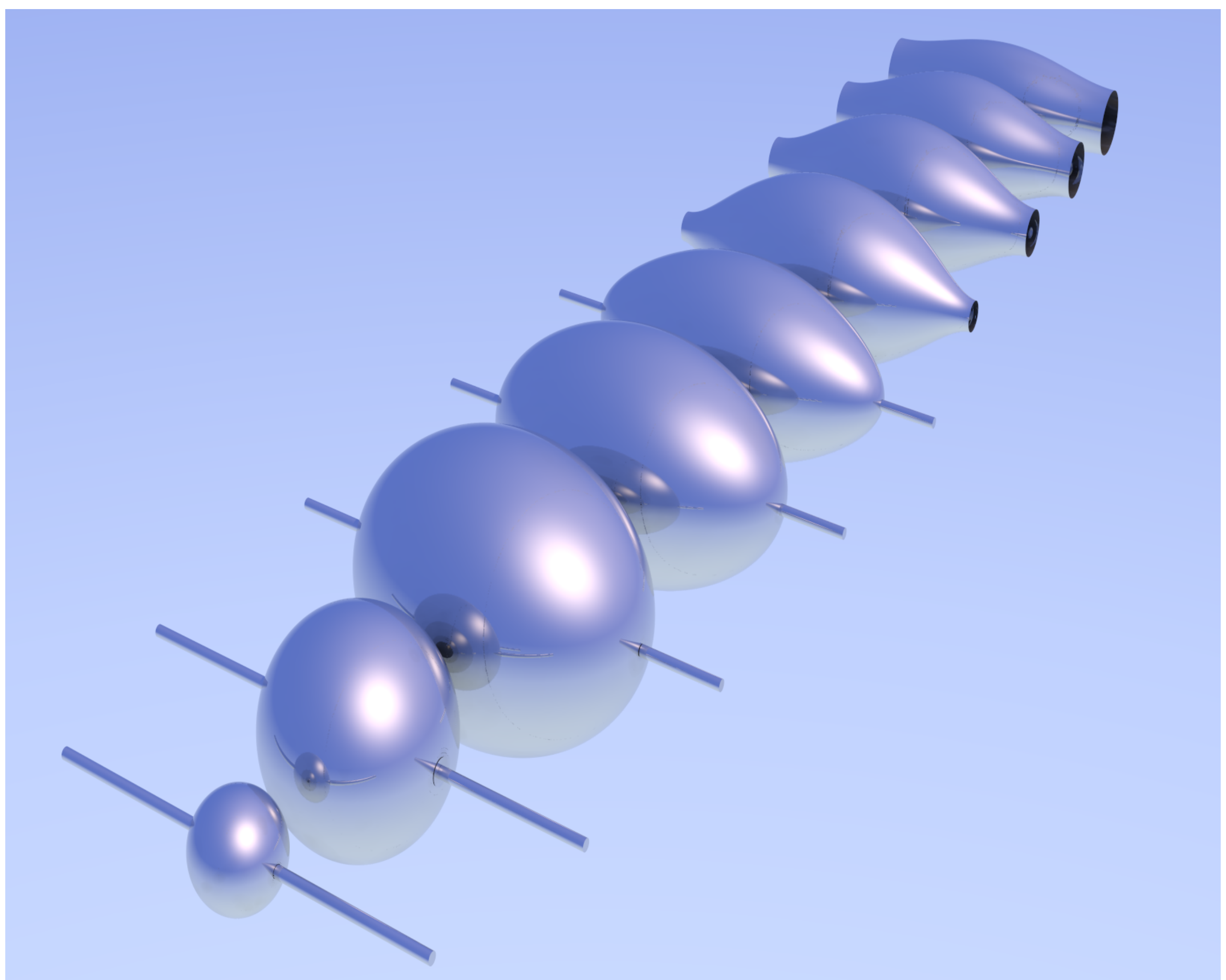
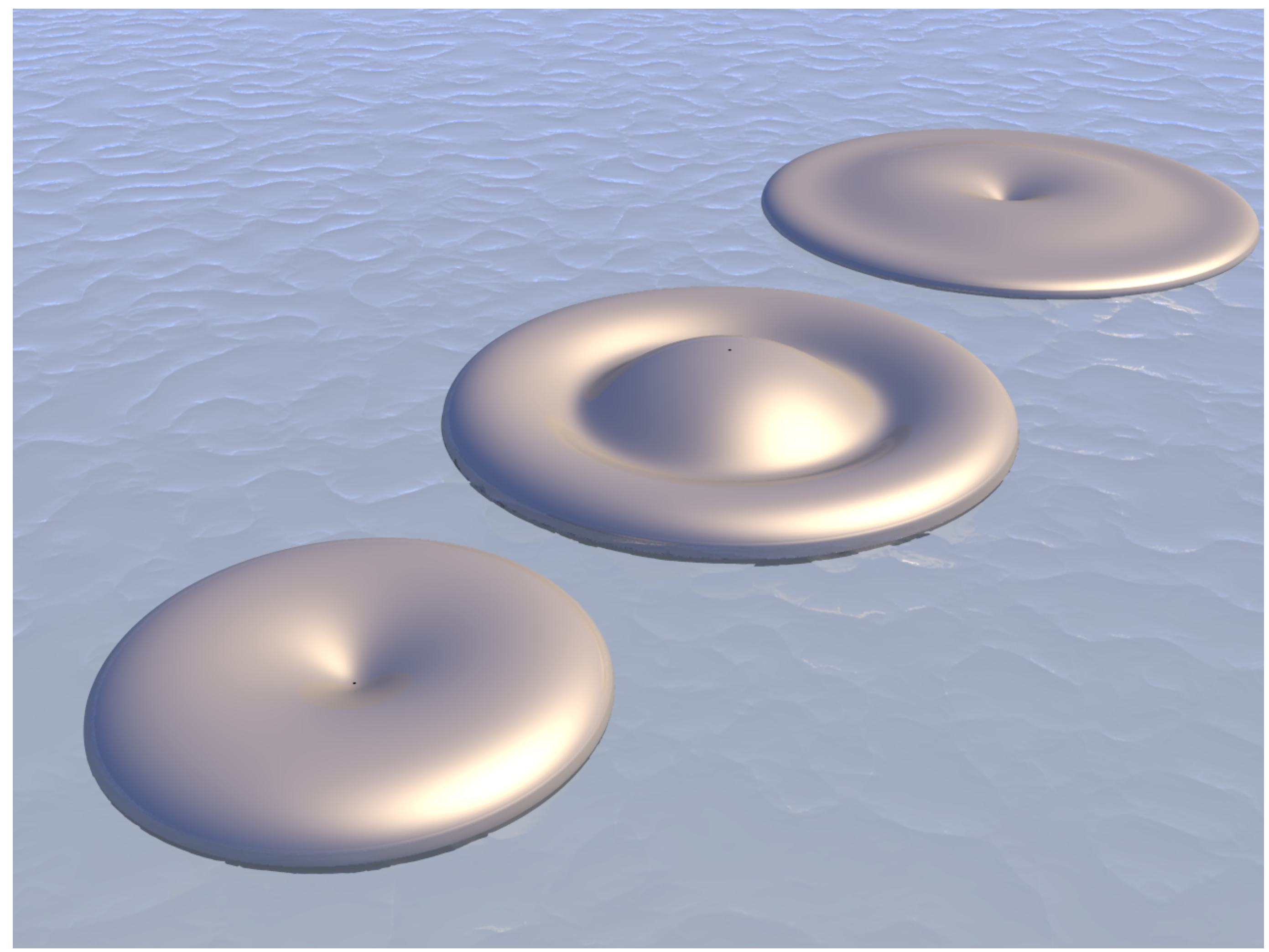


