Scattering theory for Dirac and Klein-Gordon fields on the (De Sitter) Kerr metric and the Hawking effect

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Field equations on lorentzian space-times, Hamburg, March 20, 2018
Scattering theory on black hole type spacetimes and related subjects

- **1975**: **Hawking effect**. Contributions by many others including Gibbons, Unruh, Wald,...

- **1980’s** program by Dimock, Kay on scattering theory on the Schwarzschild metric. Related work by Fredenhagen, Haag. “Conformal scattering” by Friedlander.

- **1990’s** Further developed by Alain Bachelot giving a mathematically rigorous description of the Hawking effect in the spherically symmetric setting in 1999. Further contributions by Nicolas, Melnyk, Daudé,...

- **2000’s Kerr**
  - Scattering theory on Kerr H, H-Nicolas, rigorous description of the Hawking effect for fermions (H '09).
  - Decay of the local energy for field equations. Andersson-Blue, Dafermos-Rodnianski, Shlapentokh-Rothman, Dyatlov, Finster-Kamran-Smoller-Yau, Tataru-Tohaneanu, Vasy,...

  Scattering theory for Klein-Gordon equations without positive conserved energy (Kako, Gérard, Bachelot, Georgescu-Gérard-H.), on (De Sitter) Kerr (Georgescu-Gérard-H., Dafermos-Rodnianski-Shlapentokh-Rothman), scattering theory via vector field methods (Mason, Nicolas, Joudioux, Dafermos-Rodnianski-Shlapentokh-Rothman).
The (De Sitter) Kerr metric

De Sitter Kerr metric in Boyer-Lindquist coordinates

\[ M_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega, \]

with spacetime metric

\[
g = \frac{\Delta_r - a^2 \sin^2 \theta \Delta_\theta}{\lambda^2 \rho^2} dt^2 + \frac{2a \sin^2 \theta ((r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r)}{\lambda^2 \rho^2} dtd\varphi
- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2 \theta \sigma^2}{\lambda^2 \rho^2} d\varphi^2,
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right) (r^2 + a^2) - 2Mr,
\]

\[
\Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2 \theta, \quad \sigma^2 = (r^2 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta, \quad \lambda = 1 + \frac{1}{3} \Lambda a^2.
\]

\( \Lambda \geq 0 \): cosmological constant \( (\Lambda = 0 : \text{Kerr}) \), \( M > 0 \): masse, \( a \): angular momentum per unit masse \( (|a| < M) \).

- \( \rho^2 = 0 \) is a curvature singularity, \( \Delta_r = 0 \) are coordinate singularities. \( \Delta_r > 0 \) on some open interval \( r_- < r < r_+ \). \( r = r_- \): black hole horizon, \( r = r_+ \): cosmological horizon.

- \( \partial_\varphi \) and \( \partial_t \) are Killing. There exist \( r_1(\theta), r_2(\theta) \) s. t. \( \partial_t \) is
  - timelike on \( \{(t, r, \theta, \varphi) : r_1(\theta) < r < r_2(\theta)\} \),
  - spacelike on
    \[ \{(t, r, \theta, \varphi) : r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi) : r_2(\theta) < r < r_+\} =: \mathcal{E}_- \cup \mathcal{E}_+. \]

The regions \( \mathcal{E}_- \), \( \mathcal{E}_+ \) are called ergospheres.
The Penrose diagram ($\Lambda = 0$)

- Kerr-star coordinates:
  
  \[
  t^* = t + r_*, \ r, \ \theta, \ \varphi^* = \varphi + \Lambda(r), \quad \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad \frac{d\Lambda(r)}{dr} = \frac{a}{\Delta}.
  \]

  Along incoming principal null geodesics: $\dot{t}^* = \dot{\theta} = \dot{\varphi}^* = 0$, $\dot{r} = -1$.

- Form of the metric in Kerr-star coordinates:
  
  \[
  g = g_{tt} dt^* dt^* + 2 g_{t\varphi} dt^* d\varphi^* + g_{\varphi\varphi} d\varphi^* d\varphi^* + g_{\theta\theta} d\theta^2 - 2 dt^* dr + 2a \sin^2\varphi d\varphi^* dr.
  \]

- Future event horizon: $\mathcal{H}^+ := \mathbb{R}_t^* \times \{ r = r_+ \} \times S^2_{\theta,\varphi^*}$.

- The construction of the past event horizon $\mathcal{H}^-$ is based on outgoing principal null geodesics (star-Kerr coordinates). Similar constructions for future and past null infinities $\mathcal{J}^+$ and $\mathcal{J}^-$ using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$. 

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The diagram illustrates the Penrose diagram with the following labels:

- $\Sigma_1$, $\Sigma_s$:
  - Crossing sphere
  - Space-like infinity

- $\mathcal{H}^+$, $\mathcal{J}^+$, $\mathcal{H}^-$, $\mathcal{J}^-$:
  - Future timelike infinity
  - Past timelike infinity

- $\mathcal{J}^+ \times \mathcal{J}^-$:
  - Future and past null infinities

- $\mathcal{H}^+ \times \mathcal{H}^-$:
  - Event horizons
Part 1:
Scattering theory for massless Dirac fields on the Kerr metric

1.1 The Dirac equation and the Newman-Penrose formalism

Weyl equation:

\[ \nabla_{A'} \phi_A = 0. \]

Conserved current:

\[ V^a = \phi^A \phi^{A'}, \quad C(t) = \frac{1}{\sqrt{2}} \int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} = \text{const}. \]

\( T^a \): normal to \( \Sigma_t \).

- Newman-Penrose tetrad \( l^a, n^a, m^a, \bar{m}^a \):
  \[ l_a l^a = n_a n^a = m_a m^a = \bar{l}_a \bar{m}^a = n_a \bar{m}^a = 0. \]
  - Normalization \( l_a n^a = 1, m_a \bar{m}^a = -1 \)
  - \( l^a, n^a \): Scattering directions.

- Spin frame \( o^A \bar{o}^{A'} = l^a, \bar{l}^A \bar{l}^{A'} = n^a, o^A \bar{l}^{A'} = m^a \)
  \[ \bar{l}^A \bar{o}^A = \bar{m}^a, \quad o_A \bar{l}^A = 1 \]

- Components in the spin frame: \( \phi_0 = \phi_A o^A, \phi_1 = \phi_A \bar{l}^A \)

- Weyl equation:

\[
\begin{cases}
  n^a \partial_a \phi_0 - m^a \partial_a \phi_1 + (\mu - \gamma) \phi_0 + (\tau - \beta) \phi_1 = 0, \\
  l^a \partial_a \phi_1 - \bar{m}^a \partial_a \phi_0 + (\alpha - \pi) \phi_0 + (\epsilon - \tilde{\rho}) \phi_1 = 0.
\end{cases}
\]
A new Newman Penrose tetrad

Problem : The Kerr metric is at infinity a long range perturbation of the Minkowski metric. In the long range situation asymptotic completeness is generically false without modification of the wave operators.

Dirac equation on Schwarzschild :

\[ i \partial_t \Psi = \slashed{D}_S \Psi, \quad \slashed{D}_S = \Gamma^1 D_{r*} + \frac{(1 - \frac{2M}{r})^{1/2}}{r} \slashed{D}_{S^2} + V. \]

ok because of spherical symmetry.

Tetrad adapted to the foliation : \( l^a + n^a = T^a \). Conserved quantity :

\[ \frac{1}{\sqrt{2}} \int_{\Sigma_t} (|\phi_0|^2 + |\phi_1|^2) \sigma_{\Sigma_t}. \]

\( l^a, n^a \in \text{span}\{T^a, \partial_r\} \). \( \Psi \) spinor multiplied by a certain weight :

\[ i \partial_t \Psi = \slashed{D}_K \Psi, \quad \slashed{D}_K = h \slashed{D}_{\text{sym}} h + V_\varphi D_\varphi + V. \]

Well adapted to time dependent scattering : \( h^2 - 1 \), \( V_\varphi \), \( V \) short range.
1.2 Principal results

Comparison dynamics

\[ \mathcal{H} = L^2((\mathbb{R} \times S^2); dr_* d\omega); \mathbb{C}^2), \mathbb{D}_H = \gamma D_{r_*} - \frac{a}{r_*^2 + a^2} D_\varphi, \mathbb{D}_\infty = \gamma D_{r_*}, \]
\[ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{H}^- = \{ (\psi_0, 0) \in \mathcal{H} \} \text{ (resp. } \mathcal{H}^+ = \{ (0, \psi_1) \in \mathcal{H} \} \).

