Non-Abelian gauge fields in Superstring Theory & Gauged Supergravity

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Plan of the talk:

1.- $\overbrace{\text{Gravity} + \text{YM}}^{\text{EYM}}$ + SUSY = SEYM

2.- SEYM $\subset$ Superstring Theory?

→ Heterotic superstring → d=5
→ d=4

3.- "Black-hole ansatz" $\mathcal{O}(1)$ → d=5,4

4.- "Black-hole ansatz" $\mathcal{O}(\alpha')$ Instantons

5.- α'-corrected solutions & non-Abelian BHs.

1.- Gravity + YM (EYM)

$$S = \int d^d x \sqrt{|g|} \left\{ R - \frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} \right\}$$

Very complicated problem.

Only numerical solutions known.

Non-Abelian hair with unknown interpretation:

monopoles? (Higgs needed) instantons?

Susy can help (1st order differential equations)

$$\text{EYM} \longrightarrow \text{SEYM}$$ (Gauged SUGRA)

SEYM

1.- SUSY + Gravity = Supergravity (SUGRA)

2.- Supergravity $\supset$ fields charged under some gauge symmetry (like non-Abelian YM)

$\longrightarrow$ "Gauged SUGRA"

3.- SEYM $\equiv$ the minimal gauged SUGRA that contains a given YM group

not unique! $\rightarrow$ $N = 1, 2, \ldots$

\+

Different additional fields $\left\{ \begin{array}{l} U(1)\text{s} \\ \text{Scalars (Higgs)} \end{array} \right.$

and couplings ($V(\phi)$)

4.- ~~Fermions~~ (always consistent)

5.- Powerful solution-generating techniques for supersymmetric (a.k.a. "BPS") solutions.

$\delta_\epsilon B \sim \bar{\epsilon} \cancel{F}$ $= 0$; always

$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0$; Killing Spinor Equations.

$\Downarrow$

All the Bs for which these equations can be solved have been characterized (1st order diff eqs.)

Only a few independent e.o.m. left to be solved.

In gauged SUGRA $\longrightarrow$ non-Abelian SUSY BHs, BRs, global instantons, monopoles.

6.- Some SUGRAS arise as the low-energy effective field theory of a Superstring Theory

How about gauged SUGRAS?

Non-Abelian YM fields in S.T.?

Too difficult! (eff. ac.)

- → From complicated compactifications
- → From symmetry enhancement mechanisms
- → Already present in $d=10$

Heterotic superstring.

$$\downarrow T^4$$

$d=5$ gauged SUGRA

$$\downarrow S^1$$

$d=4$ gauged SUGRA

The Heterotic Superstring Effective Action

$B = \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu$; $e^a = e^a{}_\mu\, dx^\mu$; ϕ ; A^A ;

$$H^{(0)} \equiv dB \qquad \omega^a{}_b(e) \ ; \qquad d\phi \ ;$$

$$\Omega^{(0)}_{(\pm)}{}^a{}_b = \omega^a{}_b \pm \tfrac{1}{2} H^{(0)}_\mu{}^a{}_b dx^\mu$$

$$R^{(0)}_{(\pm)}{}^a{}_b = d\Omega^{(0)}_{(\pm)}{}^a{}_b - \Omega^{(0)}_{(\pm)}{}^a{}_c \wedge \Omega^{(0)}_{(\pm)}{}^c{}_b ,$$

$$\omega^{\mathrm{L}\,(0)}_{(\pm)} = d\Omega^{(0)}_{(\pm)}{}^a{}_b \wedge \Omega^{(0)}_{(\pm)}{}^b{}_a - \tfrac{2}{3}\Omega^{(0)}_{(\pm)}{}^a{}_b \wedge \Omega^{(0)}_{(\pm)}{}^b{}_c \wedge \Omega^{(0)}_{(\pm)}{}^c{}_a$$

$$F^A = dA^A + \tfrac{1}{2}\epsilon^{ABC} A^B \wedge A^C ,$$

$$\omega^{\mathrm{YM}} = dA^A \wedge A^A + \tfrac{1}{3}\epsilon^{ABC} A^A \wedge A^B \wedge A^C$$

$$\boxed{H^{(1)} = dB + 2\alpha' \left(\omega^{\mathrm{YM}} + \omega^{\mathrm{L}\,(0)}_{(-)}\right)} \qquad \left(\longrightarrow \Omega^{(1)}_{(\pm)}{}^a{}_b \ \ldots \right)$$

T- tensors

They codify the first α' corrections

$R^{(0)}_{(-)}$

$$
\begin{aligned}
T^{(4)} &\equiv 6\alpha' \left[F^A \wedge F^A + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \right] , && \rightarrow \text{Bianchi of } H \\
T^{(2)}{}_{\mu\nu} &\equiv 2\alpha' \left[F^A{}_{\mu\rho} F^A{}_\nu{}^\rho + R_{(-)\,\mu\rho}{}^a{}_b R_{(-)\,\nu}{}^{\rho\, b}{}_a \right] , && \rightarrow \text{Einstein eq.} \\
T^{(0)} &\equiv T^{(2)\,\mu}{}_\mu \,. && \rightarrow \text{Dilaton eq.}
\end{aligned}
$$

