From Nernst branes to S-branes

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Outline

- Vector multiplet theories in four dimensions
- Dimensional reduction to three dimensions
- Construction of four-dimensional solutions by lifting three-dimensional solutions
- Nernst branes (planar solutions with vanishing entropy density in the zero temperature limit)
- S-branes/negative tension branes (cosmological solutions)
- Outlook: formal thermodynamical relations

(Mostly) based on:

- V. Cortés, P. Dempster, T.M. and O. Vaughan, *Special Geometry of Euclidean Supersymmetry IV: the local c-map*, arXiv:1507.04620, JHEP1510 (2015) 066.
- P. Dempster, D. Errington and T.M., *Nernst branes from special geometry*, arXiv:1501.07863, JHEP1505 (2015) 079.
- P. Dempster, D. Errington, J. Gutowski and T.M. *Five dimensional Nernst branes from special geometry*, arXiv:1609.05062, JHEP1611 (2016) 114
- Work in progress with G. Pope (PhD student, Liverpool), and with J. Gutowski (Surrey).

I will be sparse with references, please see above and forthcoming papers for complete references.

Four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to supergravity

Four-dimensional $\mathcal{N} = 2$ vector multiplets

Bosonic Lagrangian:

$$e_4^{-1}\mathcal{L}_4 = -\frac{1}{2}R_{(4)} - g_{A\bar{B}}(z)\partial z^A \partial \overline{z^B} + \frac{1}{4}\mathcal{I}_{IJ}(z)F^I_{\hat{\mu}\hat{\nu}}F^{J|\hat{\mu}\hat{\nu}} + \frac{1}{4}\mathcal{R}_{IJ}(z)F^I_{\hat{\mu}\hat{\nu}}\tilde{F}^{J|\hat{\mu}\hat{\nu}} - V \; .$$

Special Kähler geometry:
Couplings $g_{A\bar{B}}$, $\mathcal{I}_{IJ}$, $\mathcal{R}_{IJ}$ determined by a holomorphic prepotential $F(X^I)$, $I = 0, 1, \ldots n$, homogeneous of degree two in 'homogeneous scalars' X^I, which are subject to complex rescalings $X^I \to \lambda X^I$, $\lambda \in \mathbb{C}^*$.
n physical scalars:

$$z^A = \frac{X^A}{X^0} \; , \quad A = 1, \ldots, n \; .$$

$n + 1$ physical vector fields, including 'graviphoton.'

Electric-magnetic duality

Field equations invariant under $Sp(2n+2, \mathbb{R})$, which acts linearly on 'symplectic vectors':

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} , \quad \begin{pmatrix} F^{\pm|I}_{\hat{\mu}\hat{\nu}} \\ G^{\pm}_{I|\hat{\mu}\hat{\nu}} \end{pmatrix} , \ldots$$

where

$$F_I = \frac{\partial F}{\partial X^I} , \quad G^{\pm}_{I|\hat{\mu}\hat{\nu}} \propto \frac{1}{e} \frac{\partial \mathcal{L}}{\partial F^{\pm|I}_{\hat{\mu}\hat{\nu}}} .$$

Affine special Kähler manifolds

(N, g, J, ∇), where

- (N, g, J) Kähler with Kähler form $\omega = g(J\cdot, \cdot)$.
- ∇ is a flat, torsion-free, symplectic connection satisfying

$$d^{\nabla} J = 0 \ ,$$

equivalently:

$$\nabla g \ \text{ totally symmetric rank 3 tensor} \ .$$

Thus Kähler and Hessian.

Kähler potential has a holomorphic prepotential:

$$K = -i(X^I \bar{F}_I - \bar{X}^I F_I) \ .$$

Special real coordinates = ∇-affine coordinates which are ω-Darboux coordinates: $(q^a) = (x^I, y_I)$, where

$$\begin{aligned} X^I &= x^I + iu^I(x,y) \\ F_I &= y_I + iv_I(x,y) \ . \end{aligned}$$

Metric has a Hesse potential:

$$g_{ab} = H_{ab} := \frac{\partial H}{\partial q^a \partial q^b} \ .$$

Hesse potential $H(q^a)$ and holomorphic prepotential $F(X^I)$ are related by a Legendre transformation

$$H(x,y) = 2\left(\mathrm{Im}(F(x,u(x,y))) - y_I u^I(x,y)\right) \ .$$

Conical affine special Kähler manifolds

(N, g, J, ∇, ξ) such that

- (N, g, J, ∇) is ASK.
- ξ is a vector field such that

$$D\xi = \nabla\xi = \mathrm{Id}_{TN}$$

Vector fields

$$\xi = q^a \frac{\partial}{\partial q^a} = X^I \frac{\partial}{\partial X^I} + \text{c.c.} \quad \text{and} \quad J\xi = \frac{1}{2} H_a \Omega^{ab} \frac{\partial}{\partial q^b} = iX^I \frac{\partial}{\partial X^I} + \text{c.c.}$$

generate a homothetic, holomorphic $\mathbb{C}^*$ action.