image courtesy of Toby Wiseman

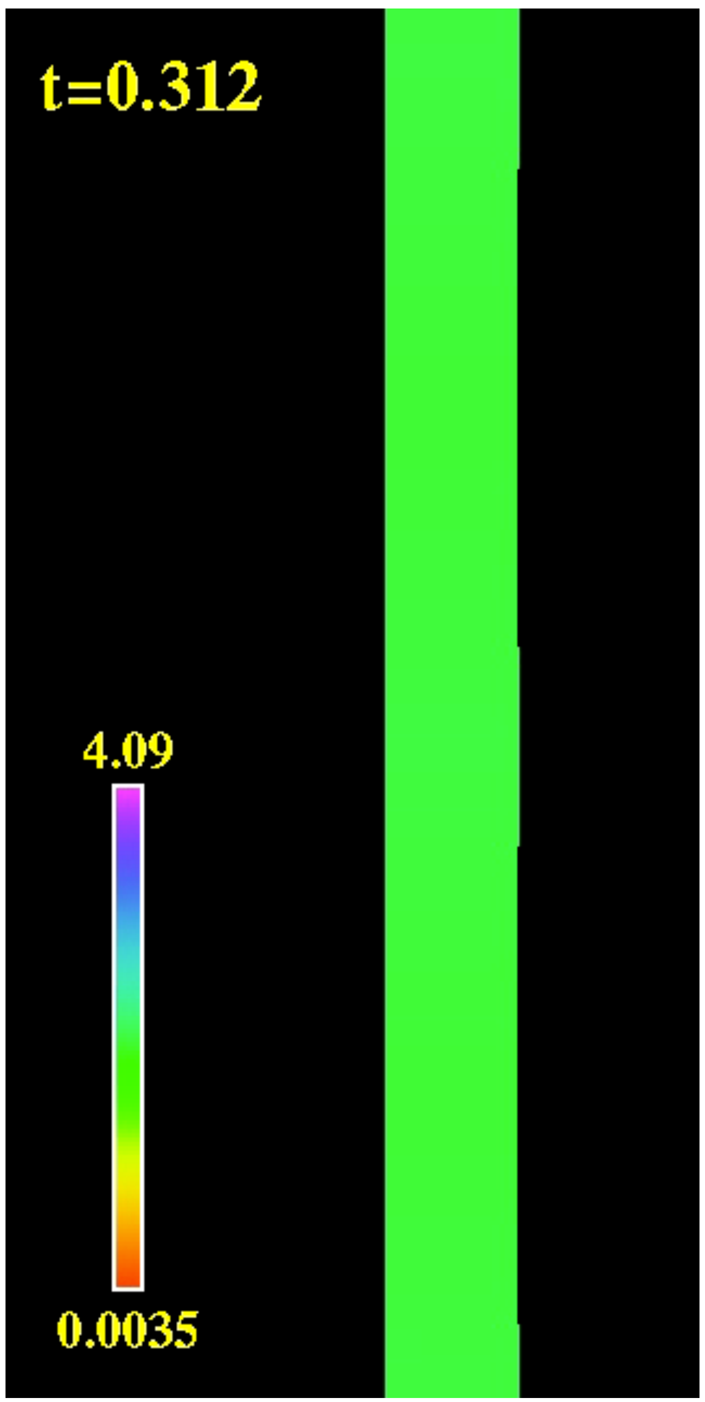


[Gregory and Laflamme; Wiseman]

