Numerical Ricci flows and black holes

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General relativity

• Provides the “correct” classical description of the gravitational force at sufficiently large scales

• Einstein equations determine the spacetime metric $g$:

\[
R_{ab}[g] - \frac{1}{2} R[g] g_{ab} = 8\pi G_N T_{ab}
\]

• Non-linear PDEs $\Rightarrow$ difficult to solve in general

• Black holes are amongst the most important solutions of Einstein’s equations
4D black holes

4D asymptotically flat stationary vacuum black holes are well-understood in classical general relativity:

- Spherical topology [Hawking]
- Rotate rigidly [Hawking]
- Uniqueness [Carter; Robinson; Bunting; Mazur]
- Methods to construct all 4D black holes explicitly are known [Kerr; Belinskii and Zakharov]
- They exist in Nature: GW150914 [LIGO]
- Stability?
Why general relativity beyond 4D?

- **Address fundamental questions** about the nature of gravity in different settings
- **Practical applications**: general relativity with certain boundary conditions gives access to certain strongly interacting (non-gravitational) physics
- **Mathematical interest**: solutions have new properties
Outline of the talk

1. General relativity and black holes beyond 4D
2. Solving the Einstein equations
3. Application: gravitational dual of a CFT in a black hole background
4. Summary and conclusions
General relativity and black holes beyond 4D
gauge/gravity correspondence

strongly interacting gauge theory $\leftrightarrow$ classical GR in anti-de Sitter space

$\Rightarrow$ GR as a calculational tool

[Maldacena]
What’s new?

• **New topologies:** black rings [Emparan and Reall], black saturns [Elvang and PF],…

• **Non-uniqueness**

• **Symmetries**

• **Rotation is different:**
  - Can have multiple spins
  - The angular momentum can be arbitrarily large
What’s new?

• Phase transitions and *generic* instabilities

image courtesy of Toby Wiseman

[Gregory and Laflamme; Wiseman]  [PF et al;Emparan, Martínez, PF]
What’s new?

• Formation of singularities in finite time

[Lehner and Pretorius] [PF, Kunesch, Tunyasuvunakool, to appear]
Many of the classical results on black holes do not hold in AdS:

- Non-spherical black holes are possible even in 4D
- Uniqueness? Symmetries?
- Black holes can move in a non-rigid way

[PF and Wiseman; Fischetti et al.]
Summary

• Black holes in $D > 4$ and/or AdS are much richer

• It is unlikely that we can explicitly construct new kinds of black holes…
Solving the Einstein equations
Solving the Einstein equations

• The Einstein equations for \((g, \mathcal{M})\)

\[
R_{ab}[g] - \frac{1}{2} R[g] g_{ab} = 8\pi G_N T_{ab}
\]

• We want to solve them in two contexts:

1. Find equilibrium configurations \(\rightarrow\) elliptic

2. Study time dependent processes \(\rightarrow\) hyperbolic

• Main difficulty: coordinate invariance
Modern approach: covariant gauge fixing

[Choquet-Bruhat; DeTurck; Headrick et al.; Lucietti et al.; Adam et al.]

• Solve the Einstein-DeTurck equations

\[ R_{ab}^{H} = 0 \quad R_{ab}^{H} = R_{ab} - \nabla_{(a} \xi_{b)} \quad \xi^{a} = g^{bc}(\Gamma_{bc}^{a} - \bar{\Gamma}_{bc}^{a}) \]

• For general stationary spacetimes,

\[ ds^2 = G_{AB}(x)(dy^A + A^A_i(x) \, dx^i)(dy^B + A^B_j(x) \, dx^j) + h_{ij}(x) \, dx^i \, dx^j \]

the equations are elliptic:

\[ R_{ab}^{H} \sim -\frac{1}{2} h^{ij} \partial_i \partial_j g_{ab} \]
Solving the Einstein equations: Ricci flow

[Headrick and Wiseman; Holzegel et al.; Kitchen et al.; Lucietti et al.; Adam et al.]

• Simulate the Ricci-DeTurck flow:

\[ \frac{\partial}{\partial \lambda} g_{ab} = -2 R^H_{ab} \]

→ evolve the metric until reaches a fixed point

• Comments:

  - Very easy to implement

  - Diffeomorphic to Ricci flow \( \Rightarrow \) existence of fixed points
Solving the Einstein equations: Ricci flow

- Given some *generic* initial data, how do we know if the flow will take us to the desired fixed point?

- The stability of the fixed point is determined by the spectrum of the Lichnerowicz operator

\[
\frac{\partial}{\partial \lambda} \delta g_{ab} = -\Delta_L \delta g_{ab}
\]

- For many black hole spacetimes, $\Delta_L$ is negative

$\rightarrow$ thermodynamic instabilities

[Gross, Perry and Yaffe]
Modern approach: covariant gauge fixing

• **Main idea**: solve the Einstein-DeTurck equation with boundary conditions compatible with

\[ \xi^a \bigg|_{\partial \mathcal{M}} = 0 \]

• **Boundary conditions**:

  - Asymptotic end: flat, (A)dS, KK
  - Regularity
  - Modified Dirichlet: \([h_{ij}], \text{tr} K, \xi = 0\)
  - Mixed Dirichlet-Neumann: \(K_{ij} = \lambda h_{ij}, \xi_n = 0\)
Modern approach: covariant gauge fixing

- Static case with AF, AdS or KK boundary conditions

\[ ds^2 = -N(x) \, dt^2 + h_{ij}(x) \, dx^i \, dx^j \]

\[ \nabla^2 \phi + \xi^a \partial_a \phi = -2 \Lambda \, \phi + 2 \left( \nabla^a \xi^b \right) \left( \nabla_a \xi_b \right) \geq 0 \quad \phi = \xi^a \xi_a \]