$\mathcal{O}(\alpha')$ effective action:

$$\boxed{S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!} H^2 - \frac{1}{2} T^{(0)} \right\}}$$

(Bergshoeff & de Roo 1989)

→ Very complicated e.o.m. but

$$\begin{aligned}
\delta S &= \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S}{\delta B_{\mu\nu}} \delta B_{\mu\nu} + \frac{\delta S}{\delta A^{A_i}{}_\mu} \delta A^{A_i}{}_\mu + \frac{\delta S}{\delta \phi} \delta\phi \\
&= \left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{\text{exp.}} \delta g_{\mu\nu} + \left.\frac{\delta S}{\delta B_{\mu\nu}}\right|_{\text{exp.}} \delta B_{\mu\nu} + \left.\frac{\delta S}{\delta A^{A_i}{}_\mu}\right|_{\text{exp.}} \delta A^{A_i}{}_\mu + \frac{\delta S}{\delta \phi} \delta\phi \\
&\quad + \frac{\delta S}{\delta \Omega_{(-)}{}^a{}_b} \left(\frac{\delta \Omega_{(-)}{}^a{}_b}{\delta g_{\mu\nu}} + \frac{\delta \Omega_{(-)}{}^a{}_b}{\delta B_{\mu\nu}} \delta B_{\mu\nu} + \frac{\delta \Omega_{(-)}{}^a{}_b}{\delta A^{A_i}{}_\mu} \delta A^{A_i}{}_\mu \right).
\end{aligned}$$

$$\frac{\delta S}{\delta \Omega_{(-)}{}^a{}_b} \sim \alpha' \frac{\delta S^{(0)}}{\delta(g_{\mu\nu}, B_{\mu\nu}, \phi)}$$

(Bergshoeff & de Roo 1989)

→ If a configuration solves the $\mathcal{O}(1)$ e.o.m. up to terms of $\mathcal{O}(\alpha')$, the e.o.m. that need to be checked are $\left.\frac{\delta S}{\delta \text{fields}}\right|_{\text{exp.}} = 0$

$$
\begin{aligned}
R_{\mu\nu} - 2\nabla_\mu \partial_\nu \phi + \tfrac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} - T^{(2)}{}_{\mu\nu} &= 0\,, \\
(\partial\phi)^2 - \tfrac{1}{2}\nabla^2\phi - \tfrac{1}{4\cdot 3!} H^2 + \tfrac{1}{8} T^{(0)} &= 0\,, \\
d\left(e^{-2\phi} \star H\right) &= 0\,, \\
\alpha' e^{2\phi} \mathfrak{D}_{(+)} \left(e^{-2\phi} \star F^{A_i}\right) &= 0\,,
\end{aligned}
$$

$i = 1, 2$
$SU(2) \times SU(2)$

If the configuration is described in terms of H, add

$$
dH - \tfrac{1}{3} T^{(4)} = 0\,,
$$

REMARKS:

1.- Ignoring all the terms in $\Omega_{(-)}$ $\Rightarrow$ exact SUGRA + compactification on $T^{5,6}$ + truncation

$\longrightarrow$ gauged $\mathcal{N}=2$, $d=5,4$ SUGRA coupled to vector multiplets.

+ solution-generating techniques

$\longrightarrow$ black holes, rings, global monopoles, instantons, multi-center solutions } with non-Abelian hair in fully analytic form.

2.- $\Omega_{(-)}$ occurs as another gauge field

$\longrightarrow$ its presence should modify the $\mathcal{O}(1)$ solutions $\boxed{\text{just as } A^A \text{ does.}}$

The Ansatz

$i = 2,3,4,5$

1.- Metric:

$$ds^2 = \frac{2}{\mathcal{Z}_-} du \left[dv - \tfrac{1}{2} \mathcal{Z}_+ du \right] - \mathcal{Z}_0 d\sigma^2 - dy^i dy^i \,,$$

Fundamental strings

pp - waves

Solitonic 5- branes

$$d\sigma^2 = h_{\underline{mn}} dx^m dx^n \,,$$

$m = \sharp, 1, 2, 3$

hyper-Kähler 4-manifold (KK monopoles)

	u,v		y^i				x^m			
	0	1	2	3	4	5	#	1	2	3
F1	X	X								
W	X	X								
S5	X	X	X	X	X	X				
KK	X	X	X	X	X	X	O			

$$\mathcal{Z}_{0,+,-} = \mathcal{Z}_{0,+,-}(x)$$

2.- KALB-RAMOND 3-FORM:

$$H = d\mathcal{Z}_{-}^{-1} \wedge du \wedge dv + \star_{(4)} d\mathcal{Z}_0\,,$$

B ?