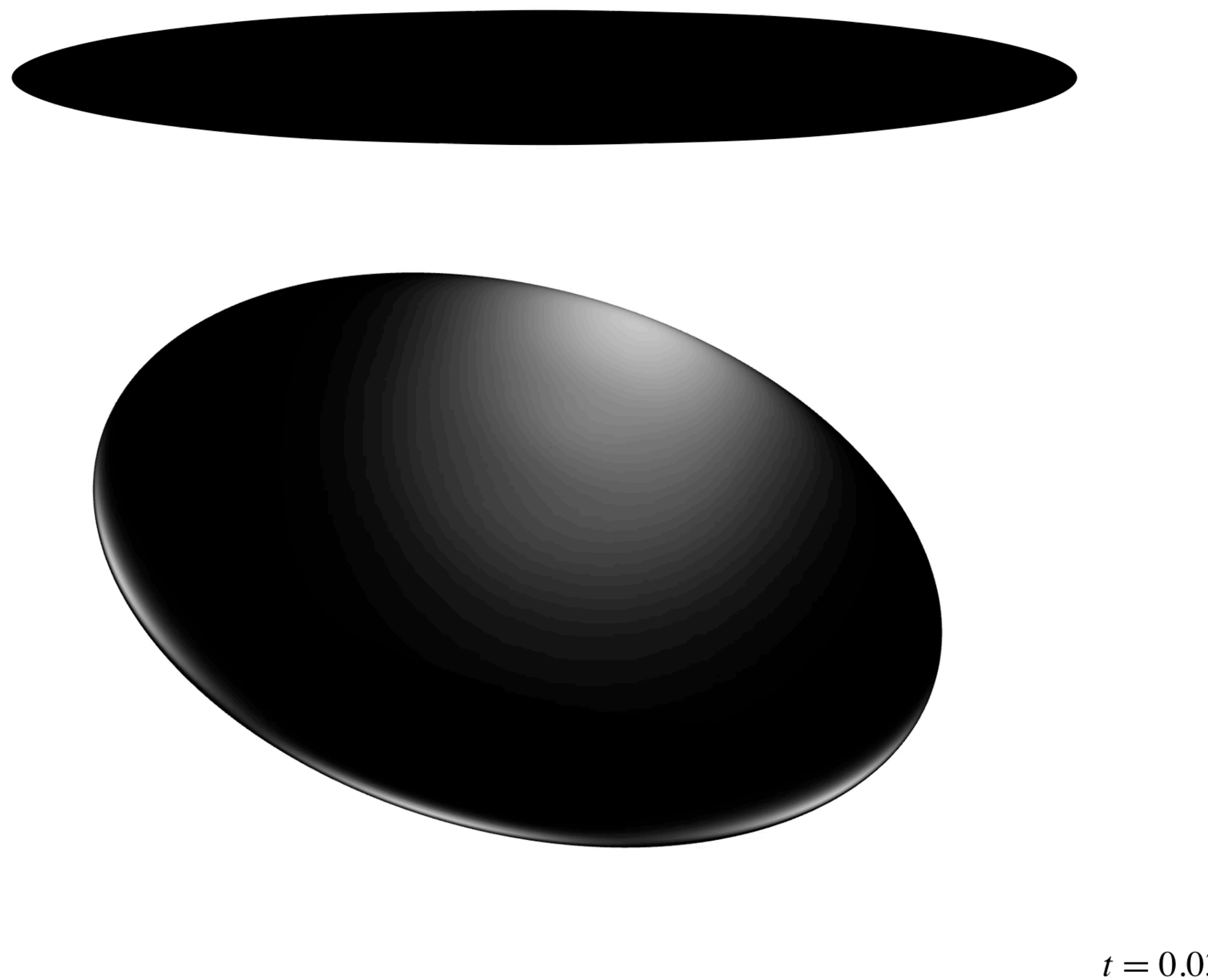
[PF et al;Emparan, Martínez, PF]

What's new?

- Formation of singularities in finite time



[Lehner and Pretorius]



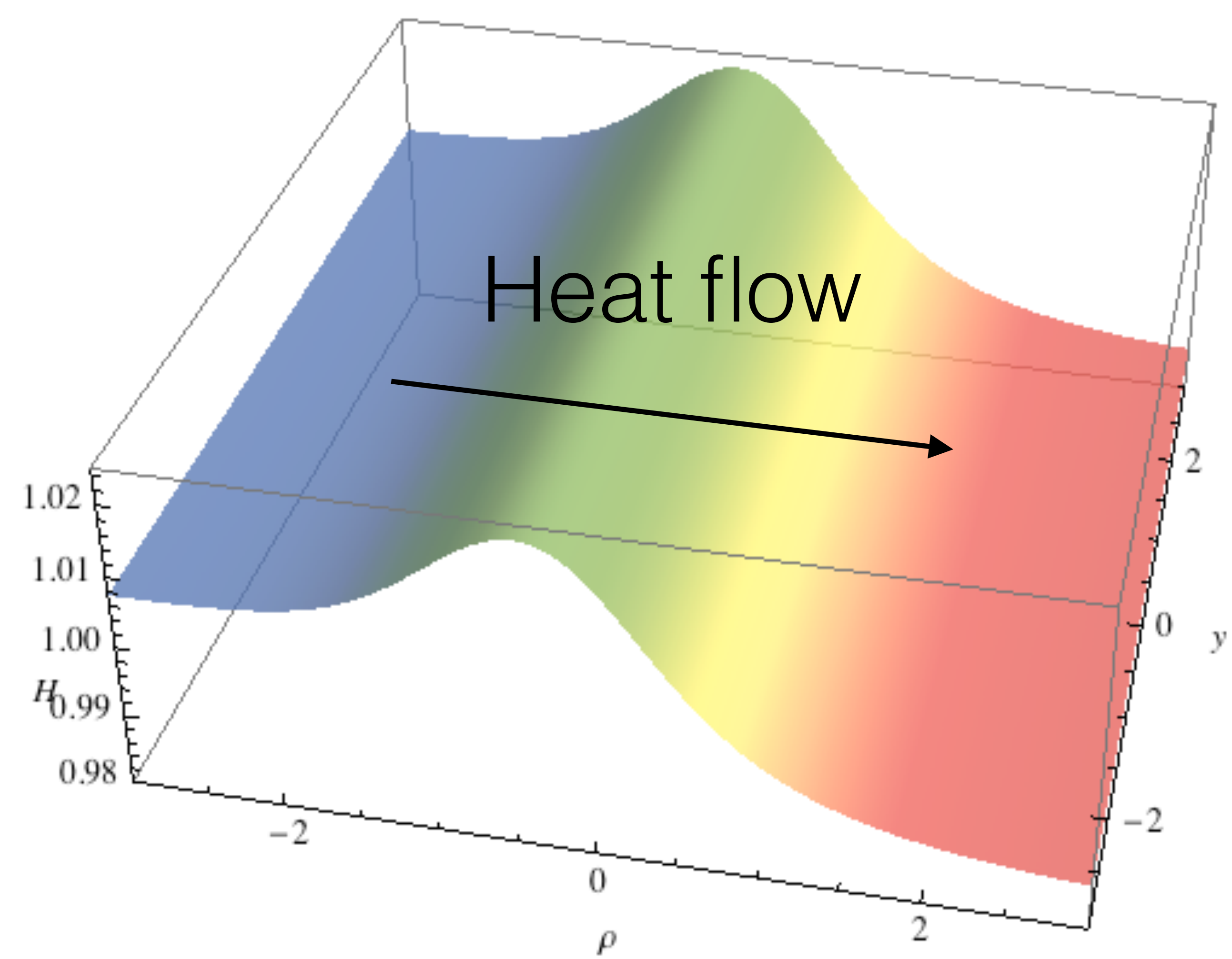
[PF, Kunesch, Tunyasuvunakool, *to appear*]

black holes in AdS

Many of the classical results on black holes do not hold in AdS:

- Non-spherical black holes are possible even in 4D
- Uniqueness? Symmetries?
- Black holes can move in a non-rigid way

[PF and Wiseman; Fischetti et al.]