Theorem (Asymptotic velocity)

There exist bounded selfadjoint operators s.t. for all \( J \in C_\infty(\mathbb{R}) : \)
\[
J(P^\pm) = s - \lim_{t \to \pm \infty} e^{-it\mathbb{D}_K} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_K},
\]
\[
J(\mp \gamma) = s - \lim_{t \to \pm \infty} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H}
\]
\[
= s - \lim_{t \to \pm \infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}.
\]

In addition we have:
\[ \sigma(P^+) = \{-1, 1\}. \]
Theorem (Asymptotic completeness)

The classical wave operators defined by the limits

\[ W^\pm_H := s - \lim_{t \to \pm \infty} e^{-it\psi_K} e^{itH} P_{H^\pm} , \]

\[ W^\pm_\infty := s - \lim_{t \to \pm \infty} e^{-it\psi_K} e^{it\infty} P_{H^\pm} , \]

\[ \Omega^\pm_H := s - \lim_{t \to \pm \infty} e^{-itD_H} e^{it\psi_K} 1_{\mathbb{R}^-}(P^\pm) , \]

\[ \Omega^\pm_\infty := s - \lim_{t \to \pm \infty} e^{-it\infty} e^{it\psi_K} 1_{\mathbb{R}^+}(P^\pm) \]

exist.

Remark

1. Proof based on Mourre theory.
2. The same theorem holds with more geometric comparison dynamics.
3. Generalized by Daudé to the massive charged case.
4. Results valid for quite general perturbations of Kerr.
5. Schwarzschild: Nicolas (95), Melnyk (02), Daudé (04).
1.3 Geometric interpretation

- $\mathcal{J}^{\pm}$ are constructed using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$.
- The Weyl equation is conformally invariant: $\hat{\nabla}^{AA'} \hat{\phi}_A = 0$, where $\hat{\phi}_A = r \phi_A$. 

**Figure** — Penrose compactification of block I
1.3 Geometric interpretation

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The Weyl equation is conformally invariant:

$\hat{\nabla}^{AA'} \hat{\phi}_A = 0$, where $\hat{\phi}_A = r \phi_A$. 

**Figure** — Penrose compactification of block $I$
\[ \lim_{r \to r_+} \Psi_0(\gamma^-, \theta, \phi^#(r)) =: \psi_0|_{\mathcal{S}^+}(0, V, \theta, \phi^#), \]
\[ \lim_{r \to r_+} \Psi_1(\gamma^-, \theta, \phi^#(r)) = 0. \]
\[ \Psi \text{ is solution of the Dirac equation. } \gamma^-, \theta, \phi^# \text{ is the principal incoming null geodesic meeting } \mathcal{S}^+ \text{ at } (0, V, \theta, \phi^#). \]

\[ \text{Trace operators :} \]
\[ \mathcal{T}_{\mathcal{S}^+} : C^\infty(\Sigma_0, \mathbb{C}^2) \to C^\infty(\mathcal{S}^+, \mathbb{C}) \]
\[ \psi_{\Sigma_0} \mapsto \psi_0|_{\mathcal{S}^+}. \]

\[ \mathcal{H} : \text{Hilbert space associated to } \Sigma_0, \mathcal{H}_{\mathcal{S}^\pm} \text{ Hilbert spaces associated to } \mathcal{S}^\pm. \]

**Theorem**

The trace operators \( \mathcal{T}_{\mathcal{S}^\pm} \) extend in a unique manner to bounded operators from \( \mathcal{H} \) to \( \mathcal{H}_{\mathcal{S}^\pm} \).

**Remark**

Let \( \mathcal{S}^\pm_{\mathcal{S}^+} \) be the \( C^\infty \) diffeomorphisms from \( \mathcal{S}^\pm \) onto \( \Sigma_0 \) defined by identifying points along incoming (resp. outgoing) principal null geodesics and \( \Omega^\pm_{H, pn} \) inverse wave operators with comparison dynamics given by the principal null directions. Then \( \mathcal{T}_{\mathcal{S}^\pm} = (\mathcal{S}^\pm_{\mathcal{S}^+})^* \Omega^\pm_{H, pn} \). Comparison dynamics \( P_N = \gamma_{D_{r^*}} - \frac{a^2}{r^2 + a^2} D_{\phi}. \)
\[
\lim_{r \to r_+} \psi_0(\gamma_{V, \theta, \varphi^#}(r)) =: \psi_0|_{S_0^+}(0, V, \theta, \varphi^#),
\]
\[
\lim_{r \to r_+} \psi_1(\gamma_{V, \theta, \varphi^#}(r)) = 0.
\]
\[
\psi \text{ is solution of the Dirac equation. } \gamma_{V, \theta, \varphi^#} \text{ is the principal incoming null geodesic meeting } S_0^+ \text{ at } (0, V, \theta, \varphi^#).
\]

Trace operators:
\[
\mathcal{T}_{S_0^+}^+ : C^\infty_0(\Sigma_0, C^2) \to C^\infty(S_0^+, C) \psi_{S_0} \mapsto \psi_0|_{S_0^+}.
\]

\[
\mathcal{H} : \text{Hilbert space associated to } \Sigma_0, \mathcal{H}_{S_0^\pm} \text{ Hilbert spaces associated to } S_0^\pm.
\]

**Theorem**

The trace operators \( \mathcal{T}_{S_0^\pm} \) extend in a unique manner to bounded operators from \( \mathcal{H} \) to \( \mathcal{H}_{S_0^\pm} \).

**Remark**

Let \( \tilde{\mathcal{T}}_{S_0^\pm} \) be the \( C^\infty \) diffeomorphisms from \( S_0^\pm \) onto \( \Sigma_0 \) defined by identifying points along incoming (resp. outgoing) principal null geodesics and \( \Omega_{H, pn}^\pm \) inverse wave operators with comparison dynamics given by the principal null directions. Then \( \mathcal{T}_{S_0^\pm} = (\tilde{\mathcal{T}}_{S_0^\pm})^* \Omega_{H, pn}^\pm \). Comparison dynamics \( P_N = \gamma D_{r_\ast} - \frac{a^2}{r^2 + a^2} D_\varphi \).
\[ \lim_{r \to r_+} \psi_0(\gamma_{V,\theta,\varphi^#}(r)) =: \psi_0|_{\mathcal{S}^+}(0, V, \theta, \varphi^#), \]
\[ \lim_{r \to r_+} \psi_1(\gamma_{V,\theta,\varphi^#}(r)) = 0. \]
\( \psi \) is solution of the Dirac equation. \( \gamma_{V,\theta,\varphi^#} \) is the principal incoming null geodesic meeting \( \mathcal{S}^+ \) at \( (0, V, \theta, \varphi^#) \).

**Trace operators**:

\[ \mathcal{T}_{\mathcal{S}^+}^{\pm}: C^\infty_0(\Sigma_0, \mathbb{C}^2) \to C^\infty(\mathcal{S}^+\mathcal{N}, \mathbb{C}) \]

\[ \psi_{\Sigma_0} \mapsto \psi_0|_{\mathcal{S}^+}. \]

**\( \mathcal{H} \)**: Hilbert space associated to \( \Sigma_0 \), \( \mathcal{H}_{\mathcal{S}^\pm} \) Hilbert spaces associated to \( \mathcal{S}^\pm \).