Assuming group action can take Kähler quotient to define the projective special Kähler manifold $\bar{N} = N/\mathbb{C}^* = N//U(1)$.

$F(X^I)$ is homogeneous of degree two in the special holomorphic coordinates X^I.
$H(q^a)$ is $U(1)$ invariant and homogeneous of degree two in the special real coordinates q^a.

Superconformal calculus uses gauge equivalence between:

- $n+1$ vector multiplets with local superconformal symmetry, scalar manifold N is conical affine special Kähler.
- n vector multiplets coupled to Poincaré supergravity, scalar manifold $\bar{N} = N//U(1)$.

Scalar potential

Potential:

$$V(X,\bar{X}) = N^{IJ}\partial_I W \partial_J \bar{W} - 2\kappa^2|W|^2, \quad (N^{IJ}) = (2\mathrm{Im}F_{IJ})^{-1} \ ,$$

Superpotential:

$$W = 2\left(g^I F_I - g_I X^I\right) \ .$$

(g^I, g_I) parameters of magnetic/electric FI gauging.

Potential (real coordinates):

$$V = g^a g^b \left[H_{ab} + \frac{H_a H_b + 4\,(\Omega q)_a\,(\Omega q)_b}{H}\right] \ , \quad -2H \stackrel{D}{=} \kappa^{-2} \ .$$

Superpotential (real coordinates)

$$W = W(q^a) = i g^a \left(H_{ab} - 2i\Omega_{ab}\right) q^b, \quad (\Omega_{ab}) = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \ ,$$

where $(g^a) := (g^I, g_I)$.

ε-complex structures

Almost complex structure:

$$J \in \Gamma(\mathrm{End}(TM)) \,, \quad J^2 = -\mathrm{Id}_{TM} \,.$$

Almost para-complex structure:

$$J \in \Gamma(\mathrm{End}(TM)) \,, \quad J^2 = \mathrm{Id}_{TM} \,,$$

with the eigendistributions having equal dimension.
Unified notation: ε-complex structure:

$$J \in \Gamma(\mathrm{End}(TM)) \,, \quad J^2 = \varepsilon \mathrm{Id}_{TM} \,, \quad \varepsilon = \pm 1 \,.$$

Various concepts of complex geometry (Hermitian, Kähler, hyper-Kähler, quaternionic-Kähler, affine and projective special Kähler) can be adapted to para-complex geometry.

Euclidean vector multiplets

Remark: The special geometry of $\mathcal{N} = 2$ vector multiplets in Euclidean space-time signature is (affine/projective) special para-Kähler.

Reduction to three dimensions

Dimensional reduction to three dimensions

Metric	$g_{\hat{\mu}\hat{\nu}}$	Metric	$g_{\mu\nu}$
		KK vector	$A_\mu \sim \tilde{\phi}$
		KK scalar	ϕ
$n+1$ Vector fields	$A^I_{\hat{\mu}}$	$n+1$ Vector fields	$A^I_\mu \sim \tilde{\zeta}_I$
		$n+1$ scalars	$A^I_\star = \zeta^I$
n complex scalars	z^A	n complex scalars	z^A

$4n+4$ independent real scalar fields: $z^A, \zeta^I, \tilde{\zeta}_I, \phi, \tilde{\phi}$.

Observation: an alternative parametrization based on using the four-dimensional special real coordinates provides new insights into scalar geometry of the reduced theory, and helps to find explicit solutions.

Re-packaging: use homogeneous variables X^I or q^a to encode the physical scalars z^A, and absorbe the KK-scalar ϕ by a field redefinition:

$$Y^I = e^{\phi/2} X^I \ , \quad q^a_{\rm new} = e^{\phi/2} q^a_{\rm old}$$

$4n+5$ real scalar fields $(q^a, \hat{q}^a, \tilde{\phi})$, subject to $U(1)$ transformations $= 4n+4$ independent fields. Advantage of keeping $U(1)$: covariance with respect to symplectic transformations is maintained.

3d Lagrangian

$$
\begin{aligned}
e_3^{-1}\mathcal{L}_3 &= -\frac{1}{2}R_{(3)} - \tilde{H}_{ab}\left(\partial_\mu q^a \partial^\mu q^b - \epsilon \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b\right) + \frac{1}{2H}V \\
&\quad -\frac{1}{H^2}(q^a \Omega_{ab} \partial_\mu q^b)^2 + \epsilon \frac{2}{H^2}(q^a \Omega_{ab} \partial_\mu \hat{q}^b)^2 \\
&\quad -\frac{1}{4H^2}(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b)^2 .
\end{aligned}
$$

where

$$
\Omega_{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \quad \tilde{H}_{ab} = \partial^2_{a,b}\tilde{H} , \quad \tilde{H} = -\frac{1}{2}\log(-2H)
$$

Hesse potentials $H, \tilde{H}$ are functions of the scalars q^a.
$\epsilon = -1$ ($\epsilon = 1$) for space-like (time-like)reduction.
$\mathcal{L}_3$ is locally $U(1)$-invariant, only $4n + 4$ propagating scalar fields.