Summary

- Black holes in $D > 4$ and/or AdS are much richer

- It is unlikely that we can explicitly construct new kinds of black holes…

Solving the Einstein equations

Solving the Einstein equations

- The Einstein equations for $(g, \mathcal{M})$

$$R_{ab}[g] - \frac{1}{2} R[g]\, g_{ab} = 8\pi\, G_N\, T_{ab}$$

- We want to solve them in two contexts:

  1. Find equilibrium configurations → elliptic

  2. Study time dependent processes → hyperbolic

- Main difficulty: coordinate invariance

Modern approach: covariant gauge fixing

[Choquet-Bruhat; DeTurck; Headrick et al.; Lucietti et al.; Adam et al.]

- Solve the Einstein-DeTurck equations

$$R^H_{ab} = 0 \qquad R^H_{ab} = R_{ab} - \nabla_{(a}\xi_{b)} \qquad \xi^a = g^{bc}(\Gamma^a{}_{bc} - \bar{\Gamma}^a{}_{bc})$$

- For general stationary spacetimes,

$$ds^2 = G_{AB}(x)\big(dy^A + A^A_i(x)\,dx^i\big)\big(dy^B + A^B_j(x)\,dx^j\big) + h_{ij}(x)\,dx^i\,dx^j$$

the equations are elliptic:

$$R^H_{ab} \sim -\frac{1}{2}\,h^{ij}\partial_i\partial_j\,g_{ab}$$

Solving the Einstein equations: Ricci flow

[Headrick and Wiseman; Holzegel et al.; Kitchen et al.; Lucietti et al.; Adam et al.]

- Simulate the Ricci-DeTurck flow:

$$\frac{\partial}{\partial\lambda}\, g_{ab} = -2\, R^H_{ab}$$

➡ evolve the metric until reaches a fixed point

- Comments:

  - Very easy to implement

  - Diffeomorphic to Ricci flow ⇒ existence of fixed points

Solving the Einstein equations: Ricci flow

- Given some ***generic*** initial data, how do we know if the flow will take us to the desired fixed point?

- The stability of the fixed point is determined by the spectrum of the Lichnerowicz operator

$$\frac{\partial}{\partial \lambda}\delta g_{ab} = -\Delta_L \delta g_{ab}$$

- For many black hole spacetimes, Δ_L is negative

[Gross, Perry and Yaffe]

➡ thermodynamic instabilities

Modern approach: covariant gauge fixing

[Lucietti, PF, Wiseman]

- Main idea: solve the Einstein-DeTurck equation with boundary conditions compatible with

$$\xi^a\Big|_{\partial\mathcal{M}} = 0$$

- Boundary conditions:

  - Asymptotic end: flat, (A)dS, KK

  - Regularity

  - Modified Dirichlet: $[h_{ij}], \mathrm{tr}K,\, \xi = 0$ [Anderson]

  - Mixed Dirichlet-Neumann: $K_{ij} = \lambda\, h_{ij},\, \xi_n = 0$

Modern approach: covariant gauge fixing

[Lucietti, PF and Wiseman]

- Static case with AF, AdS or KK boundary conditions

$$ds^2 = -N(x)\,dt^2 + h_{ij}(x)\,dx^i\,dx^j$$

➡ no Ricci solitons

$$\nabla^2\phi + \xi^a\partial_a\phi = -2\,\Lambda\,\phi + 2\,(\nabla^a\xi^b)(\nabla_a\xi_b) \geq 0 \qquad \phi = \xi^a\xi_a$$

- Stationary case: Now also proven

[PF and Wiseman, *to appear*]

Modern approach: covariant gauge fixing

- Stationary case: [PF and Wiseman, *to appear*]

  - Restricted to spacetimes of the form:

$$ds^2 = G_{AB}(x)\, dy^A\, dy^B + \hat{h}_{ij}(x)\, dx^i\, dx^j$$

  - Consider the scalar quantity: $\omega = \phi + \nabla^\mu v_\mu \,, \quad \phi = v^\mu v_\mu$

  - This satisfies: $\nabla^2 \omega + v^\mu \partial_\mu \omega = -2\Lambda\omega + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \geq 0 \quad F_{\mu\nu} = 2\, \partial_{[\mu} v_{\nu]}$

  - Use maximum principle to show: $\omega \leq 0$

Modern approach: covariant gauge fixing

- Stationary case: [PF and Wiseman, *to appear*]

  - Integrate ω over the whole manifold:

$$\int_{\mathcal{M}} dx\,\sqrt{\hat{h}}\,\sqrt{|G|}\,\omega = \int_{\mathcal{M}} dx\,\sqrt{\hat{h}}\,\sqrt{|G|}\left(\phi + \hat{\nabla}^i \hat{v}_i + \frac{1}{2\sqrt{|G|}}\,\hat{v}^i \partial_i \sqrt{|G|}\right) \leq 0$$

  - Integrate by parts the second and third terms:

$$\int_{\mathcal{M}} dx\,\sqrt{\hat{h}}\,\sqrt{|G|}\phi \leq -\int_{\partial\mathcal{M}} dS^i \sqrt{|G|}\,\hat{v}_i - \lim_{\hat{R}\to\infty}\int_{\hat{R}} dS^i \sqrt{|G|}\,\hat{v}_i$$

$$\Longrightarrow \int_{\mathcal{M}} dx\,\sqrt{\hat{h}}\,\sqrt{|G|}\,\phi = 0 \quad \Longrightarrow \quad \phi = 0 \quad \Longrightarrow \quad v_\mu = 0$$

Application:

strongly coupled CFT
in the Schwarzschild
Unruh vacuum

=

an interesting new class of ALH Einstein metrics with a good physical motivation

[w/ Lucietti and Wiseman]

CFT on a black hole background

- QFT's in black hole backgrounds exhibit interesting physical effects: Hawking radiation, vacuum polarisation…

- Can we study this in AdS/CFT?

