Ehlers-Kundt conjecture and the initial value problem for gravitational waves

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Based on joint work with JL FLORES, arxiv: 1706.03855

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3. Assuming $V(z, u)$ polynomially bounded for finite values of u,
   $\rightsquigarrow$ we will give a positive answer
4. In general, including autonomous case $V(z, u) \equiv V(z)$, open
   Basic question in potential theory!!

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1. Background:
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    - Initial value problem in this framework
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2. To sketch the ideas for
   1. Technical setup
   2. Main steps of the solved polynomial case.

GW: (too) short experimental summary

Gravitational waves: prediction by Einstein

- Hulse and Taylor (1974): undirect evidence
- LIGO experiment: direct measurement '15 reported in '16
  ...just the first step (eLISA, etc.)

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Some highlights:

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- Historic meeting at Chapel Hill'57 (Feynmann explained his sticky bead argument)
- End 50's waves in mainstream: Bondi, Pirani, Robinson '59

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- ...had been studied by the mathematician Brinkmann'25

Parallelly propagated plane-fronted waves

pp-wave: $\mathbb{R}^4$ endowed with:

$$g = dx^2 + dy^2 + 2\, du\, dv + H(z,u)\, du^2, \quad z := (x,y), \quad (x,y,u,v) \in \mathbb{R}^4$$

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plane wave: at each $u \in \mathbb{R}$, H polynomial in x, y of degree ≤ 2.

Penrose's *remarkable property*

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> *no spacelike hypersurface exists in the spacetime which is adequate for the global specification of Cauchy data*

So:

> at what extent plane waves are physically meaningful or just idealizations of the model

About global hyperbolicity

(M, g) **globally hyperbolic** if any of the following holds:

1. It is (necessarily strongly) causal and has no naked singularities, i.e. $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$.
2. It admits a (topological) Cauchy hypersurface S (S: subset crossed exactly once by any inextensible timelike curve)
3. It admits a Cauchy temporal function τ (τ smooth with spacelike and Cauchy levels $\tau =$ constant).
   So, orthogonal Cauchy splitting:
   $M = \mathbb{R} \times S, g = -\Lambda(\tau, x) d\tau^2 + g_\tau$

[Topological assertions: Geroch '70.
Weakening of strong causality: Bernal & S. '07
Smoothening: Bernal & S. '03,'05
(improvements in Müller & S, '11, Müller'16; other approaches: Fathi & Siconolfi '13, Chrusciel et al. '16, Bernard & Suhr '18.)]

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Moreover, in this case (Bernal & S. '06):

- Any *compact acausal spacelike hyp. with boundary* is extensible to a spacelike Cauchy hyp.
- Any *spacelike Cauchy hyp.* S is the level of some τ

Relaxing to conformal boundaries

$(\bar{M}, g)$ globally hyp. with **timelike boundary**
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This suggests *mixed hyperbolic problems* with:

- Initial conditions on $\{0\} \times \bar{S}$ (slice $\tau \equiv 0$) +
- Boundary conditions on $\mathbb{R} \times \partial S$ $(= \partial M$) +
- Compatibility on $\{0\} \times \partial S$ ($= (\{0\} \times \bar{S}) \cap (\mathbb{R} \times \partial S)$)

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- Marolf & Ross '02 '03: examples with no conformal bd. (bad even for holographic principle)
  $\rightsquigarrow$ extended study including the causal boundary

Conformal vs causal boundary

Reasonable conformal boundaries are **not** always available:

- Alternative causal boundary $\partial_c M$ (starting at a seminal idea by Geroch, Kronheimer and Penrose '72):
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  intrinsic and general for strongly causal spacetimes
- **Historic problems:** redefinition by Marolf-Ross
  (in order to study pp-waves)
- Boundary points: (P, F)
  - $P = I^-(\gamma)$, where γ inextendible, future-directed timelike
  - $F = I^+(\tilde{\gamma})$, where $\tilde{\gamma}$ inextendible, past-directed timelike
  - (P, F) constitute a pair when they are *S-related* (Szabados'88)
    $\rightsquigarrow$ pairs $(P, \emptyset)$ or $(\emptyset, F)$ are allowed otherwise

Conformal vs causal boundary

Redefinition and systematic study (Flores, Herrera, S '11):