**Theorem**

*The trace operators \( \mathcal{T}_{\mathcal{S}^\pm} \) extend in a unique manner to bounded operators from \( \mathcal{H} \) to \( \mathcal{H}_{\mathcal{S}^\pm} \).*

**Remark**

*Let \( \mathcal{S}_{\mathcal{S}^\pm}^{\pm} \) be the \( C^\infty \) diffeomorphisms from \( \mathcal{S}^\pm \) onto \( \Sigma_0 \) defined by identifying points along incoming (resp. outgoing) principal null geodesics and \( \Omega_{\mathcal{H},pn}^{\pm} \) inverse wave operators with comparison dynamics given by the principal null directions. Then \( \mathcal{T}_{\mathcal{S}^\pm} = (\mathcal{S}_{\mathcal{S}^\pm})^* \Omega_{\mathcal{H},pn}^{\pm} \). Comparison dynamics \( P_N = \gamma D_{r^*} - \frac{a^2}{r^2+a^2} D_{\varphi} \).*
\[ \lim_{r \to r^+} \psi_0(\gamma_{V,\theta,\varphi^#}(r)) =: \psi_0|_{\Sigma^+}(0, V, \theta, \varphi^#), \]
\[ \lim_{r \to r^+} \psi_1(\gamma_{V,\theta,\varphi^#}(r)) = 0. \]
\[ \psi \text{ is solution of the Dirac equation. } \gamma_{V,\theta,\varphi^#} \text{ is the principal incoming null geodesic meeting } \Sigma^+ \text{ at } (0, V, \theta, \varphi^#). \]

\begin{itemize}
  \item Trace operators :
  \[ \mathcal{T}_{\Sigma^+} : C^\infty(\Sigma_0, \mathbb{C}^2) \to C^\infty(\Sigma^+, \mathbb{C}) \]
  \[ \psi_{\Sigma_0} \mapsto \psi_0|_{\Sigma^+}. \]

  \item \[ \mathcal{H} : \text{Hilbert space associated to } \Sigma_0, \mathcal{H}_{\Sigma^\pm} \text{ Hilbert spaces associated to } \Sigma^\pm. \]
\end{itemize}

**Theorem**

The trace operators \( \mathcal{T}_{\Sigma^\pm} \) extend in a unique manner to bounded operators from \( \mathcal{H} \) to \( \mathcal{H}_{\Sigma^\pm} \).

**Remark**

Let \( \mathfrak{T}_{\Sigma^\pm} \) be the \( C^\infty \) diffeomorphisms from \( \Sigma^\pm \) onto \( \Sigma_0 \) defined by identifying points along incoming (resp. outgoing) principal null geodesics and \( \Omega_{H,\text{pn}}^\pm \) inverse wave operators with comparison dynamics given by the principal null directions. Then \( \mathcal{T}_{\Sigma^\pm} = (\mathfrak{T}_{\Sigma^\pm})^* \Omega_{H,\text{pn}}^\pm \). Comparison dynamics \( P_N = \gamma D_{r^*} - \frac{a^2}{r^2 + a^2} D_\varphi \).
\[
\lim_{r \to r^+} \psi_0(\gamma_{V,\theta,\varphi^\#}(r)) =: \psi_0|_{\mathcal{S}_0^+}(0, V, \theta, \varphi^\#), \\
\lim_{r \to r^+} \psi_1(\gamma_{V,\theta,\varphi^\#}(r)) = 0.
\]

\(\psi\) is solution of the Dirac equation. \(\gamma_{V,\theta,\varphi^\#}\) is the principal incoming null geodesic meeting \(\mathcal{S}_0^+\) at \((0, V, \theta, \varphi^\#)\).

- **Trace operators**:

\[
\mathcal{T}_{\mathcal{S}_0}^+ : \quad \mathcal{C}_0^\infty(\Sigma_0, \mathbb{C}^2) \rightarrow \mathcal{C}^\infty(\mathcal{S}_0^+, \mathbb{C}) \\
\psi_{\Sigma_0} \mapsto \psi_0|_{\mathcal{S}_0^+}.
\]

- **\(\mathcal{H}\)**: Hilbert space associated to \(\Sigma_0\), \(\mathcal{H}_{\mathcal{S}_0^\pm}\) Hilbert spaces associated to \(\mathcal{S}_0^\pm\).

**Theorem**

The trace operators \(\mathcal{T}_{\mathcal{S}_0}^\pm\) extend in a unique manner to bounded operators from \(\mathcal{H}\) to \(\mathcal{H}_{\mathcal{S}_0^\pm}\).

**Remark**

Let \(\mathcal{F}_{\mathcal{S}_0}^\pm\) be the \(\mathcal{C}^\infty\) diffeomorphisms from \(\mathcal{S}_0^\pm\) onto \(\Sigma_0\) defined by identifying points along incoming (resp. outgoing) principal null geodesics and \(\Omega_{H,\text{pn}}^\pm\) inverse wave operators with comparison dynamics given by the principal null directions. Then \(\mathcal{T}_{\mathcal{S}_0}^\pm = (\mathcal{F}_{\mathcal{S}_0}^\pm)^* \Omega_{H,\text{pn}}^\pm\). Comparison dynamics \(P_N = \gamma D r_* - \frac{a^2}{r^2 + a^2} D \varphi\).
Same construction for $T_j^\pm$ and $\mathcal{H}_j^\pm$. $T_j^\pm$ can be extended to bounded operators from $\mathcal{H}$ to $\mathcal{H}_j^\pm$.

$$\Pi_F : \mathcal{H} \ni \psi_{\Sigma_0} \mapsto (T_j^+ \psi_{\Sigma_0}, T_j^- \psi_{\Sigma_0}).$$

**Theorem (Goursat problem)**

$\Pi_F$ is an isometry. In particular for all $\Phi \in \mathcal{H}_F$, there exists a unique solution of the Dirac equation $\psi \in C(\mathbb{R}_t, \mathcal{H})$ s.t. $\Phi = \Pi_F \psi(0)$.

**Remark**

1) First constructions of this type: Friedlander (Minkowski, 80, 01), Bachelot (Schwarzschild, 91).
2) The inverse is possible: Mason, Nicolas (04), Joudioux (10) (asymptotically simple space-times), Dafermos-Rodnianski-Shlapentokh-Rothman (Kerr).
Part 2 : The Hawking effect as a scattering problem

D. H., Creation of fermions by rotating charged black holes, Mémoires de la SMF 117 (2009), 158 pp.
2.1 The collapse of the star

\[ M_{col} = \bigcup_t \Sigma_t^{col}, \Sigma_t^{col} = \{ (t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_\omega; \hat{r} \geq \hat{z}(t, \theta) \}. \]

Assumptions:

- For \( \hat{r} > \hat{z}(t, \theta) \), the metric is the Kerr Newman metric.
- \( \hat{z}(t, \theta) \) behaves asymptotically like certain timelike geodesics in the Kerr-Newman metric. We suppose for the conserved quantities \( L \) (angular momentum), \( Q \) (Carter constant) and \( \tilde{E} \) (rotational energy) : \( L = Q = \tilde{E} = 0 \). We also suppose an asymptotic condition on the surface of the star:

\[
\hat{z}(t, \theta) = -t - \hat{\Lambda}(\theta)e^{-2\kappa_- t} + O(e^{-4\kappa_- t}), \quad t \to \infty.
\]

\( \kappa_- > 0 \) is the surface gravity of the outer horizon, \( \hat{\Lambda}(\theta) > 0 \).

Remark:

1. \( \hat{r} \) is a coordinate adapted to simple null geodesics.
2. Dirac in \( M_{col} \) : we add a boundary condition (MIT)

\( \Psi(t) = U(t, 0)\Psi_0 \).
2.1 The collapse of the star

\[ \mathcal{M}_{col} = \bigcup_{t} \Sigma_{col}^{t}, \Sigma_{col}^{t} = \{(t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_{\omega}; \hat{r} \geq \hat{z}(t, \theta)\}. \]

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\[
\hat{z}(t, \theta) = -t - \hat{A}(\theta) e^{-2\kappa_- t} + \mathcal{O}(e^{-4\kappa_- t}), \quad t \to \infty.
\]

\( \kappa_- > 0 \) is the surface gravity of the outer horizon, \( \hat{A}(\theta) > 0 \).