- Stationary case: Now also proven

[Lucietti, PF and Wiseman]

[PF and Wiseman, to appear]
Modern approach: covariant gauge fixing

• Stationary case:

- Restricted to spacetimes of the form:

\[ ds^2 = G_{AB}(x) \, dy^A \, dy^B + \hat{h}_{ij}(x) \, dx^i \, dx^j \]

- Consider the scalar quantity: \( \omega = \phi + \nabla^\mu v_\mu \), \( \phi = v^\mu v_\mu \)

- This satisfies: \( \nabla^2 \omega + v^\mu \partial_\mu \omega = -2\Lambda \omega + \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \geq 0 \)

- Use maximum principle to show: \( \omega \leq 0 \)
Modern approach: covariant gauge fixing

- Stationary case:

  - Integrate $\omega$ over the whole manifold:

    $$\int_{\mathcal{M}} dx \sqrt{h} \sqrt{|G|} \omega = \int_{\mathcal{M}} dx \sqrt{h} \sqrt{|G|} \left( \phi + \hat{\nabla}^i \hat{v}_i + \frac{1}{2\sqrt{|G|}} \hat{v}^i \partial_i \sqrt{|G|} \right) \leq 0$$

  - Integrate by parts the second and third terms:

    $$\int_{\mathcal{M}} dx \sqrt{h} \sqrt{|G|} \phi \leq - \int_{\partial \mathcal{M}} dS^i \sqrt{|G|} \hat{v}_i - \lim_{\hat{R} \to \infty} \int_{\hat{R}} dS^i \sqrt{|G|} \hat{v}_i$$

    $$\implies \int_{\mathcal{M}} dx \sqrt{h} \sqrt{|G|} \phi = 0 \quad \implies \quad \phi = 0 \quad \implies \quad v_\mu = 0$$
Application:

strongly coupled CFT
in the Schwarzschild
Unruh vacuum

= 

an interesting new class of ALH Einstein
metrics with a good physical motivation

[w/ Lucietti and Wiseman]
CFT on a black hole background

- QFT’s in black hole backgrounds exhibit interesting physical effects: Hawking radiation, vacuum polarisation…

- Can we study this in AdS/CFT?

- Free field theory intuition:
  
  Hawking radiation: $\hbar O(N^2)$

  $\Rightarrow$ black holes may not be static in classical GR!

[Emparan et al.]
Spacetime

\[ \Lambda = -\frac{6}{\ell^2} \]

Schwarzschild

\partial \text{AdS}

\text{AdS}
Spacetime: construction

• Choose the most general metric ansatz with the desired isometries: $\mathbb{R}_t \times SO(3)$

$$ds^2 = \frac{\ell^2}{f(x)^2} \left( -4 r^2 f(r)^2 e^T dt^2 + x^2 g(x)e^S d\Omega^2_{(2)} + \frac{4}{f(r)^2} e^{T+r^2 f(r) A} dr^2 
+ \frac{4}{g(x)} e^{S+x^2 B} dx^2 + \frac{2 r x}{f(r)} F dxdx \right)$$

$$f(x) = 1 - x^2$$
$$g(x) = 2 - x^2$$

• Unknowns: $T, S, A, B, F$

• Reference metric:

$$T = S = A = 0, \quad B = -\frac{18}{5} f(r)^2 f(x)^2, \quad F = -\frac{6}{5} f(r)^2 f(x)^3$$
Spacetime: construction

$r=0$: horizon

$x=1$: conformal boundary

$r=1$: AdS Poincaré horizon

$x=0$: symmetry axis
Spacetime: construction

• Remarks

1. Smoothness with this choice of BCs and reference metric

2. No Ricci solitons

3. No free parameters
Spacetime: construction

- Numerical solution:
  1. Spatial discretisation: pseudospectral
  2. Simulate diffusion: forward Euler
Numerical Ricci flow

Embedding of the spatial metric on the horizon into

\[ ds^2_H = \frac{1}{z} \left( dz^2 + dy^2 + y^2 \, d\Omega^2_{(2)} \right) \]

\[ y = y(z) \]
Numerical Ricci flow

- Convergence:
Dual stress tensor

- Fefferman-Graham expansion:

\[ ds^2 = \frac{\ell^2}{z^2} (dz^2 + h_{ij}(z, x) \, dx^i \, dx^j) \quad h_{ij}(z, x) \approx g_{ij}^{\text{Schw}}(x) + z^4 \, t_{ij}(x) + \ldots \]

- AdS/CFT:

\[ \langle T_{ij} \rangle_{\text{CFT}} = \frac{\ell^3}{16\pi \, G_N} \, t_{ij} \]
Dual stress tensor

- Gravitational dual of the $\mathcal{N}=4$ SYM on the background of Schwarzschild in the Unruh vacuum

- The dual classical geometry captures the $O(N_c^2)$ for the full quantum stress-energy tensor. It is **static and regular**

- Negative energy density everywhere $\rightarrow$ vacuum polarisation
TO DO:

• Prove existence of a smooth solution to the Einstein equation with these boundary conditions
Summary and conclusions
• Ricci flows naturally arise in black hole physics

• These a flows on stationary non-compact Lorentzian manifolds with a variety of boundary conditions: AF, dS, AdS, KK

• Black holes spacetimes can be unstable fixed points of Ricci flow

• Having an elliptic system of PDEs one can use other standard techniques (e.g., Newton’s method) to find solutions very efficiently

• Prove existence of solutions rigorously
Thank you!!!