Hypermultiplet geometry

Three-dimensional fields organise into hypermultiplets. Scalar geometry is quaternionic-Kähler for spacelike reduction and para-quaternionic Kähler for timelike reduction.

ε-quaternionic structures

$J_1, J_2, J_3 \in \mathrm{End}(V)$, pairwise anti-commuting, $J_1 J_2 = J_3$.

- Quaternionic structure:

$$J_1^2 = J_2^2 = J_3^2 = -\mathrm{Id}\ .$$

- Para-quaternionic structure:

$$J_1^2 = J_2^2 = -J_3^2 = \mathrm{Id}\ .$$

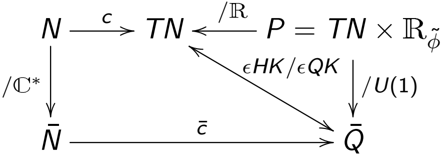
- Unified notation: ε-quaternionic structure:

$$J_1^2 = J_2^2 = -\varepsilon J_3^2 = \varepsilon \mathrm{Id}\ .$$

ε-hyper Kähler manifold: J_α (anti-)isometric, and parallel ($\Rightarrow$ integrable).
ε-quaternionic Kähler manifold: J_α (anti-)isometric, and distribution spanned by them is parallel (J_α in general not integrable).

The supergravity c-map

$\bar{N}$, $\bar{Q}$: scalar manifolds of the 4d/3d theory.
N scalar manifold of auxiliary 4d superconformal theory.



$\mathcal{L}_3$ defines a projectable symmetric tensor field on $P \to \bar{Q}$, which induces the same ε-quaternionic-Kähler metric on $\bar{Q}$ as direct reduction in terms of physical scalars.

Solutions

PI field configurations

For a certain class of field configurations, interesting solutions can be found by integrating the field equations elementarily.

For today, impose the following conditions:

- 4d field configuration is static.
- Impose that 4d scalars are 'purely imaginary' ('axion-free').
- Impose analogous conditions on gauge fields (and, in presence of a potential, gauging parameters).

This sets half of the three-dimensional scalars constant, while the remaining scalars parametrize a para-Kähler submanifold.

$$
\begin{aligned}
(q^a)|_{\mathrm{PI}} &= (x^0, 0, \ldots, 0; 0, y_1, \ldots, y_n), \\
(\partial_\mu \hat{q}^a)|_{\mathrm{PI}} &= \frac{1}{2}(\partial_\mu \zeta^0, 0, \ldots, 0; 0, \partial_\mu \tilde{\zeta}_1, \ldots, \partial_\mu \tilde{\zeta}_n), \\
(g^a)|_{\mathrm{PI}} &= (g^0, 0, \ldots, 0; 0, g_1, \ldots, g_n) \,.
\end{aligned}
$$

Additional assumption: prepotential is of 'very special type' $\Leftrightarrow$ can lift to five dimensions:

$$F = \frac{f(Y^1, \ldots, Y^n)}{Y^0}\ , \quad f \quad \text{homogeneous of degree 3.}$$

(This can be relaxed, essential point is to have some factorization of variables and some homogeneity property.)

Then one can obtain an explicit formula for Hesse potential

$$H = -\frac{1}{4}\left(-q_0 f(q_1, \ldots, q_n)\right)^{-\frac{1}{2}}\ , \quad \text{dual scalars} \quad q_a := \tilde{H}_a := \frac{\partial \tilde{H}}{\partial q^a}\ .$$

(Have shifted indices $a = 0, n+2, n+3, \ldots, 2n+1 \rightarrow$
$n = 0, 1, \ldots n$.)

Integrating the equations of motion

- Rewrite equations of motion in terms of dual real variables $q_a, \hat{q}_a$. $(\partial_\mu \hat{q}_a := \tilde{H}_{ab} \partial_\mu \hat{q}^a)$
- $\hat{q}_a$ equations are trivial to integrate.
- Einstein equation can be solved in terms of q_a.
- Block decomposition of $\tilde{H}_{ab}$ leads to partial decoupling of the scalar equations of motion.
- Homogeneity always allows to solve the scalar equations of motion by taking fields q_a which appear in the same block to be proportional to one another.

General observations

- Solutions are generically neither supersymmetric (not BPS, no Killing spinors), nor extremal (Killing horizons have finite surface gravity)
- We solve the second order field equations directly, without imposing a reduction to first order field equations, as with other methods (BPS squares, fake/pseudo-supersymmetry, etc.)
- By imposing regularity of the solution at the Killing horizon, half of the intergration constants get fixed, so that the number of undetermined integration constants corresponds to a first order system.