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2. Under general conditions
(applicable to globally hyp. s.t. with timelike bd, up to $i^{\pm}$)
the conformal and causal boundaries agree
3. When a conf. bound. ∂M is C^1 + *chronologically complete*
(inextensible timelike curves have an endpoint at ∂M):
M is glob. hyp. $\iff$ $T(\partial M)$ is nowhere timelike
(consistency with globally hyp. s.t. with timelike boundary)

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- Marolf & Ross '02 '03: interesting particular cases
- Flores & S. '08: systematic study c-boundary of pp-waves
  - Characterization possible dimensions of $\partial_c M$: $1, \ldots, n-1$.
  - Analysis conformal $\partial M \neq$ causal $\partial_c M$
  - many problematic examples
    including pp-waves non-strongly causal (nor distinguishing)!
    ...neither the conformal nor the c-boundary make sense!

EK conjecture: physical statement

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  Prove the plane waves to be the only complete [gravitational] ***pp-waves****, no matter which topology one chooses.*

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Under their viewpoint:

- Complete and Ricci flat pp-waves would represent a graviton field independent of any matter by which it would be generated
- Such gravitons (allegedly plane waves), would correspond to source-free photons in electrodynamics.

EK conjecture: the loose end

Recall:

- Source-free photons, represented by monocromatic sine waves, constitute an idealization (very useful: the basis of the Fourier analysis of homogeneous electromagnetic waves).
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In connection with the commented ideas, EK conjecture:

1. assigns the idealized role suggested by Penrose
2. circumvents the initial/mixed value problem for plane waves
3. for pp-waves, problem transferred to incompleteness
   $\rightsquigarrow$ something is missed in the modelling (so, add a source!)

Precise mathematical problem

$$\left.\begin{array}{c}\text{Ricci-flat}\\ (\Delta_z H = 0)\\ +\\ \text{Complete}\end{array}\right\}\ \text{pp-wave (any } H) \Longrightarrow \text{plane wave (quad. polyn. } x, y)$$

Reformulation in Lagrangian (Newtonian) Mechanics

Theorem

Put $V = -H$. A pp-wave is complete $\Leftrightarrow$ all the trajectories of

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are complete.

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Conjecture (Alternative EK Lagrangian conjecture)

Let $V(z,u)$ non-autonomous potential on $\mathbb{R}^2$, harmonic in z.
The dynamical system (1) is complete $\Longleftrightarrow$
$V(z,u)$ is a (at most) quadratic polynomial in z, $\forall u \in \mathbb{R}$.

Nomenclature for known results

• H polynomially bounded at finite u-times or just polynomially u-bounded:
$\forall u_0 \in \mathbb{R}, \exists \epsilon_0 > 0$ and polynomial q_0:

$$H(z,u) \leq q_0(z) \qquad\qquad \forall (z,u) \in \mathbb{R}^2 \times (u_0 - \epsilon_0, u_0 + \epsilon_0)$$

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- H quadratically polynomially u-bounded: q_0 degree ≤ 2 $\forall u_0 \in \mathbb{R}$.

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(c) **Leistner-Schliebner'16:** focus on the vague assertion "No matter which topology one chooses".
It can be extended rigorously (by taking a locally pp-wave metric on a compact manifold) and:
the problem becomes equivalent to the standard one on $\mathbb{R}^4$.
($+$ further extensions by **Costa-Flores-Herrera'16**)

Our goal: proof of polynomial EK conjecture

Theorem (Flores-S., arxiv 1706.03855)

EK conjecture is true when $H(z, u)$ is polynomially u-bounded.

Discussion

1. Necessity of the polynomial bound:
   - z-harmonicity $\Rightarrow$ z-analiticity $\Rightarrow$ z-polynomial series
     $\rightsquigarrow$ but the technique crashes for infinite series.
   - Physically: EK conj holds at any finite perturbative order

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2. Significative (even in the autonomous case $V(z,u) \equiv V(z)$)
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   - Links with the theory of polynomial holomorphic vector on $\mathbb{C}^2$
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   - Open case of physical and mathematical interest:
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3. Natural smoothness in the non-autonomous case: C^1 (existence of geodesics)
   $\rightsquigarrow$ Analiticity will not be used in our techniques.