Fundamental strings

Solitonic 5-branes

HK

3.- Dilaton:

$$e^{-2\phi} = \underbrace{e^{-2\phi_\infty}}_{1/g_s^2} \frac{\mathcal{Z}_-}{\mathcal{Z}_0}\,,$$

This configuration is an exact, supersymmetric solution of the $O(1)$ theory if

$$\boxed{\nabla^2_{(4)} \mathcal{Z}_{0,+,-} = 0}$$

HK

$T^5 \longrightarrow d=5$ BHs

$T^6 \longrightarrow d=4$ BHs

(HK $\longrightarrow$ Gibbons-Hawking)

4.— Yang-Mills fields:

$$F^{A_{1,2}} = \pm \star_{(4)} F^{A_{1,2}}$$

HK

(Papadopoulos 0809:1156)

Same selfduality as HK: $R^{mn} = * R^{mn}$

We are going to need an explicit construction of $A^{A_{1,2}}$ because we want to compute the T-tensors explicitely: Typically the Bianchi identity is solved via the "anomaly - cancellation mechanism" (Green-Schwarz)

$$T^{(4)} = 6\alpha' \left[F^{A_1} \wedge F^{A_1} + F^{A_2} \wedge F^{A_2} + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \right] = 0; \Rightarrow dH = 0;$$

Relations between SU(2) bundles

We want independent Yang-Mills fields.

Solving the Bianchi identity of H

For our Ansatz

$$dH = d\star_{(4)} dZ_0 = -\nabla^2_{(4)} Z_0 |v| d^4x \,.$$

$$\left(h_{\underline{mn}} = v^p{}_{\underline{m}} v^p{}_{\underline{n}} \,. \qquad \det(v^p{}_{\underline{m}}) = |v| \right)$$

$$\Rightarrow \nabla^2_{(2)} Z_0 |v| d^4x + 2\alpha' \left[F^{Ai} \wedge F^{Ai} + R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a \right] = 0$$

We know from gauged $N=2, d=5,4$ SUGRA how $F\wedge F$ contributes to Z_0! $\Rightarrow F^{Ai} \wedge F^{Ai} \sim \nabla^2_{(4)} (\text{Something}) |v| d^4x$

We have found two constuctions of selfdual SU(2) bundles on HK spaces with this property:

1.- 't Hooft ansatz

2.- Kronheimer-Bogomol'nyi-Protogenov

't Hooft Ansatz

Based on $so(4) \simeq su^{+}(2) \times su^{-}(2)$

$$\left\{(\mathbb{M}_{mn})^{pq} \equiv 2\delta_{mn}{}^{pq}\right\}, \quad \text{generate } so(4)$$

$$\left\{(\mathbb{M}^{\pm}_{mn})^{pq} = \delta_{mn}{}^{pq} \pm \tfrac{1}{2}\varepsilon_{mn}{}^{pq} = (\mathbb{M}^{\pm}_{pq})^{mn}\,,\ \tfrac{1}{2}\varepsilon_{mn}{}^{pq}\mathbb{M}^{\pm}{}_{pq} = \pm\mathbb{M}^{\pm}{}_{mn}\right\}, \quad \text{generate } su^{\pm}(2)$$

$$\left([\mathbb{M}^{\pm}_{\sharp i}, \mathbb{M}^{\pm}_{\sharp j}] = \mp\varepsilon_{ijk}\mathbb{M}^{\pm}_{\sharp k}\,. \qquad su^{\pm}(2)\right)$$

$$J^{i}_{mn} \equiv 2(\mathbb{M}^{-}_{\sharp i})^{mn}$$ 't Hooft symbols

$\left\{ \begin{array}{l} su^{-}(2) \text{ generators in 4 rep.} \\ \text{hypercomplex structure} \end{array} \right.$

in $so(4)$ indices

HK $\Longleftrightarrow$ $\nabla_m J^{i}{}_{np} = 0\,,$ $\longrightarrow$ $[\omega, J^{i}] = 0\,, \quad \Rightarrow \quad \omega = \omega^{+}\,,$

$$[\nabla_m, \nabla_n] J^{i}{}_{pq} = 0\,, \qquad [R, J^{i}] = 0\,, \quad \Rightarrow \quad R = R^{+}\,,$$

Bianchi $\longrightarrow$ as 2-form

$\Rightarrow$ Ricci flat.