Example so far include: black holes and black strings in four and five dimensions, Nernst branes, and most recently planar solutions with static patches containing a timelike singularity (interpreted as a negative tension brane) related by analytic continuation to cosmological patches asymptotic to Kasner solutions.

Solutions with planar symmetry

Metric:

$$\begin{aligned} ds_4^2 &= -e^{\phi}(dt + V_\mu dx^\mu)^2 + e^{\phi} ds_3^2 \\ ds_3^2 &= e^{4\psi} d\tau^2 + e^{2\psi}(dx^2 + dy^2) \end{aligned}$$

$\phi = \phi(\tau)$ (absorbed into scalars), $V_\mu = 0$, and $\psi = \psi(\tau)$.

Scalars $q_a(\tau), \hat{q}_a(\tau)$.

$\hat{q}_a$-equations (four-dimensional gauge field equations) trivial:

$$\ddot{\hat{q}}_a = 0 \Rightarrow \dot{\hat{q}}_a = K_a \ .$$

No further integration required as this determines the four-dimensional field strengths.

Cases where the field equations have been integrated

One charge solutions ('Nernst branes')

Charges: $(-Q_0, 0, \ldots, 0 | 0, \ldots, 0)$

Gauging: $(0, \ldots, 0 | 0, g_1, g_2, \ldots, g_n)$

Hesse potential: $H = -\frac{1}{4}(-q_0 f(q_1, \ldots, q_n))$

Two charge solutions:

Charges: $(-Q_0, 0, \ldots, 0 | 0, P^1, 0, \ldots, 0)$

Gauging: $(0, \ldots, 0 | 0, 0, g_2, \ldots, g_n)$

Hesse potential: $H = -\frac{1}{4}(-q_0 q_1 f(q_2, \ldots, q_n))$

Three charge solutions (gauged STU model):

Charges: $(-Q_0, 0, 0, 0|0, P^1, P^2, 0)$

Gauging: $(0, 0, 0, 0|0, 0, 0, g_3)$

Hesse potential: $H = -\frac{1}{4}(-q_0 q_1 q_2 q_3)$

Four charge solutions (ungauged STU model):

Charges: $(-Q_0, 0, 0, 0|0, P^1, P^2, P^3)$

Gauging: $(0, 0, 0, 0|0, 0, 0, 0)$

Hesse potential: $H = -\frac{1}{4}(-q_0 q_1 q_2 q_3)$

Three charge and four charge solutions show the same qualitative behaviour. We focus on the four charge solution.

One charge solutions: Nernst branes

One charge solution in 3 dimensions

$$
\begin{aligned}
q_0 &= \pm - \frac{Q_0}{B_0} \sinh\left(B_0 \tau + B_0 \frac{h_0}{Q_0} \right), \\
q_A &= \pm \frac{1}{8 g_A} B_0^{-\frac{1}{2}} e^{\frac{1}{2} B_0 \tau} \left(\sinh(B_0 \tau) \right)^{\frac{1}{2}} \quad \text{for} \quad A = 1, \ldots, n, \\
\dot{\hat{q}}_0 &= -Q_0, \\
e^{-4\psi} &= \frac{1}{B_0^3} \sinh^3(B_0 \tau) e^{B_0 \tau}, \\
e^{\phi} &= \frac{1}{2} (-q_0)^{-\frac{1}{2}} (f(q_1, \ldots, q_n))^{-\frac{1}{2}}.
\end{aligned}
$$

4d regularity $\Rightarrow$ two integration constants (apart from Q_0): $B_0 \geq 0$, extremality parameter (temperature), h_0 (chemical potential).

One charge solution in 4 dimensions

New transverse coordinate:

$$e^{-2B_0\tau} = 1 - \frac{2B_0}{\rho} =: W(\rho)$$

Asymptotic region: $\rho \to \infty$, horizon: $\rho = 2B_0$.

4d metric:

$$ds_4^2 = -\mathcal{H}^{-\frac{1}{2}} W \rho^{\frac{3}{4}} dt^2 + \mathcal{H}^{\frac{1}{2}} \rho^{-\frac{7}{4}} \frac{d\rho^2}{W} + \mathcal{H}^{\frac{1}{2}} \rho^{\frac{3}{4}} (dx^2 + dy^2),$$

where

$$\mathcal{H}(\rho) \equiv \pm 4 \left(\frac{1}{8}\right)^3 f\left(\frac{1}{g_1}, \dots, \frac{1}{g_n}\right) \mathcal{H}_0(\rho) \,, \quad \mathcal{H}_0(\rho) = -\left[\frac{Q_0}{B_0} \sinh\left(\frac{B_0 h_0}{Q_0}\right) + \frac{Q_0 e^{-\frac{B_0 h_0}{Q_0}}}{\rho}\right].$$

Black brane thermodynamics

Temperature (surface gravity or Euclidean method):

$$4\pi T = Z^{-1/2}(2B_0)^{3/4} e^{-\frac{B_0 h_0}{2Q_0}} .$$

$Z =$ combination of constants.