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- Classically u is the "external time"
  (but u is not a time function in the relativistic sense for the pp-wave)

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4. ... choose $h_4(z) = ih_3$ holomorphic and un-switch the coordinates $\rightsquigarrow$ **but this is again** h_1

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Incompatibility, complex variable does not seem to work!

Explicit form of $V = -H$

Result 1: expression of V in polar coordinates

- Autonomous case: z-harmonic + polynomially u-bounded $\Longrightarrow$

$$\begin{aligned} V(\rho,\theta) = \quad & -\lambda \rho^n \cos n(\theta+\alpha) \\ & - \textstyle\sum_{m=0}^{n-1} \lambda_m \rho^m \cos m(\theta+\alpha_m), \end{aligned} \qquad (2)$$

$\lambda > 0$ and $\lambda_m, \alpha, \alpha_m$ constants.

Tool: harmonic polynomials on $\mathbb{C} = \mathbb{R}^2$

1. V harmonic $\Leftrightarrow$ real part of an entire function V_C
2. V **(upper)** bounded by a polynomial $\Leftrightarrow$ V polynomial
3. V harmonic polynomial of degree n $\Leftrightarrow$
   terms of degree m ($\in \{1, \ldots, n\}$) harmonic p_m
4. Homogeneous harmonic polynomials degree $m > 0$: 2-dim
   $p_m(\rho, \theta) = A_m \rho^m \cos(m\theta) + B_m \rho^m \sin(m\theta)$
   $= \lambda_m \rho^m \cos m(\theta + \alpha_m)$
5. Thus, V harmonic polynomial p of degree $n > 0$:
   $p = \sum_{m=0}^{n} p_m$

u-dependence

- Non autonomous case: expected explicit form

$$V(\rho,\theta,u) = \begin{array}{l} -\lambda(u)\rho^n \cos n(\theta+\alpha(u)) \\ -\sum_{m=0}^{n-1} \lambda_m(u)\rho^m \cos m(\theta+\alpha_m(u)), \end{array} \tag{3}$$

$\lambda(u) > 0$, C^1-smooth $\lambda(u), \lambda_m(u), \alpha(u), \alpha_m(u)$.

u-dependence

- Non autonomous case: expected explicit form

$$V(\rho,\theta,u) = \begin{array}{l} -\lambda(u)\rho^n \cos n(\theta+\alpha(u)) \\ -\sum_{m=0}^{n-1} \lambda_m(u)\rho^m \cos m(\theta+\alpha_m(u)), \end{array} \tag{3}$$

$$\lambda(u)>0,\ C^1\text{-smooth } \lambda(u),\lambda_m(u),\alpha(u),\alpha_m(u).$$

...but only in some (dense) intervals $I=(u_0-\epsilon_0,u_0+\epsilon_0)$

$$V(z,u) = \begin{cases} e^{-1/u^2}\rho^n\cos(n\theta+1/u) & u\neq 0 \\ 0 & u=0 \end{cases}$$

$\rightsquigarrow$ non-continuous $\alpha(u)=1/u$

EK conjecture vs Lagrangian EK conjecture

Theorem

Put $V = -H$. A pp-wave is complete $\Leftrightarrow$ all the trajectories of

$$\ddot{z}(s) = -\nabla_z V(z(s), s) \tag{4}$$

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Recall: V complete $\not\Longleftarrow\!\!\!\Longrightarrow$ $-V$ complete

Criterion for completeness: upper quadratic bound of $-V$

Proposition

$(-V)$ *quadratically polynomially u-bounded, that is,*

$$-V(z,u) \leq a(u)|z|^2 + b(u)$$

$\Rightarrow$ *complete trajectories*

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Rough idea. Autonomous case: constant energy
$E = \frac{1}{2}\dot{z}^2(s) + V(z(s))$

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Rough idea. Autonomous case: constant energy
$E = \frac{1}{2}\dot{z}^2(s) + V(z(s)) \Rightarrow |\dot{z}(s)| \lesssim |z(s)| \Rightarrow \text{length}(z|_{[0,s]}) \lesssim e^{C|s|}$
$\Rightarrow$ in a finite time, $z(s)$ covers a finite length
(Lagrangian viewpoint) $\rightsquigarrow$ completeness
(Non-autonomous bound too $\rightsquigarrow$ restrict to compact $[u_0, u_1]$) $\Box$