Consider A^A $SU(2)$ connection on HK

$$A^A \longrightarrow A^{\#i} \longrightarrow A^{mn} = -\frac{1}{2}\varepsilon^{mnpq}A^{pq};$$

$$F^A = dA^A + \tfrac{1}{2}\epsilon^{ABC}A^B \wedge A^C, \qquad \omega^{\mathrm{YM}} = dA^A \wedge A^A + \tfrac{1}{3}\epsilon^{ABC}A^A \wedge A^B \wedge A^C.$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$F^{mn} = dA^{mn} + A^{mp} \wedge A^{pn}, \qquad \omega^{\mathrm{YM}} \equiv - (dA^{mn} \wedge A^{nm} + \tfrac{2}{3}A^{mn} \wedge A^{np} \wedge A^{pm}),$$

↳ relevant sign!

In this notation, the 't Hooft Ansatz reads

$$A = \mathbb{M}^{-}_{mp} V^p v^m .$$

(2 $su^-(2) \subset so(4)$ indices not written)

$\mathbb{M}^-_{mp} \to J^i_{mn}$, $v^m \to$ HK

$\nabla_{(4)m} M^-_{pq} = 0$

$F = +\star_{(4)} F \iff V_p = \partial_p \log P; \quad \nabla^2_{(4)} P = 0$

Then :

$$\omega^{\rm YM} = -\star_{(4)} dV^2 = -\star_{(4)} d(\partial \log P)^2 ,$$

$$F^A \wedge F^A = d\omega^{\rm YM} = -d\star_{(4)} d(\partial \log P)^2 = \nabla^2_{(4)} \left[(\partial \log P)^2 \right] |v| d^4x ,$$

For this Ansatz, it just happens that

$$\Omega_{(-)+-m} \;=\; \Omega_{(-)m+-} = \mathcal{Z}_0^{-1/2} \partial_m \log \mathcal{Z}_- , \qquad \Omega_{(-)++m} \;=\; \tfrac{1}{2} \mathcal{Z}_- \mathcal{Z}_0^{-1/2} \partial_m \mathcal{Z}_+ ,$$

$$\Omega_{(-)mnp} \;=\; \mathcal{Z}_0^{-1/2} \Big[\underset{\substack{\downarrow \\ \text{HK} \\ \omega^+}}{\varpi_{mnp}} + \underset{\substack{\downarrow \\ \text{'t Hooft-like}}}{(\mathbb{M}^-_{mq})_{np}} \partial_q \log \mathcal{Z}_0 \Big] ,$$

$$\omega^{\rm L}_{(-)} \;\equiv\; d\Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_a - \tfrac{2}{3} \Omega_{(-)}{}^a{}_b \wedge \Omega_{(-)}{}^b{}_c \wedge \Omega_{(-)}{}^c{}_a$$

$$\;=\; d\Omega_{(-)mn} \wedge \Omega_{(-)nm} + \tfrac{2}{3} \Omega_{(-)mn} \wedge \Omega_{(-)np} \wedge \Omega_{(-)pm} ,$$

$$= \omega^{\rm LHK} + \omega^{\rm LS5} ;$$

$$\omega^{\rm LS5} = \star_{(4)} d(\log \mathcal{Z}_0)^2 ;$$

How about $\omega^{\rm LHK}$?

"Twisted" 't Hooft Ansatz

If the HK space is a Gibbons-Hawking space

$$d\sigma^2 = H^{-1}(d\eta + \chi)^2 + H dx^x dx^x\,, \quad \partial_{\underline{x}} H = \varepsilon_{xyz}\partial_{\underline{y}}\chi_{\underline{z}}\,.$$

Simplest frame

$$\left\{\begin{array}{lcl} v^\sharp & = & H^{-\frac{1}{2}}[d\eta + \chi_{\underline{x}} dx^x]\,, \\ v^x & = & H^{\frac{1}{2}} dx^x\,, \end{array}\right. \longrightarrow \varpi_{mn} = (\mathbb{N}^+_{mn})_{pq}\partial_q \log H v^p\,:$$

where we are using the twisted $su(2)$ generators

$$(\mathbb{N}^\pm_{mn})^{pq} \equiv \eta_{mr}\eta_{ns}(\mathbb{M}^\mp_{rs})^{pq} \qquad \Rightarrow \qquad (\mathbb{N}^\pm_{mn})^{pq} = (\mathbb{N}^\mp_{pq})^{mn}\,,$$

$$\eta = \mathrm{diag}(-+++)$$

Then a calculation similar to that of the Yang-Mills one

$$\omega^{\mathrm{LHK}} = \star_{(4)} d(\partial \log H)^2\,,$$

$$R^{mn}\wedge R^{nm} = d\omega^{\mathrm{LHK}} = d\star_{(4)} d(\partial\log H)^2 = -\nabla^2\left[(\partial\log H)^2\right]|v|d^4x\,.$$