Remark

1. \( \hat{r} \) is a coordinate adapted to simple null geodesics.
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2.1 The collapse of the star

\[ \mathcal{M}_{\text{col}} = \bigcup_t \Sigma_t^{\text{col}}, \quad \Sigma_t^{\text{col}} = \{(t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_\omega; \, \hat{r} \geq \hat{z}(t, \theta)\}. \]

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**Remark**

1. \( \hat{r} \) is a coordinate adapted to simple null geodesics.
2. Dirac in \( \mathcal{M}_{\text{col}} \) : we add a boundary condition (MIT) 
   \[ \Psi(t) = U(t, 0)\psi_0. \]
2.2 Dirac quantum fields

Dimock ’82.

\[ \mathcal{M}_{col} = \bigcup_{t \in \mathbb{R}} \Sigma_t^{col}, \quad \Sigma_t^{col} = \{(t, \hat{r}, \theta, \varphi); \hat{r} \geq \hat{z}(t, \theta)\}. \]

Dirac quantum field \( \Psi_0 \) and the CAR-algebra \( \mathcal{U}(\mathcal{H}_0) \) constructed in the usual way. Fermi-Fock representation.

\[ S_{col} : \quad \left( C_0^\infty(\mathcal{M}_{col}) \right)^4 \quad \mapsto \quad \mathcal{H}_0 \]
\[ \Phi \mapsto S_{col} \Phi := \int_{\mathbb{R}} U(0, t) \Phi(t) \, dt \]

Quantum spin field:

\[ \psi_{col} : \quad \left( C_0^\infty(\mathcal{M}_{col}) \right)^4 \quad \rightarrow \quad \mathcal{L}(\mathcal{F}(\mathcal{H}_0)) \]
\[ \Phi \mapsto \psi_{col}(\Phi) := \Psi_0(S_{col} \Phi) \]

\( \mathcal{U}_{col}(\mathcal{O}) = \) algebra generated by \( \psi_{col}^*(\Phi^1) \psi_{col}(\Phi^2) \), \( \text{supp} \Phi^j \subset \mathcal{O} \).

\[ \mathcal{U}_{col}(\mathcal{M}_{col}) = \bigcup_{\mathcal{O} \subset \mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O}). \]

Same procedure on \( \mathcal{M}_{BH} \):

\[ S : \Phi \in \left( C_0^\infty(\mathcal{M}_{BH}) \right)^4 \mapsto S\Phi := \int_{\mathbb{R}} e^{-itH} \Phi(t) \, dt. \]
States

\[ \mathcal{U}_{\text{col}}(\mathcal{M}_{\text{col}}) \]

Vacuum state:

\[
\omega_{\text{col}}(\psi_{\text{col}}^*(\Phi_1)\psi_{\text{col}}(\Phi_2)) := \omega_{\text{vac}}(\psi_0^*(S_{\text{col}}\Phi_1)\psi_0(S_{\text{col}}\Phi_2)) \\
= \langle 1_{[0,\infty)}(H_0)S_{\text{col}}\Phi_1, S_{\text{col}}\Phi_2 \rangle.
\]

\[ \mathcal{U}_{\text{BH}}(\mathcal{M}_{\text{BH}}) \]

Vacuum state

\[
\omega_{\text{vac}}(\psi_{\text{BH}}^*(\Phi_1)\psi_{\text{BH}}(\Phi_2)) = \langle 1_{[0,\infty)}(H)S\phi_1, S\phi_2 \rangle.
\]

Thermal Hawking state

\[
\omega^{\eta,\sigma}_{\text{Haw}}(\psi_{\text{BH}}^*(\Phi_1)\psi_{\text{BH}}(\Phi_2)) = \langle \mu e^{\sigma H}(1 + \mu e^{\sigma H})^{-1}S\Phi_1, S\Phi_2 \rangle_{\mathcal{H}} \\
= \omega^{\eta,\sigma}_{\text{KMS}}(\psi^*(S\Phi_1)\psi(S\Phi_2)), \\
T_{\text{Haw}} = \sigma^{-1}, \mu = e^{\sigma\eta}, \sigma > 0.
\]

\( T_{\text{Haw}} \) Hawking temperature, \( \mu \) chemical potential.
The Hawking effect

\[ \Phi \in (C_0^\infty (\mathcal{M}_{col}))^4, \quad \Phi^T(t, \hat{r}, \omega) = \Phi(t - T, \hat{r}, \omega). \]

**Theorem (Hawking effect)**

Let \( \Phi_j \in (C_0^\infty (\mathcal{M}_{col}))^4, \ j = 1, 2. \) We have

\[
\lim_{T \to \infty} \omega_{\text{col}}(\psi_{\text{col}}^*(\Phi_1^T)\psi_{\text{col}}(\Phi_2^T)) \\
= \omega_{\text{Haw}}^{\eta, \sigma}(\psi_{\text{BH}}^*(1_{\mathbb{R}^+}(P^-)\Phi_1)\psi_{\text{BH}}(1_{\mathbb{R}^+}(P^-)\Phi_2)) \\
+ \omega_{\text{vac}}(\psi_{\text{BH}}^*(1_{\mathbb{R}^-}(P^-)\Phi_1)\psi_{\text{BH}}(1_{\mathbb{R}^-}(P^-)\Phi_2)),
\]

\[
T_{\text{Haw}} = 1/\sigma = \kappa_- / 2\pi, \quad \mu = e^{\sigma \eta}, \quad \eta = \frac{qQr_-}{r_+^2 + a^2} + \frac{aD_\varphi}{r_+^2 + a^2}.
\]
2.3 Explanation

**Figure**—Collapse of the star

Change in frequencies: mixing of positive and negative frequencies.
2.4 The analytic problem

\[ \lim_{T \to \infty} \left\| 1_{[0, \infty)} (D_0) U(0, T) f \right\|_0^2 \]

\[ = \langle 1_{\mathbb{R}^+} (P^-) f, \mu e^{\sigma P} (1 + \mu e^{\sigma P})^{-1} 1_{\mathbb{R}^+} (P^-) f \rangle \]

\[ + \left\| 1_{[0, \infty)} (\mathcal{D}) 1_{\mathbb{R}^-} (P^-) f \right\|^2 . \]

Remark

1) Hawking 1975,
2) Bachelot (99), Melnyk (04).
3) Schwarzschild : Moving mirror, equation with potential.
2.5 Toy model: The moving mirror

\[ z(t) = -t - Ae^{-2\kappa t}; \ A > 0, \ \kappa > 0, \]

\[
\begin{aligned}
\partial_t \psi &= i\mathcal{D}\psi, \\
\psi_1(t, z(t)) &= \sqrt{\frac{1-z}{1+z}} \psi_2(t, z(t)), \quad \mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_x. \\
\psi(t = s, .) &= \psi_s(.)
\end{aligned}
\]

Solution given by a unitary propagator \( U(t, s) \). Conserved \( L^2 \) norm:

\[ \|\psi\|_{\mathcal{H}_t}^2 = \int_{z(t)}^\infty |\psi|^2(t, x)dx. \]

Explicit calculation:

\[
\lim_{T \to \infty} \|1_{[0, \infty)}(\mathcal{D}_0) U(0, T) f\|_0^2 = \langle e^{\frac{2\pi}{\kappa} p} \left( 1 + e^{\frac{2\pi}{\kappa} p} \right)^{-1} P_2 f, P_2 f \rangle + \|1_{[0, \infty)}(\mathcal{D}) P_1 f\|^2.
\]

Scattering problem: show that the real system behaves the same way.
2.6 Some remarks on the proof