Criterion for incompleteness: lower radial quadratic bound

$\rightsquigarrow$ **For EK, focus on** $n \geq 3$ **and incompleteness**

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Trajectory in polar coordinates, $(\rho(s), \theta(s))$

Lemma (Criterion incompleteness of ρ)

For $n > 2$, assume:

$$\ddot{\rho}(s) \geq n\lambda_0 \rho^{n-1}(s), \qquad \rho(0) > 0, \qquad \dot{\rho}(0) \geq 0.$$

$\lambda_0 > 0 \Rightarrow$ *all the solutions are incomplete (to the right)*

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Idea of the proof. After some manipulations

$$\int_{\rho(0)}^{\rho} \frac{d\bar{\rho}}{\sqrt{2\lambda_0(\bar{\rho}^n - \rho(0)^n) + \dot{\rho}(0)^2}} \geq s(\rho)$$

and the integral is finite for $\rho = \infty$. $\square$

Precise result

For $n \geq 3$, autonomous case, assuming with no loss of generality:

$$V(\rho,\theta) = -\lambda\rho^n \cos n\theta - \sum_{m=0}^{n-1} \lambda_m \rho^m \cos m(\theta + \alpha_m),$$

Proposition

For any $0 < \theta_0 < \theta_+ < \pi/(2n)$ $\exists \rho_0 > 0$*: any trajectory* γ *with*

$$\gamma(0) = (\rho(0), \theta(0)) \in D[\rho_0, \theta_0] := \{(\rho,\theta) : \rho > \rho_0, |\theta| < \theta_0\}$$

and $\dot\rho(0) \geq 0, \dot\theta(0) = 0$, *satisfies:*

(a) γ *remains in* $D[\rho_0, \theta_+]$

(b) *whenever in this region,* $\ddot\rho(s) \geq \lambda_0 n \rho^{n-1}(s)$,

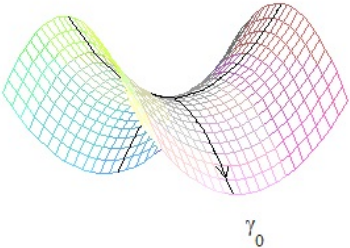
Thus, γ *is incomplete*

The role of harmonicity

- If $V(\rho, \theta) = -\lambda \rho^n \cos n\theta$ (only leading term): incomplete trajectory γ_0 reparametrizing positive x-axis.
- In general: γ_0 is not a trajectory but a direction of maximum asymptotic decreasing

The role of harmonicity

- If $V(\rho,\theta) = -\lambda\rho^n \cos n\theta$ (only leading term):
  incomplete trajectory γ_0 reparametrizing positive x-axis.
- In general: γ_0 is not a trajectory but a direction of maximum asymptotic decreasing
- Region $D[\rho_0,\theta_0]$: by harmonicity each point of γ_0 is a strict minimum of V under θ-variations $\rightsquigarrow$ oscillations
  $\rightsquigarrow$ confinement in $D[\rho_0,\theta_+]$ (where V decreases enough fast)



Summing up: claims

In general:

- No hope to find an (incomplete) radial trajectory
- ...but a radial direction γ_0 so that V decreases fast on γ_0 and:

1. Trajectories $(\rho(s), \theta(s))$ starting at some $D[\rho_0, \theta_0]$ will have bounded angular oscillations around γ_0
2. ...so that they remain in some bigger $D[\rho_0, \theta_+]$, $\theta_0 < \theta_+ < \pi/(2n)$
3. and $\rho(s)$ satisfies the differential inequality ensuring incompleteness on $D[\rho_0, \theta_+]$

Preliminary step: steepest V-decreasing direction

- Identify a radial direction γ_0 with steepest decreasing V (provided by the leading mononomial of V)

Potential and its gradient

Identification of θ_k for radial γ_k:

$$V(\rho,\theta) = -\rho^n \cos(n\theta) - \sum_{m=1}^{n-1} \lambda_m \rho^m \cos m(\theta + \alpha_m),$$