Chemical potential:

$$\mu \equiv A_t(\tau = 0) = \frac{1}{2}\left(\frac{B_0}{Q_0}\right)\left[\coth\left(\frac{B_0 h_0}{Q_0}\right) - 1\right],$$

diverges for $h_0 \to 0$.

Entropy density:

$$s = Z^{1/2}(2B_0)^{1/4} e^{\frac{B_0 h_0}{2Q_0}}$$

Note limits: $T = 0 \Leftrightarrow B_0 = 0$ and $\mu = \infty \Leftrightarrow h_0 = 0$.

Can eliminate B_0:

$$B_0 = 2\pi s T.$$

Equation of state:

$$s^3 = 4\pi Z^2 T \left(1 + \frac{2\pi s T}{Q_0 \mu}\right).$$

Nernst law:

$$s \xrightarrow[T\to 0]{} 0 \,, \quad \mu, Q_0 \text{ fixed}$$

Scaling regimes:

$$s \sim T^{1/3} \quad \text{for} \quad T/\mu \ll 1$$
$$s \sim T \quad \text{for} \quad T/\mu \gg 1$$

Remark: for $T \to 0$ we recover the extremal Nernst brane solution of S. Barisch, G. Lopes Cardoso, M. Haack, S. Naampuri and N.A. Obers, JHEP 1111 (2011) 090, [arXiv: 1108.02960].

hvLif geometries

Hyperscaling violating Lifshitz geometries hvLif$_{z,\theta}$ with d transverse spatial dimensions:

$$ds^2_{d+2} = r^{-\frac{2(d-\theta)}{d}} \left(-r^{-2(z-1)}dt^2 + dr^2 + dx_i^2\right) ,$$

Scaling behaviour:

$$(r, x_i) \mapsto \lambda(r, x_i) , \quad t \mapsto \lambda^z t , \quad ds^2_{d+2} \mapsto \lambda^{2\theta/d} ds^2_{d+2} .$$

$z =$ Lifshitz exponent, measures deviations from relativistic symmetry ($\lambda \neq 1$).
$\theta =$ hyperscaling violating exponent, measures deviation from scale invariance ($\theta \neq 0$).

Thought to be dual to QFT$_{1,d}$, with above scaling behaviour, i.p.

$$s \sim T^{(d-\theta)/z} .$$

Asymptotic behaviour of 4d Nernst branes

Chem. Pot, Temp.	Infinity	Horizon
$\mu < \infty,\ T > 0$	$\text{hvLif}_{1,-1} = CAdS_4$ Scalars $\to \infty$ $R, K \to 0$	$\text{hvLif}_{0,2} = \text{Rindler} \times \mathbb{R}^2$.
$\mu < \infty,\ T = 0$	$\text{hvLif}_{1,-1} = CAdS_4$ as above	$\text{hvLif}_{3,1}$ Scalars $\to \infty$ infinite tidal forces
$\mu = \infty,\ T > 0,$	$\text{hvLif}_{3,1}$ Scalars $\to 0$ $R, K \to \infty$	$\text{hvLif}_{0,2} = \text{Rindler} \times \mathbb{R}^2$.
$\mu = \infty,\ T = 0$	$\text{hvLif}_{3,1}$ as above	$\text{hvLif}_{3,1}$ as above

For $\rho \to \infty$ the solution degenerates, and the equation of state we found does not show the asymptotic behaviour $s \sim T^3$ expected for $z = 1, \theta = -1$.

Interpretation: decompactification limit, solution must be interpreted from a 5d perspective. Clue AdS_5 has $d = 3, z = 1, \theta = 0$ and therefore $s \sim T^3$.

One charge solution in five dimensions

Boosted AdS Schwarzschild Black Brane:

$$ds^2_{(5)} = \frac{l^2 dr^2}{r^2 W(r)} + \frac{r^2}{l^2}\left[-W(r)(u_t dt + u_z dz)^2 + (u_z dt + u_t dz)^2 + dx^2 + dy^2\right]$$

where

$$W(r) = 1 - \frac{r_+^4}{r^4}\,, \quad r_+^4 := 2B_0\,, \quad u_t = \sqrt{1+\tilde{\Delta}}\,, \quad u_z = \sqrt{\tilde{\Delta}}$$

and $l = \mathrm{AdS}_5$-radius.

Temperature from surfrace gravity or absence of conical singularity in Euclidean continuation:

$$\pi T = \frac{r_+}{l^2 u_t}\,, \quad r_+^4 = 2B_0\,.$$

Remark: 'linear' version of rotating black hole, i.p. ergoregion.
Remark: Generalized Carter-Novotný-Horský metric.

Mass and Momentum

Using quasilocal stress tensor obtain:

- Mass

$$M = \frac{(4u_t^2 - 1)r_+^4}{16\pi G l^5} V_3$$

- Linear momentum

$$P_z = \frac{4r_+^4 u_t u_z}{16\pi G l^5} V_3$$

Boundary stress tensor has perfect fluid form with pressure proportional to $r_+^4 \sim T^4$ (ultra-relativistic).