- We compare to a dynamics for which the radiation can be explicitly computed.
- Can’t compare dynamics on Cauchy surfaces → characteristic Cauchy problem.
- Three time intervals:
  - \([T/2 + c_0, T]\) no boundary involved → use asymptotic completeness + propagation estimates.
  - \([t_c, T/2 + c_0]\) use Duhamel formula + construction of tetrad and coordinates:
    - There exists a coordinate system \((t, \hat{r}, \omega)\) such that \(\hat{r} = -t + c_0\) along incoming simple null geodesics \(\hat{r} = c_0 = 0\).
    - There exists a Newman-Penrose tetrad such that:
      \(\mathcal{D} = \Gamma \mathcal{D} \hat{r} + P \omega + W\), \(\Gamma = \text{Diag}(1, -1, -1, 1)\). \(P\) is a differential operator with derivatives only in the angular directions and \(W\) is a potential.
  - \([0, t_c]\):
    \[\|1_{[0, \infty]}(\mathcal{D}_0)U(0, t_c)U_H(t_c, T)\Omega_H f\| \sim \|1_{[0, \infty]}(\mathcal{D}_H, 0)U_H(0, T)\Omega_H f\|\] if evolution is essentially given by the group (and not the evolution system). For this
    - \(U_H(t_c, T) \Omega_H f \rightarrow 0\).
    - The Hamiltonian flow stays outside the surface of the star for data in the given regime \(|\xi| \gg |\Theta|\).
    - Propagation of singularities, compact Sobolev embeddings.
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- Can’t compare dynamics on Cauchy surfaces $\rightarrow$ characteristic Cauchy problem.
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  - $[T/2 + c_0, T]$ no boundary involved $\rightarrow$ use asymptotic completeness+propagation estimates.
  - $[t_e, T/2 + c_0]$ use Duhamel formula + construction of tetrad and coordinates:
    - There exists a coordinate system $(t, \hat{r}, \omega)$ such that $\hat{r} = -t + c$ along incoming simple null geodesics $f_2 = f_2 = 0$.
    - There exists a Newman-Penrose tetrad such that:
      - $P_\omega$ is a differential operator with derivatives only in the angular directions and $W$ is a potential.
  - $[0, t_e]$:
    $$\|1_{[0, \infty]}(D_0) U(0, t_e) U_H(t_e, T) \Omega f\| \sim \|1_{[0, \infty]}(D_{H,0}) U_H(0, T) \Omega f\|$$
    if evolution is essentially given by the group (and not the evolution system). For this
    - $U_H(t_e, T) \Omega f \rightarrow 0$.
    - The hamiltonian flow stays outside the surface of the star for data in the given regime $|\xi| >> |\theta|$.
- Propagation of singularities, compact Sobolev embeddings.
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    - There exists a coordinate system $(t, \hat{r}, \omega)$ such that $\hat{r} = -t + c$ along incoming simple null geodesics ($L = Q = 0$).
    - There exists a Newman Penrose tetrad such that:
      $\Psi = \Gamma D_t + P_\omega + W$, $\Gamma = \text{Diag}(1, -1, -1, 1)$. $P_\omega$ is a differential operator with derivatives only in the angular directions and $W$ is a potential.
  - $[0, t_\epsilon]$:
    $\|1_{[0, \infty]}(\mathcal{D}_0)U(0, t_\epsilon)U_H(t_\epsilon, T)\Omega_H f\| \sim \|1_{[0, \infty]}(\mathcal{D}_H, 0)U_H(0, T)\Omega_H f\|$ if evolution is essentially given by the group (and not the evolution system). For this
    - $U_H(t_\epsilon, T)\Omega_H f \to 0$.
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    - There exists a coordinate system \((t, \tilde{r}, \omega)\) such that \(\tilde{r} = -t + c\) along incoming simple null geodesics \((L = Q = 0)\).
    - There exists a Newman Penrose tetrad such that:
      \(\varphi = \Gamma D_l + P_\omega + W, \Gamma = \text{Diag}(1, -1, -1, 1)\).
      \(P_\omega\) is a differential operator with derivatives only in the angular directions and \(W\) is a potential.
  - \([0, t_\epsilon]\):
    \[\|1_{[0,\infty]}(\varphi_0)U(0, t_\epsilon)U_H(t_\epsilon, T)\Omega_H^- f\| \sim \|1_{[0,\infty]}(D_{H,0})U_H(0, T)\Omega_H^- f\|\] if evolution is essentially given by the group (and not the evolution system). For this
    - \(U_H(t_\epsilon, T)\Omega_H^- f \rightarrow 0\).
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  - \([0, t_\epsilon]\):
    \[
    \|1_{[0, \infty]}(\Psi_0)U(0, t_\epsilon)U_H(t_\epsilon, T)\Omega_H f\| \sim \|1_{[0, \infty]}(\mathbb{D}_H, 0)U_H(0, T)\Omega_H f\| \text{ if evolution is essentially given by the group (and not the evolution system). For this}
    \]
    - \(U_H(t_\epsilon, T)\Omega_H f \rightarrow 0\).
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  - $[0, t_\epsilon]$:
    $\|1_{[0, \infty]}(\hat{D}_{0}) U(0, t_\epsilon) U_H(t_\epsilon, T) \Omega_H^{-1} f\| \sim \|1_{[0, \infty)}(\hat{D}_H) U_H(0, T) \Omega_H^{-1} f\|$ if evolution is essentially given by the group (and not the evolution system). For this
      - $U_H(t_\epsilon, T) \Omega_H^{-1} f \rightarrow 0$.
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      \(P_\omega\) is a differential operator with derivatives only in the angular directions and \(W\) is a potential.
  - \([0, t_\epsilon] : \|1_{[0, \infty]}(\Phi_0) U(0, t_\epsilon) U_H(t_\epsilon, T) \Omega_H^-- f\| \sim \|1_{[0, \infty]}(\mathbb{D}_{H,0}) U_H(0, T) \Omega_H^-- f\|\) if evolution is essentially given by the group (and not the evolution system). For this
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  - \([0, t_\epsilon]\):
    \[\|1_{[0, \infty]}(\mathcal{D}_0)U(0, t_\epsilon)U_H(t_\epsilon, T)\Omega_H^-f\| \sim \|1_{[0, \infty]}(\mathcal{D}_H, 0)U_H(0, T)\Omega_H^-f\|\]
    if evolution is essentially given by the group (and not the evolution system). For this
    - \(U_H(t_\epsilon, T)\Omega_H^-f \rightharpoonup 0\).
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  - $[0, t_\epsilon]$ :
    $\| 1_{[0, \infty)}(\mathcal{D}_0) U(0, t_\epsilon) U_H(t_\epsilon, T) \Omega_H^- f \| \sim \| 1_{[0, \infty)}(\mathcal{D}_H, 0) U_H(0, T) \Omega_H^- f \|$ if evolution is essentially given by the group (and not the evolution system). For this
      - $U_H(t_\epsilon, T) \Omega_H^- f \xrightarrow{\epsilon \downarrow 0} 0$.
      - The hamiltonian flow stays outside the surface of the star for data in the given regime ($|\xi| \gg |\Theta|$).
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      $\psi = \Gamma D_{\hat{r}} + P_\omega + W$, $\Gamma = \text{Diag}(1, -1, -1, 1)$. $P_\omega$ is a differential operator with derivatives only in the angular directions and $W$ is a potential.
  - $[0, t_\epsilon]$ :
    $\|1_{[0, \infty]}(\psi_0) U(0, t_\epsilon) U_H(t_\epsilon, T) \Omega^+ f\| \sim \|1_{[0, \infty)}(\mathbb{D}_H, 0) U_H(0, T) \Omega_H^- f\|$ if evolution is essentially given by the group (and not the evolution system). For this
    - $U_H(t_\epsilon, T) \Omega_H^- f \rightrightarrows 0$.
    - The hamiltonian flow stays outside the surface of the star for data in the given regime ($|\xi| > > |\Theta|$).
    - Propagation of singularities, compact Sobolev embeddings.
Part 3: Scattering theory for the Klein-Gordon equation on the De Sitter Kerr metric

3.1 The Klein-Gordon equation on the De Sitter Kerr metric

De Sitter Kerr metric in Boyer-Lindquist coordinates

\[ \mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_{\omega}, \text{ with spacetime metric} \]

\[
g = \frac{\Delta_r - a^2 \sin^2 \theta \Delta_\theta}{\lambda^2 \rho^2} dt^2 + \frac{2a \sin^2 \theta((r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r)}{\lambda^2 \rho^2} dtd\varphi \\
- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2 \theta \sigma^2}{\lambda^2 \rho^2} d\varphi^2,
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right) (r^2 + a^2) - 2Mr,
\]

\[
\Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2 \theta, \quad \sigma^2 = (r^2 + a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2 \theta, \quad \lambda = 1 + \frac{1}{3} \Lambda a^2.
\]

\( \Lambda > 0 \) : cosmological constant, \( M > 0 \) : masse, \( a \) : angular momentum per unit masse.