(up to a rotation and homothety), thus

$$\begin{aligned} -\nabla V = \quad & \left(n\rho^{n-1}\cos(n\theta) + \textstyle\sum_{m=1}^{n-1} m\lambda_m \rho^{m-1} \cos m(\theta+\alpha_m) \right) \partial_\rho \\ & - \left(n\rho^{n-2}\sin(n\theta) + \textstyle\sum_{m=1}^{n-1} m\lambda_m \rho^{m-2} \sin m(\theta+\alpha_m) \right) \partial_\theta \end{aligned}$$

$\partial_\theta V$ vanish for big ρ at n angles $\vartheta_k(\rho) \in [0, 2\pi)$:

$$\lim_{\rho\to\infty} \vartheta_k(\rho) = \hat{\theta}_k := 2\pi k/n, \qquad\qquad k = 0, \ldots, n-1.$$

($\hat{\theta}_k := (2\pi k - \alpha(u))/n$ in the non-autonomous case)

Aim

Focus on the angle $\hat{\theta}_0 = 0$, choose $0 < \theta_0 < \theta_+ < \pi/(2n)$:

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Proposition

V harmonic polynomial of degree $n \geq 3$
$\exists \rho_0 > 0$: any V-trajectory γ with

$$\gamma(0) = (\rho(0), \theta(0)) \in D[\rho_0, \theta_0] \qquad \dot{\rho}(0) \geq 0,\ \dot{\theta}(0) = 0, \tag{5}$$

remains in $D[\rho_0, \theta_+]$ and is incomplete.

Steps

- **Step 0**: for big $\rho_0 > 0$ study the region $D[\rho_0, \theta_+]$ to find suitable technical bounds for V, $\partial V/\partial\rho$ (necessary for confinement and incompleteness)

Steps

For suitably prescribed ρ_0,
and any trajectory γ ($\gamma(0) \in D[\rho_0, \theta_0] \subset D[\rho_0, \theta_+]$) as in Prop.:

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1. Bound the growth of $|\theta(s)|$-peaks in terms of $\rho(s)$.
   (the biggest starting point $\rho(s_0)$, the smallest growth)
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2. Check that, from peak to peak, ρ increases very fast, so that the increasing of the amplitudes of the oscillations will not allow γ to escape from $D[\rho_0, \theta_+]$.
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$\rightsquigarrow$ ρ will arrive ∞ (in finite time) before an oscillation moving γ outside $D[\rho_0, \theta_+]$ can occur.

Step 0: technical bounds for V from its harmonicity

Proposition

(Incompleteness) $\forall\theta_+ \in (0, \pi/(2n))$ $\delta_0 \in (0, \cos(n\theta_+))$, $\exists \rho_0 > 1$: $-\partial_\rho V(\rho,\theta) > \delta_0 n \rho^{n-1}$ $(> 0), \forall \rho \geq \rho_0, \forall\theta \in (-\theta_+, \theta_+)$.

(Confinement) *Chosen* $0 < \epsilon < \theta_- < \theta_+$, $\exists \rho_0 > 1, \delta > 0$ *s. t.* $\forall\theta_1 \in [\theta_-, \theta_+]$, $\forall\rho \geq \rho_0$:

$V(\rho,\theta_1) - V(\rho,\theta) > \delta(\theta_1 - \theta)\rho^n$, $\forall\theta \in (-\theta_1 + \epsilon, \theta_1)$,

$V(\rho,-\theta_1) - V(\rho,\theta) > \delta(\theta_1 + \theta)\rho^n$, $\forall\theta \in (-\theta_1, \theta_1 - \epsilon)$,

$\partial_\rho V(\rho,\theta_1) - \partial_\rho V(\rho,\theta) > \delta n(\theta_1 - \theta)\rho^{n-1}$, $\forall\theta \in (-\theta_1 + \epsilon, \theta_1)$,

$\partial_\rho V(\rho,-\theta_1) - \partial_\rho V(\rho,\theta) > \delta n(\theta_1 + \theta)\rho^{n-1}$, $\forall\theta \in (-\theta_1, \theta_1 - \epsilon)$.