Entropy and First Law

Entropy:

$$S = \frac{r_+^3}{4Gl^3} u_t V_3$$

First law (important consistency check!)

$$\delta M = T\delta S - w\delta P_z$$

$w =$ boost velocity.

Smarr-type relation:

$$\frac{1}{4}M = \frac{1}{3}TS - \frac{1}{4}wP_Z$$

Stability

Mass relation:

$$M(T, w) = \frac{l^3}{16\pi G} V_3 \frac{3 + w^2}{(1 - w^2)^3} (\pi T)^4$$

Heat capacity

$$C_T = \left. \frac{\partial M}{\partial T} \right|_w > 0$$

Entropy-Temperature relation

$$S(T,w) = \frac{l^3}{4G} V_3 \frac{(\pi T)^3}{(1-w^2)^2}$$

- High temperature (small boost velocity)

  $$u_z \to 0\,, \quad r_+ \to \infty\,, \quad u_z^2 r_+^4 \to \Delta\,, \quad \Rightarrow |w| \ll 1 \Rightarrow S \sim T^3$$

  Scaling relation for AdS_5.

- Low temperature (high boost velocity)

  $$u_t \to \infty\,, \quad r_+ \to 0\,, \quad u_t^2 r_+^4 \to \Delta \quad \Rightarrow 1 - w^2 \sim T^{4/3} \Rightarrow S \sim T^{1/3}$$

  Same scaling as for 4d IR geometry.

Extremal limit

Extremal limit: zero temperature $r_+ \to 0$, infinite boost $u_t \to \infty$, with $u_t^2 r_+^4 = \Delta$ fixed.

$$w = -1 \,, \quad T = 0 \,, \quad M = |P_z| \,.$$

Ergosphere disappears.
Horizon moves with speed of light.
Kaigorodov metric, gravitational wave in AdS_5.
Solution is $\frac{1}{4}$ BPS (2 Killing spinors).

5d vs 4d solution

- 5d solution 'regularizes' 4d solution: geometry at infinity is AdS_5.
- Continuous parameters in 5d: (T, P_z). Upon compactification momentum becomes (discrete!) charge Q. Continuous parameters in 4d (T, μ). Where does the chemical potential come from.
- Answer: the radius of compactified dimension varies along the transverse coordinate. Chemical potential determined by minimal value of the radius.
- Can recover 4d thermodynamic relations from 5d.

Four charge solution: Negative tension branes and cosmological solutions

Four charge solution in three dimensions

Three-dimensional scalars

$$
\begin{aligned}
q_0(\tau) &= \mp \frac{Q_0}{B_0} \sinh\left(B_0 \tau + B_0 \frac{h_0}{Q_0} \right) \\
q_a(\tau) &= \pm \frac{P^a}{B_a} \sinh\left(B_a \tau + B_a \frac{h_a}{P^a} \right) \ , \quad a = 1, 2, 3 \ .
\end{aligned}
$$

8 integration constants B_0, B_a, h_0, h_a.

3d metric:

$$
e^{-4\psi} = A \exp\left(2\sqrt{B_0^2 + B_1^2 + B_2^2 + B_3^2}\ \tau \right)
$$

Four-dimensional physical scalars:

$$z^A = -i \left(\frac{q_0 q_A^2}{q_1 q_2 q_3} \right)^{1/2}$$

Four-dimensional metric:

$$ds_4^2 = -e^{\phi} dt^2 + e^{-\phi+4\psi} d\tau^2 + e^{-\phi+2\psi}(dx^2 + dy^2) \ ,$$

where

$$e^{\phi} = \frac{1}{2}(-q_0 q_1 q_2 q_3)^{-1/2} \ .$$

Regularity of 4d scalars and metric for $\tau \to \infty$ (Killing horizon) requires: $B_0 = B_1 = B_2 = B_3 = B$. Reduction of number of integration constants to $4 + 1$ (initial conditions for the scalars + non-extremality parameter).

Introduce new transverse coordinate

$$W(\zeta) := 1 - \alpha\zeta := e^{-2B\tau} \ .$$

Define:

$$H_a(\zeta) := \bar{K}_a \left[\frac{2}{\alpha} \sinh\left(\frac{\alpha h_a}{2K_a} \right) + e^{-\frac{\alpha h_a}{2K_a}} \ \zeta \right]$$

Metric:

$$ds_4^2 = -\frac{W(\zeta)}{H(\zeta)} dt^2 + \frac{H(\zeta)}{W(\zeta)} d\zeta^2 + H(\zeta)(dx^2 + dy^2) \ .$$

where $H(\zeta) = 2\sqrt{H_0 H_1 H_2 H_3}$.