\( \rho^2 = 0 \) is a curvature singularity, \( \Delta_r = 0 \) are coordinate singularities.

\( \Delta_r > 0 \) on some open interval \( r_- < r < r_+ \). \( r = r_- \) : black hole horizon, \( r = r_+ \) : cosmological horizon.

\( \partial_\varphi \) and \( \partial_t \) are Killing. There exist \( r_1(\theta), \ r_2(\theta) \) s. t. \( \partial_t \) is

\( \text{timelike on } \{(t, r, \theta, \varphi) : r_1(\theta) < r < r_2(\theta) \} \),

\( \text{spacelike on } \{(t, r, \theta, \varphi) : r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi) : r_2(\theta) < r < r_+ \} =: \mathcal{E}_- \cup \mathcal{E}_+ \).

The regions \( \mathcal{E}_- \), \( \mathcal{E}_+ \) are called ergospheres.
3.1 The Klein-Gordon equation on the De Sitter Kerr metric

We now consider the unitary transform

\[ U : \quad L^2(\mathcal{M}; \frac{\sigma^2}{\Delta r \Delta \theta} dr d\omega) \rightarrow L^2(\mathcal{M}; dr d\omega) \]
\[ \psi \mapsto \frac{\sigma}{\sqrt{\Delta r \Delta \theta}} \psi \]

If \( \psi \) fulfills \((\Box_g + m^2)\psi = 0\), then \( u = U\psi \) fulfills

\[ (\partial_t^2 - 2ik\partial_t + h)u = 0. \]

with

\[ k = \frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2} D\varphi, \]
\[ h = -\frac{(\Delta_r - a^2 \sin^2 \theta \Delta_\theta)}{\sin^2 \theta \sigma^2} \partial^2 \varphi - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sigma} \partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sigma} \]
\[ - \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sin \theta \sigma} \partial_\theta \sin \theta \Delta_\theta \partial_\theta \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda \sigma} + \frac{\rho^2 \Delta_r \Delta_\theta}{\lambda^2 \sigma^2} m^2. \]

\( h \) is not positive inside the ergospheres. This entails that the natural conserved quantity

\[ \tilde{\mathcal{E}}(u) = \|\partial_t u\|^2 + (hu|u) \]

is not positive.
3+1 decomposition, energies, Killing fields

Let \( v = e^{-ikt} u \). Then \( u \) is solution of (2) if and only if \( v \) is solution of

\[
(\partial_t^2 + h(t))v = 0, \quad h(t) = e^{-ikt} h_0 e^{ikt}, \quad h_0 = h + k^2 \geq 0.
\]

Natural energy :

\[
\|\partial_t v\|^2 + (h(t)v|v).
\]

Rewriting for \( u \):

\[
\dot{\mathcal{E}}(u) = \|(\partial_t - ik)u\|^2 + (h_0 u|u).
\]

This energy is positive, but may grow in time \( \rightarrow \) superradiance.

Remark

- \( \partial_t - ik = \nabla_t \left( \frac{\nabla t}{\nabla_b t \nabla^b t} \right) \).

- \( k = \Omega D_\varphi \) and \( \Omega \) has finite limits \( \Omega_{-/+} \) when \( r \to r_{\mp} \). These limits are called *angular velocities* of the horizons. The Killing fields \( \partial_t - \Omega_{-/+} \partial_\varphi \) on the De Sitter Kerr metric are timelike close to the black hole (-) resp. cosmological (+) horizon.
3.2 The abstract equation

\( \mathcal{H} \) Hilbert space. \( h, k \) selfadjoint, \( k \in \mathcal{B}(\mathcal{H}) \).

\[
\begin{cases}
(\partial_t^2 - 2ik\partial_t + h)u &= 0, \\
u|_{t=0} &= u_0, \\
\partial_t u|_{t=0} &= u_1.
\end{cases}
\]

(3)

Hyperbolic equation

(A1) \( h_0 := h + k^2 \geq 0 \).

Formally \( u = e^{izt} v \) solution if and only if

\[ p(z)v = 0 \]

with \( p(z) = h_0 - (k - z)^2 = h + z(2k - z), z \in \mathbb{C} \). \( p(z) \) is called the quadratic pencil.

Conserved quantities

\[ \langle u|u \rangle_\ell := \|u_1 - \ell u_0\|^2 + (p(\ell)u_0|u_0), \]

where \( p(\ell) = h_0 - (k - \ell)^2 \). Conserved by the evolution, but in general not positive definite, because none of the operators \( p(\ell) \) is in general positive.
Spaces and operators

\( \mathcal{H}^i \) : scale of Sobolev spaces associated to \( h_0 \).

\[(A2)\quad 0 \notin \sigma_{pp}(h_0); \ h_0^{1/2}kh_0^{-1/2} \in B(\mathcal{H}).\]

Homogeneous energy spaces

\[\dot{\mathcal{E}} = \Phi(k)h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \Phi(k) = \begin{pmatrix} \mathbb{I} & 0 \\ k & \mathbb{I} \end{pmatrix}.\]

where \( \dot{\mathcal{E}} \) is equipped with the norm \( \|(u_0, u_1)\|_{\dot{\mathcal{E}}}^2 = \|u_1 - ku_0\|^2 + (h_0 u_0 | u_0) \).

Klein Gordon operator

\[\psi = (u, \frac{1}{i}\partial_t u), \quad (\partial_t - iH)\psi = 0, \quad H = \begin{pmatrix} 0 & \mathbb{I} \\ h & 2k \end{pmatrix},\]

\[(H - z)^{-1} = p^{-1}(z)\begin{pmatrix} z - 2k & \mathbb{I} \\ h & z \end{pmatrix}.\]

We note \( \dot{H} \) the Klein-Gordon operator on the homogeneous energy space.
3.3 Results in the De Sitter Kerr case

Uniform boundedness of the evolution

(4) \( \mathcal{H}^n = \{ u \in L^2(\mathbb{R} \times S^2) : (D\varphi - n)u = 0 \}, \ n \in \mathbb{Z}. \)

We construct the homogeneous energy space \( \dot{\mathcal{E}}^n \) as well as the Klein-Gordon operator \( \dot{H}^n \) as in Sect. 3.2.

**Theorem**

There exists \( a_0 > 0 \) such that for \( |a| < a_0 \) the following holds: for all \( n \in \mathbb{Z} \), there exists \( C_n > 0 \) such that

(5) \( \| e^{-it\dot{H}^n} u \|_{\dot{\mathcal{E}}^n} \leq C_n \| u \|_{\dot{\mathcal{E}}^n}, \ u \in \dot{\mathcal{E}}^n, \ t \in \mathbb{R}. \)

**Remark**

1. Note that for \( n = 0 \) the Hamiltonian \( \dot{H}^n = \dot{H}^0 \) is selfadjoint, therefore the only issue is \( n \neq 0 \).
2. Different from uniform boundedness on Cauchy surfaces crossing the horizon.
Asymptotic dynamics

Regge-Wheeler type coordinate $\frac{dx}{dr} = \frac{r^2+a^2}{\Delta r}$.

$x \pm t = \text{const.}$ along principal null geodesics.