Moreover, chosen $0 < \theta_- < \theta_+ < \pi/(2n)$, *one can find* $A > 0$, $\rho_0 > 1$ *such that all the previous inequalities hold for* $\rho \geq \rho_0$, *by replacing* ϵ *and* δ *by* A/ρ *and* $1/\rho$, *resp.*,

$\epsilon > 0 \rightarrow$ lower order asymmetry of V $\epsilon\rho \sim A > 0$;

$\theta_1 \rightarrow$ use to bound the amplitude of oscillation

Step 1: Radial projection

- Trajectory $\gamma(s) \equiv (\rho(s), \theta(s))$, $s \in [0, b)$: and $\theta_1 \in \mathbb{R}$:
  θ_1-projection: $\gamma_{\theta_1}(s) \equiv (\rho(s), \theta_1)$
- Energy $\gamma_{\theta_1}(s)$= radial kinetic $\gamma(s)$ + potential $V(\rho(s), \theta_1)$
- In particular, choosing $s_1 \in [0, b)$, energy $\theta(s_1)-$projection:
  Energy[$\theta(s_1)$-proj]: $F(s) = \frac{1}{2}\dot{\rho}(s)^2 + V(\rho(s), \theta(s_1))$

Step 1: bounding angular peaks by the radial distance

First estimate of the (angular) peaks.
Key: whenever $\theta(s)$ is monotonous, the energy F cannot decrease.

Proposition

Choose $0 < \theta_- < \theta_+ < \pi/(2n)$ *and let* $A, \rho_0 > 0$ *as above.*
Let $(\rho(s), \theta(s))$*, trajectory with*
$(\rho(s_0), \theta(s_0)) \in D[\rho_0, \theta_+]$*,* $\dot{\theta}(s_0) = 0$ *and* $\dot{\rho}(s_0) \geq 0$
for some $s_0 \in [0, b)$*.*
If $s_1 \in (s_0, b)$ *satisfies* $|\theta(s_0)| < |\theta(s_1)| < \theta_+$*, and* $\theta(s)$ *is (non-necessarily strictly)* **monotonous on** (s_0, s_1)**,** *then*
$|\theta(s_1)| - |\theta(s_0)| \leq A/\rho(s_0)$*.*

Step 2: Angular length

- Such an inequality is not enough to confine γ in $D[\rho, \theta_+]$: we must assure that the radial coordinate grows enough fast in each oscillation.

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Angular length:

$$\overline{\theta}(s) := \int_{s_0}^{s} |\dot{\theta}(\sigma)| d\sigma, \quad s \in [s_0, b) \tag{6}$$

Step 2: Bound of the angular length

Lemma

For some ρ_0 big enough, any solution $\gamma : [s_0, b) \to \mathbb{R}^2$ starting at $D[\rho_0, \theta_+]$ with $\dot{\rho}(s_0) > 0, \dot{\theta}(s_0) = 0$ satisfies (for $\Lambda := 8/\cos(n\theta_+) > 1$):

$$\rho(s) > \rho(s_0)e^{\overline{\theta}(s)/\Lambda}$$

for all $s \in (s_0, b)$ such that $\gamma([s_0, s]) \subset D[\rho_0, \theta_+]$.

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Discussing the possible cases (none/finite/infinite oscillations), this bound combined with the previous one, implies the confinement in $D[\rho_0, \theta_+]$ $\square$

Conclusion

Achievements:

- Clarify physical and mathematical grounds of EK conjecture
- Solve the significative polynomial case

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Interest:

1. **Relativity**: foundational basis of gravitational waves.
2. **Classical Mechanics**: the forces under consideration (coming from a divergence free gradient potential) are the most standard ones in Mechanics!!
3. **Dynamical Systems**: proof completely original, no standard tool on stability and attractors seem to be appliable
4. **Complex variable (theory of holomorphic vector fields)**: also in this framework, only the polynomial case has been fully developed

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Obvious open lines of research:

- Non-polynomial case
- Beyond the original motivation: higher dimensions, impulsive waves, Finslerian modifications of the waves...

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What is more EK conjecture introduces the pattern

$$\text{Source-free dynamics} \Longrightarrow \begin{cases} \text{Natural (mathematical) vacuum, or} \\ \text{Incompleteness (eventually missed source),} \end{cases}$$

which may serve as a paradigm for other parts of Physics, as well as for its mathematical modelling.

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Thank you
for your attention