Scalars:

$$z^A = -iH_A \left(\frac{H_0}{H_1 H_2 H_3} \right)^{1/2} \ .$$

Expectation from previous spherical and planar solutions:
Killing horizon at $\tau \to \infty \Leftrightarrow \zeta = \alpha^{-1}$, and asympotic spacetime at $\tau \to 0 \Leftrightarrow \zeta = 0$.
Instead, first zero of any H_a at $\zeta = \zeta_S < \alpha^{-1}$ gives rise to a curvature singularity at finite distance.

$\zeta = \zeta_S$	curvature singularity
$\zeta_S < \zeta < \alpha^{-1}$	static patch
$\zeta = \alpha^{-1}$	Killing horizon
$\alpha^{-1} < \zeta < \infty$	time dependent, cosmological patch
$\zeta \to \infty$	asymptotic to vacuum typ D Kasner solution

'Extremal limit' $\alpha \to 0$ moves the Killing horizon to infinity and removes the cosmologcial patch.

Conformal diagram

'Schwarzschild rotated by 90 degrees,' and 'inside-out': patches with singularities are static, asymptotic regions are time-dependent.

This type of conformal diagram has appeared before in Einstein and Einstein-Maxwell theory and more recently in Einstein-Maxwell-Dilaton theories, `C. Grojean, F. Quevedo, G. Tasinato, I. Zavala, hep-th/0106120, JHEP08 (2001) 005`, and discussed in `C.P. Burgess, F. Quevedo, I. Zavala, S.-J. Rey, G. Tasinato, hep-th/0207104, JHEP10 (2002) 028` . (See there for earlier references).

Our solutions generalize previous solutions to the case of multiple vector and scalar fields, and allow an embedding into string theory. They reduce to Einstein-Maxwell solutions upon choosing the scalars constant.

Solutions with constant scalars

Set scalars constant by

$$Q_0 = P^1 = P^2 = P^3 = K \,, \quad h_0 = h^1 = h^2 = h^3 = h$$

Further rewriting.
Static patch $r < \frac{e^2}{m}$:

$$ds_4^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2) \,, \quad f(r) = -\frac{m}{r} + \frac{e^2}{r^2} \,.$$

Cosmological patch $t > \frac{e^2}{m}$, (relabel $r \leftrightarrow t$):

$$ds_4^2 = -\frac{dt^2}{f(t)} + f(t)dt^2 + t^2(dx^2 + dy^2) \,, \quad f(t) = \frac{m}{t} - \frac{e^2}{r^2} \,.$$

Planar Reissner-Nordström/Schwarzschild ($e = 0$) solution and its analytic continuations (asymptotic to Kasner).