Unitary transform:

$$V : L^2(\mathbb{R}(r_-,r_+) \times S^2) \rightarrow L^2(\mathbb{R} \times S^2, dx d\omega),$$

$$v(r,\omega) \mapsto \sqrt{\frac{\Delta r}{r^2+a^2}} v(r(x),\omega).$$

Asymptotic equations:

$$(\partial_t^2 - 2\Omega_{-/+} \partial \varphi \partial_t + h_{-/+})u_{-/+} = 0,$$

$$h_{-/+} = \Omega_{-/+}^2 - \partial_x^2 - \partial^2 _\varphi.$$

The conserved quantities:

$$\|(\partial_t - i\Omega_{-/+} D_\varphi)u_{-/+}\|^2 + ((h_{-/+} - \Omega_{-/+}^2 \partial^2 _\varphi)u_{-/+} | u_{-/+})$$

$$= \|(\partial_t - i\Omega_{-/+} D_\varphi)u_{-/+}\|^2 + (-\partial_x^2 u_{-/+} | u_{-/+})$$

are positive.
Asymptotic profiles

Let $\ell_{-/+} = \Omega_{-/+} n$. Also let $i_{-/+} \in C^\infty(\mathbb{R})$, $i_- = 0$ in a neighborhood of $\infty$, $i_+ = 0$ in a neighborhood of $-\infty$ and $i_-^2 + i_+^2 = 1$. Let

$$h_{-/+}^n = -\partial_x^2 - \ell_{-/+}^2, \quad k_{-/+} = \ell_{-/+}, \quad H_{-/+}^n = \begin{pmatrix} 0 & \mathbb{I} \\ h_{-/+} & 2k_{-/+} \end{pmatrix}$$

acting on $\mathcal{H}^n$ defined in (4).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}_{l/r}^n$. Let $\{q(q+1) : q \in \mathbb{N}\} = \sigma(-\Delta_{S^2})$ and $Z_q = \mathbb{I}_{\{q(q+1)\}}(-\Delta_{S^2}) \mathcal{H}$. Let

$$W_q := (Z_q \otimes L^2(\mathbb{R})) \oplus (Z_q \otimes L^2(\mathbb{R})), \quad \mathcal{E}_{-/+}^{q,n} := \mathcal{E}_{-/+}^n \cap W_q,$$

$$\mathcal{E}_{-/+}^{\text{fin},n} := \left\{ u \in \mathcal{E}_{-/+}^n : \exists Q > 0, \ u \in \bigoplus_{q \leq Q} \mathcal{E}_{-/+}^{q,n} \right\}.$$
Theorem

There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z} \setminus \{0\}$ the following holds:

- i) For all $u \in E_{\text{fin}, n}$ the limits
  
  $$W_{-/+} u = \lim_{t \to \infty} e^{itH^n_{-/+}} i_{-/+} e^{-itH^n_{-/+}} u$$

  exist in $\dot{E}^n$. The operators $W_{-/+}$ extend to bounded operators
  $W_{-/+} \in \mathcal{B}(\dot{E}^n_{-/+}; \dot{E}^n)$.

- ii) The inverse wave operators
  
  $$\Omega_{-/+} = s-lim_{t \to \infty} e^{itH^n_{-/+}} i_{-/+} e^{-itH^n}$$

  exist in $\mathcal{B}(\dot{E}^n; \dot{E}^n_{-/+})$.

i), ii) also hold for $n = 0$ if $m > 0$.

Remark

1. We can also compare to comparison dynamics given by a product of transport equations along principal null geodesics. The appropriate energy space is the energy space of this comparison dynamics.

2. Results uniform in $n$ recently obtained by Dafermos, Rodnianski, Shlapentokh-Rothman for the wave equation on Kerr.
3.4 Basic resolvent estimates and existence of the dynamics

Lemma (Basic resolvent estimates)

Let $\epsilon > 0$. We have

$$\|p^{-1}(z)u\| \lesssim |z|^{-1} |\text{Im} z|^{-1} \|u\|,$$

$$\|h_{1/2}^{1/2} p^{-1}(z)u\| \lesssim |\text{Im} z|^{-1} \|u\|.$$ 

uniformly in $|z| \geq (1 + \epsilon) \|k\|_{\mathcal{B}(\mathcal{H})}$, $|\text{Im} z| > 0$.

Remark

i) Interpretation : superradiance does not occur for $|z| \geq (1 + \epsilon) \|k\|$.

ii) Explanation : $p(z) = h_0 - (k - z)^2$, $h_0 \geq 0$.

Lemma (Existence of the dynamics)

$(\dot{H}, D(\dot{H}))$ is the generator of a $C_0$- group $e^{-it\dot{H}}$ on $\dot{\mathcal{E}}$. 
3.5 Klein-Gordon operators with “two ends”

$\mathcal{M} = \mathbb{R} \times S^2_\omega$, $h$ second order differential operator, $k$ bounded multiplication operator. We suppose

\[
\begin{align*}
&\{ \ w = w(x), \ w \in C^\infty(\mathbb{R}), \\
&\quad \text{w}_{i_+} k_{i_+} w, \ \text{w}_{i_-} (k - \ell) i_- w \in \mathcal{B}(\mathcal{H}).
\end{align*}
\]

\[
\left\{ \begin{array}{l}
  k_\pm = k \mp \ell j_\pm^2, \\
  h_\pm = h_0 - k_\pm^2 \\
  \tilde{h}_- = h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2.
\end{array} \right.
\]

(TE) For $\epsilon > 0$ $(h_+, k_+), (\tilde{h}_-, k_- - \ell)$ satisfy

\[
h_+ \geq 0, \ \tilde{h}_- \geq 0, \ \text{w}^{-\epsilon} (h_+ - z^2)^{-1} \text{w}^{-\epsilon}, \ \text{w}^{-\epsilon} (\tilde{h}_- - z^2)^{-1} \text{w}^{-\epsilon}
\]

extend meromorphically to $\text{Im} z > -\delta_\epsilon$.

Remark

*In the De Sitter Kerr case the meromorphic extension follows from a result of Mazzeo-Melrose.*
Construction of the resolvent

\[ \dot{\mathcal{E}}_+ = h_+^{-1/2} \mathcal{H} \oplus \mathcal{H}, \quad \dot{\mathcal{E}}_- = \Phi(\ell) \tilde{h}_-^{-1/2} \mathcal{H} \oplus \mathcal{H}. \]

\[ \dot{\mathcal{H}}_\pm = \begin{pmatrix} 0 & 1 \\ h_\pm & 2k_\pm \end{pmatrix}. \]

are selfadjoint. We note \( \dot{R}_\pm(z) := (\dot{\mathcal{H}}_\pm - z)^{-1} \).

**Proposition**

Let \( \epsilon > 0 \). Then \( w^\epsilon \dot{R}_\pm(z) w^{-\epsilon} \) extends finite meromorphically to \( \text{Im} z > -\delta_{\epsilon/2} \) as an operator valued function with values in \( \mathcal{B}(\dot{\mathcal{E}}_\pm) \).

**Proposition**

There exists a finite set \( Z \subset \mathbb{C} \setminus \mathbb{R} \) with \( \overline{Z} = Z \) such that the spectrum of \( \dot{H} \) is included in \( \mathbb{R} \cup Z \) and such that the resolvent \( \dot{R}(z) \) is a finite meromorphic function on \( \mathbb{C} \setminus \mathbb{R} \). Moreover the set \( Z \) consists of eigenvalues of finite multiplicity of \( \dot{H} \).

Idea of the proof.

\[ Q(z) := i_- (\dot{\mathcal{H}}_- - z)^{-1} i_- + i_+ (\dot{\mathcal{H}}_+ - z)^{-1} i_. \]

Then computation of \((H - z)Q(z)\) + meromorphic Fredholm theory.
Smooth functional calculus

\[ \| f \|_m := \sup_{\lambda \in \mathbb{R}, \alpha \leq m} |f^{(\alpha)}(\lambda)|. \]

**Proposition**

(i) Let \( f \in C_0^\infty(\mathbb{R}) \). Let \( \tilde{f} \) be an almost analytic extension of \( f \) such that \( \text{supp}\tilde{f} \cap \sigma_{pp}(\dot{H}) = \emptyset \). Then the integral

\[ f(\dot{H}) := \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\partial\tilde{f}}{\partial \bar{z}}(z) \dot{R}(z) dz \wedge d\bar{z} \]

is norm convergent in \( B(\hat{\mathcal{E}}) \) and independent of the choice of the almost analytic extension of \( f \).