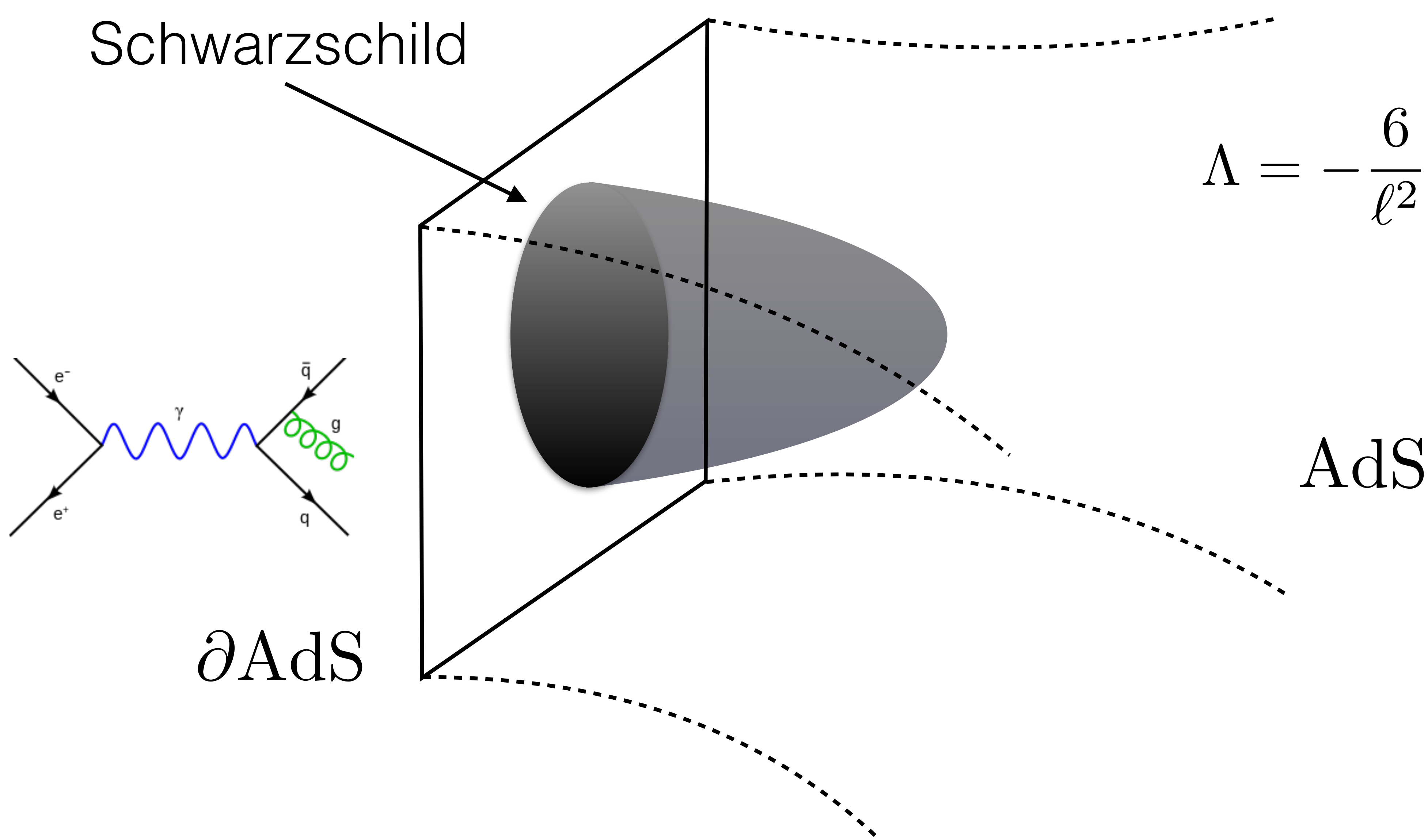
- Free field theory intuition:

  Hawking radiation: $\hbar O(N^2)$

  $\Rightarrow$ black holes may not be static in classical GR!

[Emparan et al.]

Spacetime



Spacetime: construction

- Choose the most general metric ansatz with the desired isometries: $\mathbb{R}_t \times SO(3)$

$$ds^2 = \frac{\ell^2}{f(x)^2}\bigg(-4\,r^2 f(r)^2 e^T dt^2 + x^2 g(x) e^S d\Omega_{(2)}^2 + \frac{4}{f(r)^2}\, e^{T+r^2 f(r) A} dr^2 + \frac{4}{g(x)} e^{S+x^2 B}\, dx^2 + \frac{2\,r\,x}{f(r)}\, F\, dr dx\bigg)$$