Then, using the 't Hooft Ansatz for the SU(2) gauge fields and using a Gibbons-Hawking HK space, the Bianchi identity takes the form

$$\nabla^2_{(4)}\left\{\mathcal{Z}_0+2\alpha'\left[(\partial\log P_1)^2+(\partial\log P_2)^2-(\partial\log\mathcal{Z}_0)^2-(\partial\log H)^2\right]\right\}|v|d^4x=\mathcal{O}(\alpha'^2)\,,$$

and is solved to this order by

$$\mathcal{Z}_0=\mathcal{Z}_0^{(0)}-2\alpha'\left[\underbrace{(\partial\log P_1)^2}_{SU(2)}+\underbrace{(\partial\log P_2)^2}_{SU(2)}-\underbrace{(\partial\log\mathcal{Z}_0^{(0)})^2}_{S5}-\underbrace{(\partial\log H)^2}_{HK-GH}\right]+\mathcal{O}(\alpha'^2)\,,$$

with

$$\nabla^2_{(4)}\mathcal{Z}_0^{(0)}=0\,.$$ (Same for $P_{1,2}$ but $\nabla^2_{\mathbb{R}^3}H=0$)

Before we study the e.o.m. are there other ways of getting Laplacians? KBP for Yang-Mills. →

Kronheimer-Bogomol'nyi-Protogenov

Kronheimer (1985): $\boxed{\hat{F} = + *_{(4)} \hat{F}}$ on Gibbons - Hawking ($\hat{}$)

instantons ↕ monopoles

Define
$$\begin{cases} \Phi \equiv - H \hat{A}_{\#} \\ A_r \equiv \hat{A}_r - \chi_r \hat{A}_{\#} \,; \quad r = 1, 2, 3 \end{cases}$$

$$\Longrightarrow \quad \boxed{D_r \Phi = \frac{1}{2} \varepsilon_{rst} F_{st}} \quad \text{on } \mathbb{R}^3$$

Bogomol'nyi equation (Bogomol'nyi 1976)

In this case:
$$\hat{F}^A \wedge \hat{F}^A = \hat{F}^A \wedge * \hat{F}^A = \cdots = 2 \hat{F}^A_{\# r} \hat{F}^A_{\# r} |v| d^4x =$$

$$= 2 H^{-2} \left[D_r \Phi^A - \Phi^A \partial_r \log H \right]^2 |v| d^4x$$

$$= \cdots = \frac{1}{H} \partial_r \partial_r \left(\Phi^A \Phi^A / H \right) |v| d^4x = \nabla^2_{(4)} \left(\frac{|\Phi|^2}{H} \right) |v| d^4 x$$

$$D^2 \Phi^A = 0 \,; \quad \partial_r \partial_r H = 0 \,;$$

Solutions to the SU(2) Bogomol'nyi Eqs.

a) Spherically symmetric (Protogenov 1977)

General form
$$\begin{cases} \breve{A}^A = -h(r)\,\varepsilon^A{}_{rs}\, x^r\, dx^s ; \\ \breve{\Phi}^A = -f(r)\,\delta^A{}_r\, x^r ; \end{cases}$$

BPS 't Hooft-Polyakov magnetic monopole

$$f = -\frac{1}{g r^2}\left[1 - \mu r \coth(\mu r + s)\right];$$
$$h = \frac{1}{g r^2}\left[\frac{\mu r}{\sinh(\mu r + s)} - 1\right];$$

Protogenov's hair parameter

Coloured monopoles → BPST instantons $\left(H = \frac{1}{r}\right)$

$$f = -\frac{1}{g r^2(1 + \lambda^2 r)} \quad ; \quad h = -f ;$$

f) Multicenter solutions

Ramirez's multimonopole solution 2015

$$\Phi^A = -\delta^{Ar} \frac{1}{gP} \partial_{\underline{r}} P \; ; \qquad \breve{A}^A{}_{\underline{r}} = -\varepsilon^A{}_{rs} \frac{1}{gP} \partial_{\underline{s}} P \; ;$$

$$\partial_{\underline{r}} \partial_{\underline{r}} P = 0 \quad ; \qquad P = \lambda^2 + \frac{1}{r} \rightarrow \text{Coloured monopole}$$

No more simple solutions known

Summarizing:

$$\omega^{YM} = - *_{(4)} d \left\{ \begin{matrix} (\partial \log P_1)^2 + (\partial \log P_2)^2 \\ |\Phi_1|^2/H + |\Phi_2|^2/H \end{matrix} \right\}$$

Gibbons-Hawking only

$$\omega^{L}_{(-)} = *_{(4)} d \left[(\partial \log Z_0)^2 + (\partial \log H)^2 \right] ;$$

$$T^{(4)} = 6\alpha' \nabla^2_{(4)} \left[(\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log Z_0^{(0)})^2 + (\partial \log H)^2 \right]$$

$$Z_0 = Z_0^{(0)} - 2\alpha' \left[(\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log Z_0^{(0)})^2 + (\partial \log H)^2 \right]$$

$$H = d\frac{1}{Z_-} \wedge du \wedge dv + *_{(4)} d Z_0$$

$$= dB - 2\alpha' *_{(4)} d \left[(\partial \log P_1)^2 + (\partial \log P_2)^2 - (\partial \log Z_0^{(0)})^2 + (\partial \log H)^2 \right]$$

$$\Rightarrow \quad dB = \underbrace{d\frac{1}{Z_-} \wedge du \wedge dv + *_{(4)} d Z_0^{(0)}}_{O(1)}$$

$O(1) \to$ no α' corrections to B

The Yang-Mills equations

→ Automatically solved for this Ansatz.