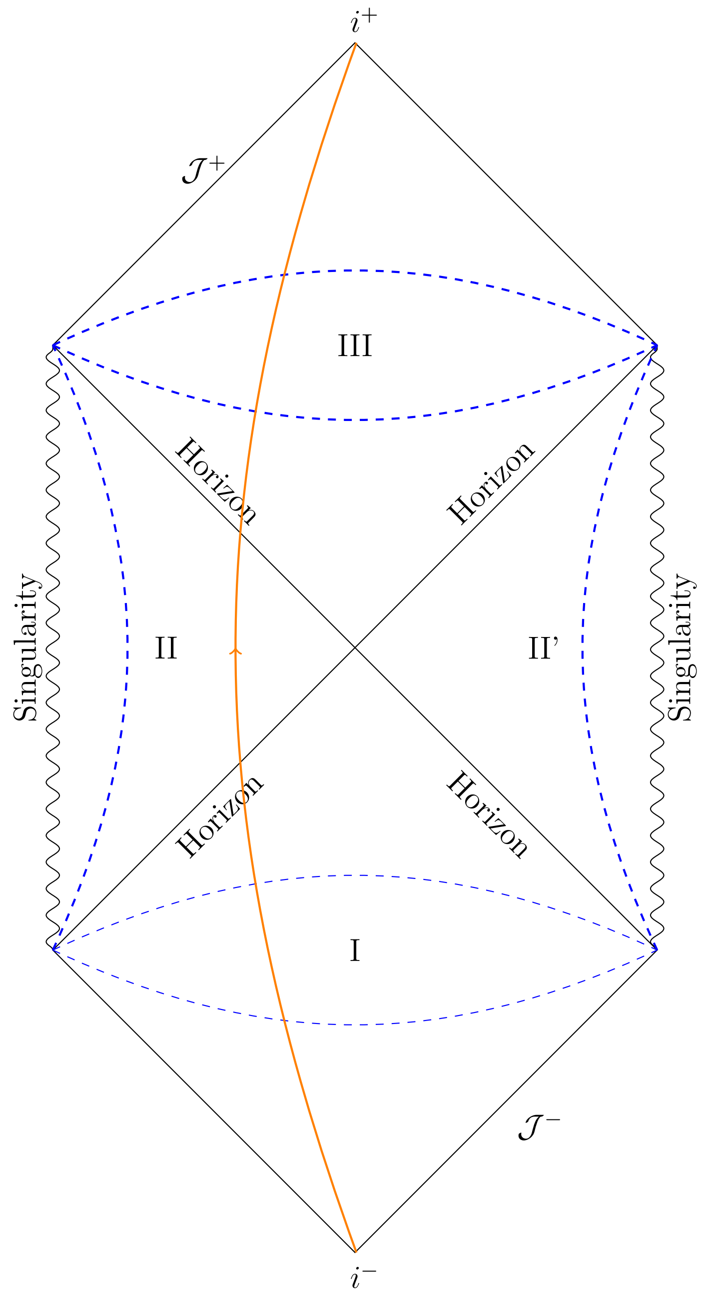


Figure 1: Conformal diagram for the four charge solution. Patches I and III are cosmological (non-stationary), Patches II and II' are static with repulsive time-like singularities ('negative tensions branes'). The orange line is a generic timelike geodesic. The solution is complete for timelike geodesics, but (at least) the past horizon $I/II, II'$ is unstable (like the inner horizon of the Reissner-Nordström solution.) The future horizon $II, II'/III$ passes some tests for stability. The metric is asymptotic to a Kasner solution at early and late times. While it remains to clarify how physical (i.p. stable) the solution is, one can establish versions of standard 'thermodynamic' relations, at least at a formal level. Solutions with the same conformal diagram have been discussed in the literature in the context of Einstein-Maxwell-Dilaton theories in 2002.

Formal thermodynamic relations.

Thermodynamics in the static patch

Burgess et al, JHEP10 (2002) 028:

- Komar integrals can be used to define a ‘position-dependent’ mass/tension and chemical potential in the static patch. Position dependent = depends on endpoint value of the transverse coordinate of the hypersurface we integrate over. Expressions diverge for $r \to 0$ (curvature singularity). Dependence on transverse ‘cut-off.’
- Mass/tension is negative, consistent with repulsive behaviour of the singularity.
- Smarr-type relation involving position dependent quantities:

$$W = -T \log Z = T I_E = TS + Q\Phi(r) - T(r)$$

Thermodynamics in the cosmological patch?

The cosmological patch has an asymptotic boundary at $t \to \infty$, and the boundary terms contributing to Komar- or Gibbons-Hawking-York type expressions for 'mass' and other charges turn out to be finite.

Mere curiosity or physically relevant?

Metric in cosmological patch

$$ds_4^2 = -\frac{dt^2}{f(t)} + f(t)dr^2 + t^2(dx^2+dy^2) \ , \quad f(t) = \frac{m}{t} - \frac{e^2}{t^2} \ , \quad t > t_h = \frac{e^2}{m} \ .$$

Temperature. Defined either through surface gravity of Killing horizon, or absence of conical singularity of Euclidean continuation $(r, x, y) \to -i(r, x, y)$.

$$T = \frac{m^3}{4\pi e^4} \ .$$

Entropy (density) defined through 'area density,' include conventional factor 1/4:

$$s = \frac{1}{4} t_h^2 = \frac{e^4}{4m^2} \ , \quad S = s \int dxdy$$

or through Euclidean action (boundary terms evaluated for $t \to \infty$)

‘Mass/Tension’ (momentum? analytic regularization?) defined using Komar integral

$$M = -\frac{1}{8\pi} \int \star d\xi \ ,$$

or Gibbons-Hawking-York mass gives

$$M = -\frac{m}{8\pi} \int dxdy \ .$$

Chemical potentials. Defined using limit

$$A_r(t \to \infty) \ , \quad \text{with boundary condition } A_r(t_h) = 0 \ .$$

We have four chemical potentials $\mu^0, \tilde{\mu}_a$, $a = 1, 2, 3$.

Electric and magnetic charges Q_0, P^A defined by flux integrals.

We seem to be close to proving that a ‘first law’ of the form

$$dM = TdS + \mu^0 dQ_0 + \tilde{\mu}_1 dP^1 + \tilde{\mu}_2 dP^2 + \tilde{\mu}_3 dP^3$$

together with other thermodynamic relations holds for the general 4-charge and 3-charge solution.

Further remarks

Extremal limit

Limit $B \to 0$:

- 'Horizon' moves to infinite distance, cosmological patch disappears.
- $T \to 0$. Extremal limit.
- $s \to \infty$. Entropy density diverges. Since also $m \to 0$ could indicate 'tensionless limit.'

Negative tension branes in string theory

- Arise in orbifold/orientifold constructions. Located at fixed points.
- Required when extending network of string dualities by time-like T-duality transformations.

Future directions

- Properties of field equations, relation 1st order formulations (BPS squares, pseudo/fake-supersymmetry), Einstein-Maxwell-Dilaton theories.
- Physical interpretation of formal thermodynamic relations. E.g. does this imply anything about stability?
- Embedding into higher-dimensional supergravity and string theory.
- Negative tension branes and string dualities.