$$f(x) = 1 - x^2$$

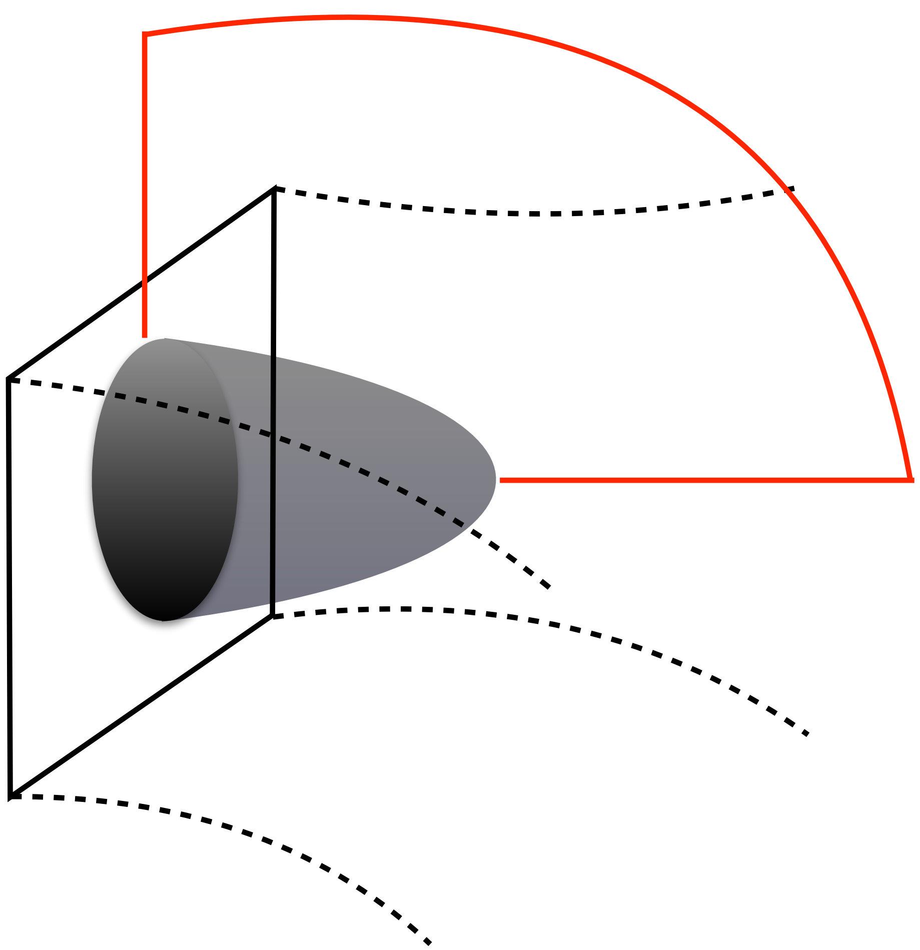
$$g(x) = 2 - x^2$$

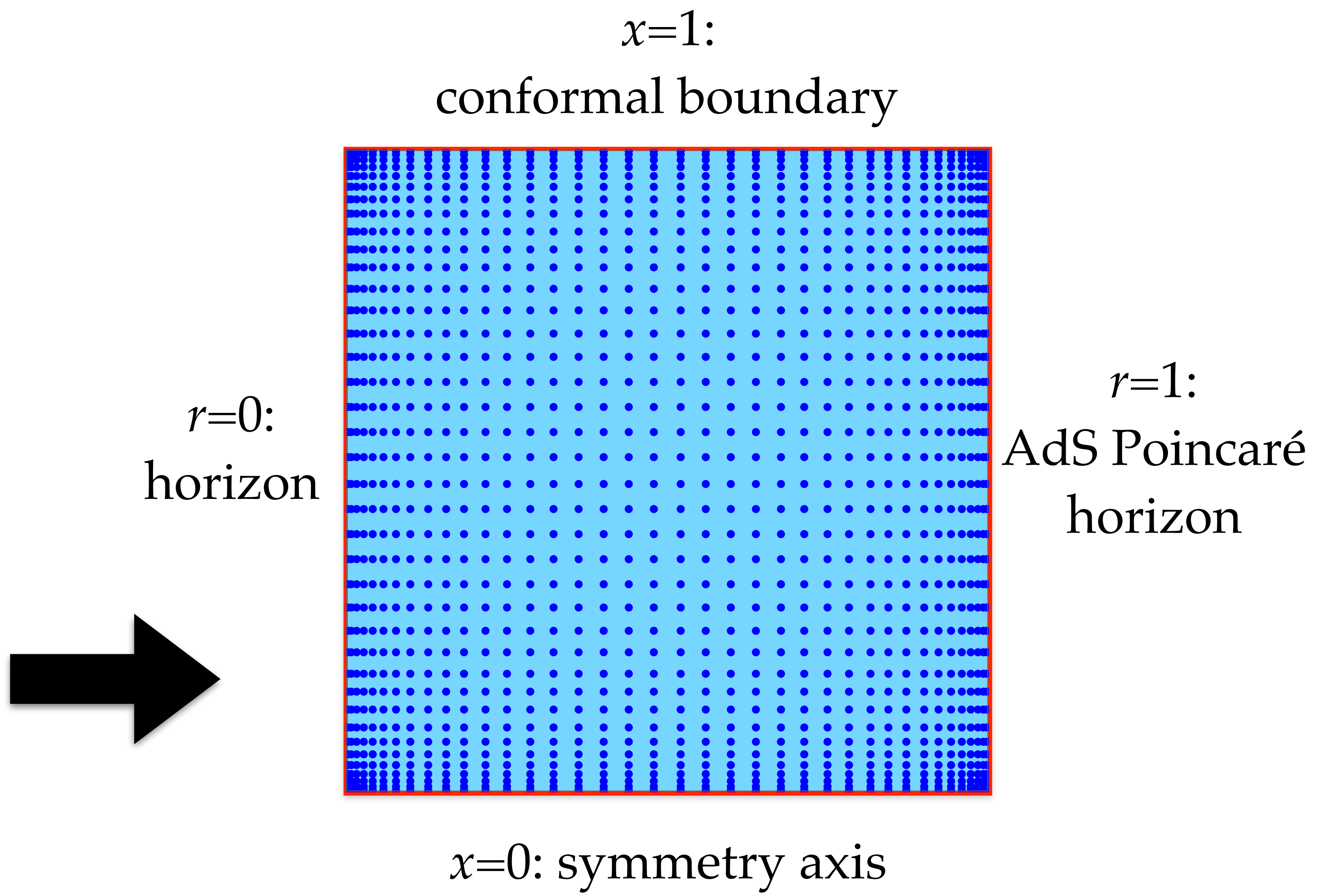
- Unknowns: *T, S, A, B, F*

- Reference metric:

$$T = S = A = 0\,,\; B = -\frac{18}{5}\, f(r)^2\, f(x)^2\,,\; F = -\frac{6}{5}\, f(r)^2 f(x)^3$$