The Kalb-Ramond equations

→ Solved for any Z_- $\boxed{\nabla^2_{(4)} Z_- = 0}$

The dilaton equations

→ Automatically solved for this Ansatz

SUSY

The Einstein equations

→ All automatically solved for this Ansatz except for the ++ component

$$T^{(2)}_{++} = -2\alpha' R_{(-)\,+abc} R_{(-)\,+}{}^{abc} = -2\alpha' \frac{\mathcal{Z}_-}{\mathcal{Z}_0} \nabla^2_{(4)} \left(\frac{\partial_n \mathcal{Z}^{(0)}_+ \partial_n \mathcal{Z}_-}{\mathcal{Z}^{(0)}_0 \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2) .$$

(This component does not contribute to $T^{(0)}$ or any other invariants!)

$$\Rightarrow \quad \mathcal{Z}_+ = \mathcal{Z}^{(0)}_+ - 4\,\alpha' \left(\frac{\partial_n \mathcal{Z}^{(0)}_+ \partial_n \mathcal{Z}_-}{\mathcal{Z}^{(0)}_0 \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2) .$$

Harmonic in HK

It can be solved but it is not clear why

Exact $\mathcal{O}(\alpha')$ solution:

$$
\begin{aligned}
\mathcal{Z}_+ &= \mathcal{Z}_+^{(0)} - 4\,\alpha' \left(\frac{\partial_n \mathcal{Z}_+^{(0)} \partial_n \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)}\, \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2)\,, \\
\mathcal{Z}_- &= \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \\
\mathcal{Z}_0 &= \mathcal{Z}_0^{(0)} \\
&\quad -2\alpha' \left[(\partial \log P_1^{(0)})^2 + (\partial \log P_2^{(0)})^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 - (\partial \log H^{(0)})^2 \right] \\
&\quad +\mathcal{O}(\alpha'^2)\,, \\
H &= H^{(0)} + \mathcal{O}(\alpha'^2)\,, \\
P_{1,2} &= P_{1,2}^{(0)} + \mathcal{O}(\alpha'^2)\,.
\end{aligned}
$$

Exact $\mathcal{O}(\alpha')$ solution:

$$
\begin{aligned}
\mathcal{Z}_+ &= \mathcal{Z}_+^{(0)} - 4\,\alpha' \left(\frac{\partial_n \mathcal{Z}_+^{(0)} \partial_n \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)}\, \mathcal{Z}_-} \right) + \mathcal{O}(\alpha'^2)\,, \\
\mathcal{Z}_- &= \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \\
\mathcal{Z}_0 &= \mathcal{Z}_0^{(0)} \\
&\quad -2\alpha' \left[(\partial \log P_1^{(0)})^2 + (\partial \log P_2^{(0)})^2 - (\partial \log \mathcal{Z}_0^{(0)})^2 - (\partial \log H^{(0)})^2 \right] \\
&\quad + \mathcal{O}(\alpha'^2)\,, \\
H &= H^{(0)} + \mathcal{O}(\alpha'^2)\,, \\
P_{1,2} &= P_{1,2}^{(0)} + \mathcal{O}(\alpha'^2)\,.
\end{aligned}
$$

$d=5$ option : $H^{(0)} = 1$; $P_2^{(0)} = 1$; harmonic functions in $\mathbb{R}^4$

EXACT $\mathcal{O}(\alpha')$ SOLUTION:

$$
\begin{aligned}
\mathcal{Z}_+ &= \mathcal{Z}_+^{(0)} + 4\,\alpha'\left(\frac{\partial_n \mathcal{Z}_+^{(0)}\partial_n \mathcal{Z}_-^{(0)}}{\mathcal{Z}_0^{(0)}\,\mathcal{Z}_-}\right) + \mathcal{O}(\alpha'^2)\,,\\
\mathcal{Z}_- &= \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,,\\
\mathcal{Z}_0 &= \mathcal{Z}_0^{(0)}\\
&\quad -2\alpha'\left[(\partial\log P_1^{(0)})^2 + (\partial\log P_2^{(0)})^2 + (\partial\log \mathcal{Z}_0^{(0)})^2 - (\partial\log H^{(0)})^2\right]\\
&\quad +\mathcal{O}(\alpha'^2)\,,\\
H &= H^{(0)} + \mathcal{O}(\alpha'^2)\,,\\
P_{1,2} &= P_{1,2}^{(0)} + \mathcal{O}(\alpha'^2)\,.
\end{aligned}
$$

"SUGRA option": only α' correction in $\mathcal{Z}^{(0)}$ due to $P_{1,2}$

Observe:

1.- The solution is exact to $O(\alpha')$ with no use of the anomaly-cancellation mechanism
($\text{now } \mathcal{P}_1^{(0)} = \mathcal{Z}_0^{(0)} \; ; \; \mathcal{P}_2^{(0)} = \mathcal{H}^{(0)}$)

2.- For BHs the corrected solution covers near-horizon and asymptotic regions.

3.- $\mathcal{Z}_+^{(0)} = \mathcal{Z}_-^{(0)} = \mathcal{H}^{(0)} = \mathcal{P}_2^{(0)} = 1 \; ; \; \mathcal{Z}_0^{(0)} = \mathcal{P}_1^{(0)}$ "symmetric 5-brane" (Callan, Harvey, Strominger (1991))

4.- Singular gauge → spurious singularities ⇔ shift in harmonic functions ⇔ change in the number of branes
⇒ important for entropy calculations
(Removable singularity theorem Uhlenbeck (1982))

α'- corrected T-duality

These solutions have 2 non-trivial isometries:

1.- u: the direction of propagation of the wave (string)

2.- z: the triholomorphic isometry of the Gibbons-Hawking space

At zeroth order in α' T-duality along

u: interchanges waves and strings (momentum an winding)

$$\mathcal{Z}_+^{(0)} \rightleftarrows \mathcal{Z}_-^{(0)}$$

z: interchanges S5-branes and KK monopoles

$$\mathcal{Z}_0^{(0)} \rightleftarrows H^{(0)}$$

α'-corrected

Now, using the α'-corrected Buscher T-duality rules (Bergshoeff, Janssen, O. (1995)) (never before used due to lack of α'-corrected solns)

$$
\begin{aligned}
g'_{\mu\nu} &= g_{\mu\nu} + \left[g_{\underline{xx}} G_{\underline{x}\mu} G_{\underline{x}\nu} - 2 G_{\underline{xx}} G_{\underline{x}(\mu} g_{\nu)\underline{x}} \right] / G^2_{\underline{xx}}\,, \qquad (\mu,\nu \neq x) \\
B'_{\mu\nu} &= B_{\mu\nu} - G_{\underline{x}[\mu} G_{\nu]\underline{x}} / G_{\underline{xx}}\,, \\
g'_{\underline{x}\mu} &= -g_{\underline{x}\mu}/G_{\underline{xx}} + g_{\underline{xx}} G_{\underline{x}\mu}/G^2_{\underline{xx}}\,, & B'_{\underline{x}\mu} &= -B_{\underline{x}\mu}/G_{\underline{xx}} - G_{\underline{x}\mu}/G_{\underline{xx}}\,, \\
g'_{\underline{xx}} &= g_{\underline{xx}}/G^2_{\underline{xx}}\,, & e^{-2\phi'} &= e^{-2\phi} |G_{\underline{xx}}|\,, \\
A'^{A}_{\underline{x}} &= -A^A_{\underline{x}}/G_{\underline{xx}}\,, & A'^{A}_{\mu} &= A^A_{\mu} - A^A_{\underline{x}} G_{\underline{x}\mu}/G_{\underline{xx}}\,,
\end{aligned}
$$

where

$$
G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\nu} - 2\alpha' \left\{ A^A_{\mu} A^A{}_{\nu} + \Omega_{(-)\,\mu}{}^{a}{}_{b} \Omega_{(-)\,\nu}{}^{b}{}_{a} \right\} .
$$

$$
\boxed{Z_0^{(0)} \rightleftarrows H^{(0)} \qquad\qquad Z_+^{(0)} \rightleftarrows Z_-^{(0)}}
$$

Highly non-trivial test passed!

EXAMPLE: d=5 BH

BPST instanton size κ

$$H^{(0)} = 1 \quad (\mathbb{R}^4) \qquad \mathcal{Z}^{(0)}_{0,+,-} = 1 + \frac{\mathcal{Q}_{0,+,-}}{\rho^2} \; ; \qquad \overbrace{P_1^{(0)} = 1 + \frac{\kappa^2}{\rho^2}}^{\text{BPST instanton size } \kappa} \; ,$$

$$P_2^{(0)} = 1$$

$$\left\{ \begin{aligned} \mathcal{Z}_0 &= \mathcal{Z}_0^{(0)} + 8\alpha' \left[\frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2} - \frac{\rho^2 + 2\mathcal{Q}_0}{(\rho^2+\mathcal{Q}_0)^2} \right] + \mathcal{O}(\alpha'^2)\,, \\ \mathcal{Z}_- &= \mathcal{Z}_-^{(0)} + \mathcal{O}(\alpha'^2)\,, \\ \mathcal{Z}_+ &= \mathcal{Z}_+^{(0)} + 16\alpha' \frac{\mathcal{Q}_+(\rho^2 + \mathcal{Q}_0 + \mathcal{Q}_-)}{\mathcal{Q}_0(\rho^2+\mathcal{Q}_0)(\rho^2+\mathcal{Q}_-)} + \mathcal{O}(\alpha'^2)\,, \end{aligned} \right.$$