Spacetime: construction





Spacetime: construction

- Remarks

  1. Smoothness with this choice of BCs and reference metric

  2. No Ricci solitons

  3. No free parameters

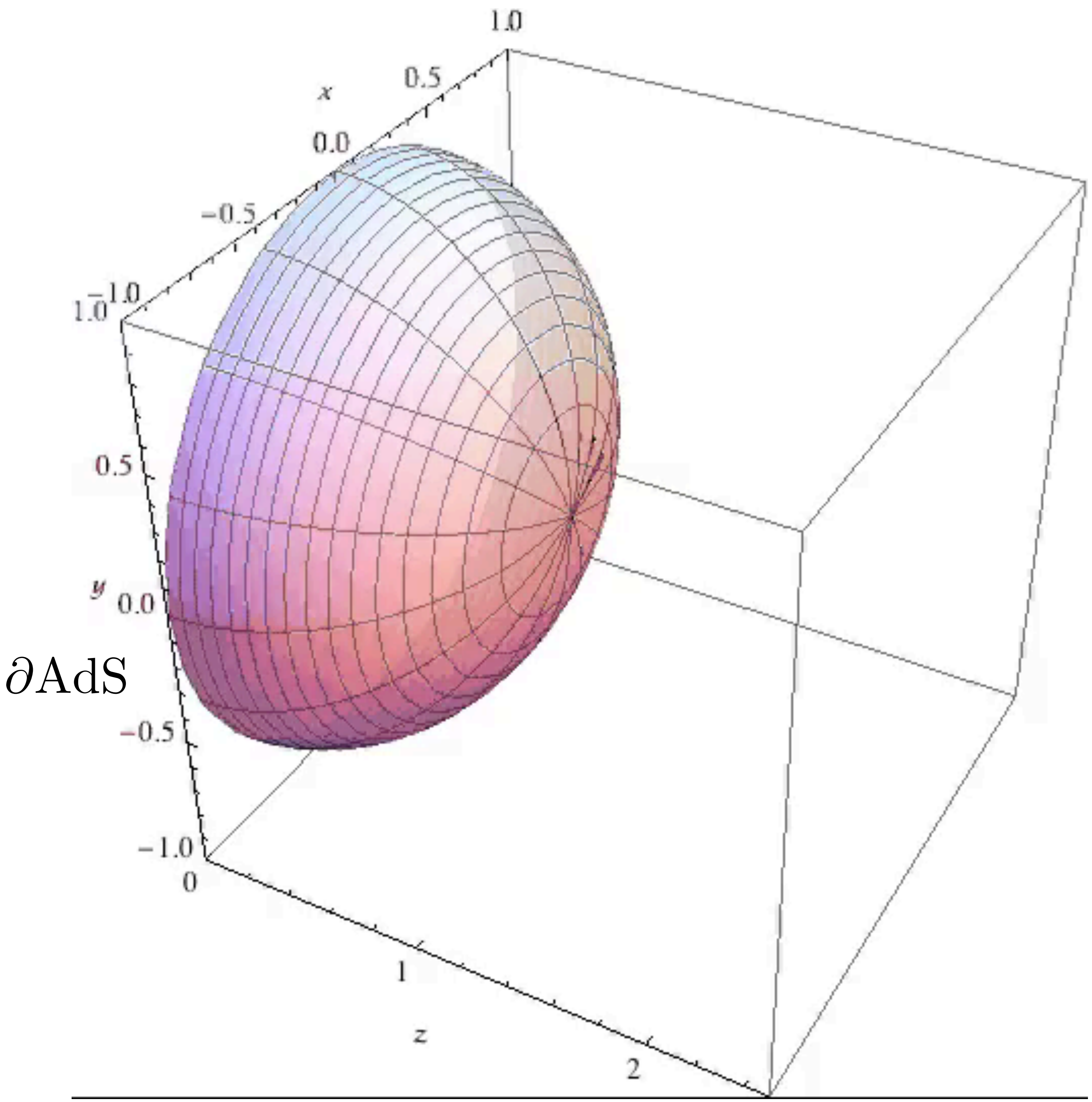
Spacetime: construction

- Numerical solution:

    1. Spatial discretisation: pseudospectral

    2. Simulate diffusion: forward Euler

Numerical Ricci flow



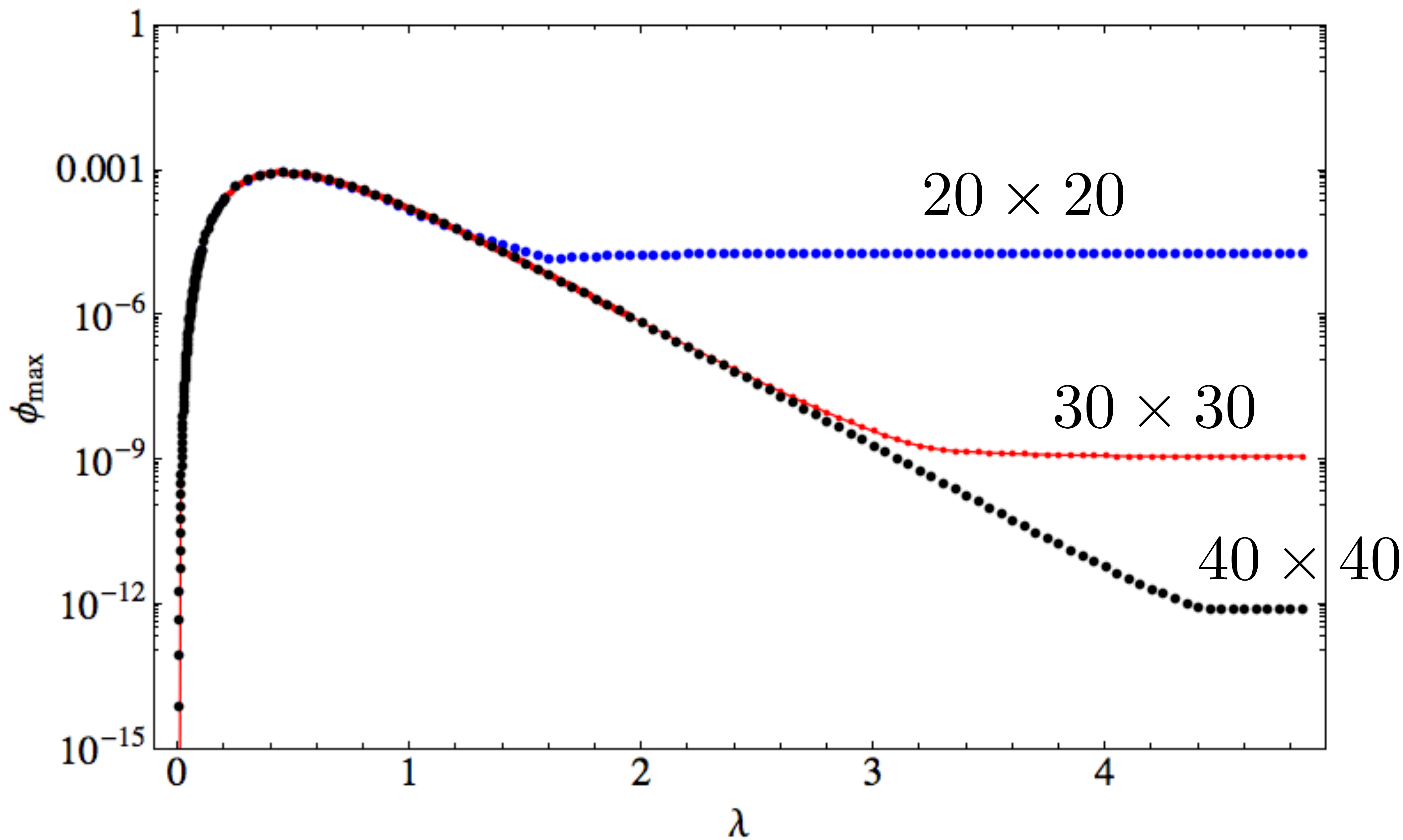
Embedding of the spatial metric on the horizon into