$$\left\{ \begin{aligned} ds^2 &= f^2 dt^2 - f^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, \\ e^{2\phi} &= e^{2\phi_\infty} \frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \\ k &= k_\infty (f\mathcal{Z}_+)^{3/4}\,, \end{aligned} \right.$$

$$\boxed{f^{-3} = \mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\,,}$$

$$+ \quad \underbrace{\text{Abelian and non-Abelian gauge fields}}_{\text{BPST instanton size } \kappa}$$

If the 3 charges $Q_{0,+,-} \neq 0 \Rightarrow$ regular (extremal) horizon

$$A_{\mathrm{H}} = 2\pi^2\sqrt{\mathcal{Q}_0\mathcal{Q}_+\mathcal{Q}_-}\,, \quad \text{for} \quad \mathcal{Q}_0 \neq 0\,,$$

$\Rightarrow$ No α' corrections in the near-horizon region

In the asymptotic region

$$\begin{aligned} M &= \frac{\pi}{4G_N^{(5)}}\left[\mathcal{Q}_0 + \mathcal{Q}_+(1 + 16\alpha'/\mathcal{Q}_0) + \mathcal{Q}_-\right] \\ &= \frac{R_z}{g_s^2\ell_s^2}N_{\mathrm{S5}} + \frac{R_z}{\ell_s^2}N_{\mathrm{F1}} + \frac{1}{R_z}N_{\mathrm{W}}(1 + 16/N_{\mathrm{S5}})\,, \quad \text{for} \quad \mathcal{Q}_0 \neq 0\,, \end{aligned}$$

The non-Abelian fields contribute to the total mass.

BUT the entropy is not simply given by $A_H/4$ (Wald (1993) Iyer & Wald (1994))

For any diff-invariant theory

Wald formula:

$$\boxed{S = -2\pi\int_{\mathrm{H}} d^3x\sqrt{|h|}\,\frac{\partial\mathcal{L}_{(5)}}{\partial R_{abcd}}\epsilon_{ab}\epsilon_{cd}\,,}$$

(Everything 5-dimensional)

Applying directly Wald's entropy formula requires knowing $\mathcal{L}_{(5)}$ $\longrightarrow$ horrible calculation!

However, it can be done in 10-dimensional language:

$$\boxed{S = -2\pi \int_{\mathrm{H}\times\mathrm{S}^1\times\mathrm{T}^4} d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\frac{\partial\mathcal{L}_{(10)}}{\partial\hat{R}_{abcd}}\epsilon_{ab}\epsilon_{cd}\,,}$$

3 terms: ① Einstein-Hilbert $\rightarrow$ $S^{(0)} = \frac{A_{\mathrm{H}}}{4G_N^{(5)}} = 2\pi\sqrt{N_{\mathrm{S5}}N_{\mathrm{F1}}N_{\mathrm{W}}}\,,$

② $R_{(-)}{}^{a}{}_{b}R_{(-)}{}^{b}{}_{a} \longrightarrow \underset{=}{0}$ because $R_{(-)\mu\nu\rho\sigma}=0$ on $AdS_3\times S^3\times T^4$

③ $\omega^{L}_{(-)}$ in H^2

$$\longrightarrow \quad S^{(1)} = \frac{\alpha'}{2G_N^{(10)}}\int d^8\hat{x}\frac{\sqrt{|\hat{g}|}}{\sqrt{f}}\, e^{-3(\phi-\phi_\infty)}(k/k_\infty)^{2/3}\hat{H}^{0\sharp\hat{g}}\hat{\Omega}_{(-)\,\hat{g}}{}^{0\sharp} = +\frac{8\alpha'}{\mathcal{Q}_0}S^{(0)}\,,$$

$$\boxed{\begin{aligned} S &= 2\pi\sqrt{N_{S5}N_{F1}N_{W}}\,(1+8/N_{S5}) \\ &\sim 2\pi\sqrt{(N_{S5}+16)N_{F1}N_{W}}\,, \quad \text{for} \quad N_{S5} \gg 16\,. \end{aligned}}$$

(in agreement with microscopic calculation.)

(Castro & Murthy (2009))

CONCLUSIONS:

1.- It is possible to compute α' corrections to physically interesting backgrounds (complete BHs)

2.- It is possible to apply Wald's entropy formula directly in $d=10$ avoiding unwarranted assumptions, incomplete actions etc.

3.- Non-Abelian solutions are an essential ingredient (but we do not know them w/o SUSY)

4.- $d=4$ BHs include a KK monopole (in progress)

Danke!