$$ds_H^2 = \frac{1}{z}\left(dz^2 + dy^2 + y^2\, d\Omega_{(2)}^2\right)$$

$$y = y(z)$$

Numerical Ricci flow

- Convergence:



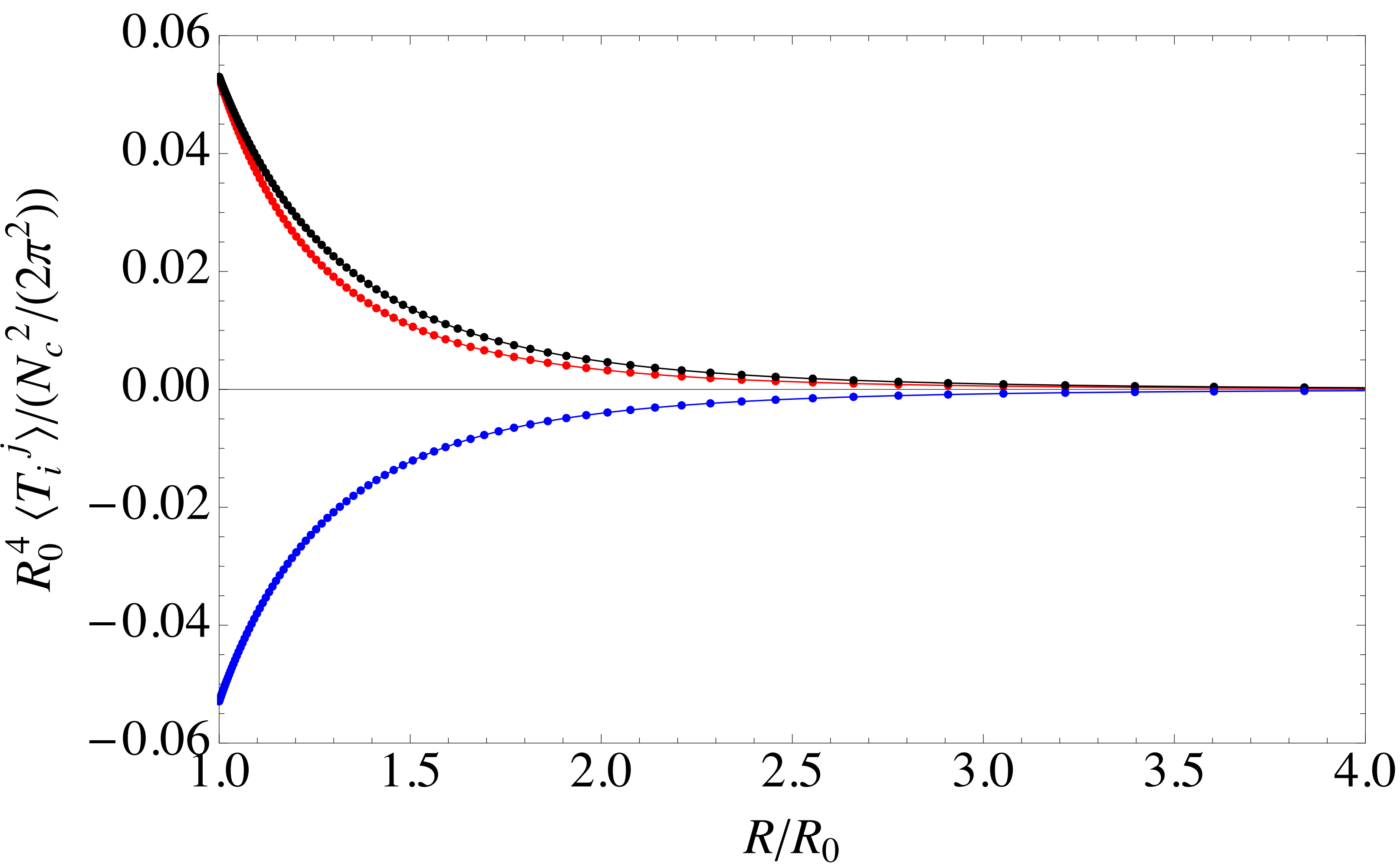
Dual stress tensor

- Fefferman-Graham expansion:

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + h_{ij}(z,x)\,dx^i\,dx^j\right) \qquad h_{ij}(z,x) \approx g_{ij}^{\rm Schw}(x) + z^4\,t_{ij}(x) + \dots$$

- AdS/CFT:

$$\langle T_{ij}\rangle^{\rm CFT} = \frac{\ell^3}{16\pi\,G_N}\,t_{ij}$$



Dual stress tensor

- Gravitational dual of the $\mathcal{N}=4$ SYM on the background of Schwarzschild in the Unruh vacuum

- The dual classical geometry captures the $O(N_c^2)$ fo the full quantum stres-energy tensor. It is ***static and regular***

- Negative energy density everywhere → vacuum polarisation

TO DO:

- Prove existence of a smooth solution to the Einstein equation with these boundary conditions

Summary and conclusions

- Ricci flows naturally arise in black hole physics

- These a flows on stationary non-compact Lorentzian manifolds with a variety of boundary conditions: AF, dS, AdS, KK

- Black holes spacetimes can be unstable fixed points of Ricci flow

- Having an elliptic system of PDEs one can use other standard techniques (e.g., Newton’s method) to find solutions very efficiently

- Prove existence of solutions rigorously

Thank you!!!