(ii) The map \( C_0^\infty(\mathbb{R}) \ni f \mapsto f(\dot{H}) \in B(\hat{\mathcal{E}}) \) is a homomorphism of algebras with

\[ f(\dot{H})^* = \overline{\tilde{f}(\dot{H}^*)}, \quad \| f(\dot{H}) \|_{B(\hat{\mathcal{E}})} \leq \| f \|_m \quad \text{for some} \quad m \in \mathbb{N}. \]

**Proposition**

Let \( \chi \in C_0^\infty(\mathbb{R}) \), \( \chi \equiv 1 \) in a neighborhood of zero. Then

\[ s - \lim_{L \to \infty} \chi \left( \frac{\dot{H}}{L} \right) = \mathbb{I} - \mathbb{I}^C_{pp}(\dot{H}). \]
3.6 Resonances and Propagation estimates

Lemma

\( w^{-\epsilon} \dot{R}(z)w^{-\epsilon} \) can be extended meromorphically from the upper half plane to \( \text{Im} z > -\delta_\epsilon, \delta_\epsilon > 0 \) with values in \( \mathcal{B}_\infty(\mathcal{E}) \). poles: resonances.

Proposition

Let \( \epsilon > 0 \). There exists a discrete closed set \( \mathcal{T}_H \subset \mathbb{R}, \nu > 0 \) such that for all \( \chi \in C^\infty_0(\mathbb{R} \setminus \mathcal{T}_H) \) we have

\[
\sup_{\|u\|_{\mathcal{E}}=1, \nu \geq \delta > 0} \int_{\mathbb{R}} (\|w^{-\epsilon} \dot{R}(\lambda+i\delta)\chi(\mathcal{H})u\|_{\mathcal{E}}^2 + \|w^{-\epsilon} \dot{R}(\lambda-i\delta)\chi(\mathcal{H})u\|_{\mathcal{E}}^2) d\lambda < \infty.
\]

Definition

We call \( \lambda \in \mathbb{R} \) a regular point of \( \mathcal{H} \) if there exists \( \chi \in C^\infty_0(\mathbb{R}), \chi(\lambda) = 1 \) such that (7) holds. Otherwise we call it a singular point.

Remark

Note that in the selfadjoint case \( \mathcal{T}_H \) is the set of real resonances by Kato’s theory of \( H \)-smoothness.
Propagation estimates

Proposition

Let $\epsilon > 0$. Then there exists a discrete closed set $\hat{T} \subset \mathbb{R}$ such that for all $\chi \in C_0^\infty (\mathbb{R} \setminus \hat{T})$ and all $k \in \mathbb{N}$ we have

$$\| w^{-\epsilon} e^{-it\hat{H}} \chi(\hat{H}) w^{-\epsilon} \|_{B(\hat{\epsilon})} \lesssim \langle t \rangle^{-k}.$$ 

Proposition

Let $\epsilon > 0$. Then we have for all $\chi \in C_0^\infty (\mathbb{R} \setminus \hat{T}_H)$:

$$\int \| w^{-\epsilon} e^{-it\hat{H}} \chi(\hat{H}) \varphi \|_{\hat{\epsilon}}^2 dt \lesssim \| \varphi \|_{\hat{\epsilon}}^2.$$

Theorem

Suppose that $\lambda_0 \in \mathbb{R}$ is neither a resonance of $w^{-\epsilon} \hat{R}(\lambda) w^{-\epsilon}$ nor of $w^{-\epsilon} Q(\lambda) w^{-\epsilon}$. Then $\lambda_0$ is a regular point of $\hat{H}$.

Proof.

$$w^{-\epsilon} \hat{R}(z) = w^{-\epsilon} Q(z) - w^{-\epsilon} \hat{R}(z) w^{-\epsilon} w^\epsilon \xi K(z).$$

$Q(z), K(z)$ constructed using only resolvents of selfadjoint operators, $\xi \in C_0^\infty$. 

3.7 Uniform boundedness of the evolution

For \( \chi \in C^\infty(\mathbb{R}) \) and \( \mu > 0 \) we put \( \chi_\mu(.) = \chi\left(\frac{.}{\mu}\right) \).

**Theorem**

i) Let \( \chi \in C^\infty(\mathbb{R}) \), \( \text{supp} \chi \subset \mathbb{R} \setminus [-1,1] \), \( \chi \equiv 1 \) on \( \mathbb{R} \setminus (-2,2) \). Then there exists \( \mu_0 > 0, C_1 > 0 \) such that we have for \( \mu \geq \mu_0 \)

\[
\| e^{-it\hat{H}} \chi_\mu(\hat{H}) u \|_{\dot{E}} \leq C_1 \| \chi_\mu(\hat{H}) u \|_{\dot{E}} \quad \forall u \in \dot{E}, \forall t \in \mathbb{R}.
\]

ii) Let \( \varphi \in C_0^\infty(\mathbb{R} \setminus \hat{T}_H) \). Then there exists \( C_2 > 0 \) such that for all \( u \in \dot{E} \) and \( t \in \mathbb{R} \) we have

\[
\| e^{-it\hat{H}} \varphi(\hat{H}) u \|_{\dot{E}} \leq C_2 \| \varphi(\hat{H}) u \|_{\dot{E}}.
\]

**Remark**

The general abstract framework, the work of Dyatlov and an hypoellipticity argument gives the uniform boundedness of the evolution and then the asymptotic completeness result.
Part 4: Convergence rate for the Hawking radiation in the De Sitter Schwarzschild case

4.1 Local energy decay for the wave equation on the De Sitter Schwarzschild spacetime (a=0)

Distribution of resonances (Sa Barreto-Zworski ’97):

Modified energy space:

\[ \| (u_0, u_1) \|_{E_{\text{mod}}}^2 = \| u_1 \|^2 + \langle Pu_0, u_0 \rangle + \left( \int_0^1 \int_{S^2} |u_0(s, \omega)|^2 ds d\omega \right). \]

Theorem (Bony-Ha ’08)

Let \( \chi \in C_0^\infty(M) \). There exists \( \varepsilon > 0 \) such that \( \chi e^{-itH} \chi u = \gamma \left( \begin{array}{c} r \chi \langle r, \chi u_2 \rangle \\ 0 \end{array} \right) + R_2(t)u, \quad \| R_2(t)u \|_{E_{\text{mod}}} \lesssim e^{-\varepsilon t} \| -\Delta \omega u \|_{E_{\text{mod}}}. \)

Remark

1. No resonance 0 for Klein Gordon equation with positive masse of the field \( m > 0 \).
2. Similar picture in much more general situations, see Vasy ’13.
Consequence for asymptotic completeness

**Theorem (Alexis Drouot ’15)**

*Consider u solution in $\mathcal{M}$ of $(m > 0)$

$$(\Box + m^2)u = 0, \ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1$$

with $u_0, u_1$ in $C^1$. There exists $C^1$ functions (called radiation fields of u) $u^*_\pm : \mathcal{M} \to \mathbb{R}$ and $C \in \mathbb{R}$ (depending only on $\text{supp}(u_0; u_1)$) such that

$$u^*_\pm(x, \omega) = 0 \text{ for } x \leq C; \quad u^*_\pm = O_C(\mathcal{O}_C(e^{-\nu_0(x)})),$$

and

$$u(t, x, \omega) = u^*_+(-(t + x), \omega) + u^*_-(t + x, \omega) + O_C(e^{ct}), \ c > 0.$$  

Proof uses results of Bony-H. ’08 and Melrose-Sa-Barreto-Vasy ’14.
Convergence rate for the Hawking effect

**Theorem (Alexis Drouot ’15)**

There exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$ the following is true. Let

$$E_T(u_0, u_1) = E_{H_0, T_0}(u(0), \partial_t u(0)),$$

where $u$ solves for $m > 0$

$$\begin{cases} (\Box_g + m^2) &= 0, \\ u|_{\partial B} &= 0, \\ u(T) &= u_0, \\ \partial_t u(T) &= u_1 \end{cases}$$

Then

$$E_T(u_0, u_1) = E_{+}^{D_x, T_0}(u^*_+, D_x u^*_+) + E_{-}^{D_x, T_{\text{Haw}}}(u^*_-, D_x u^*_-) + O(e^{-cT}), \quad T \to \infty.$$ 

for some $c > 0$. 
Scattering theory

- The fact that the mixed term has two different limits makes it more complicated than for the Klein-Gordon equation coupled to an electric field. Mourre theory on Krein spaces: Georgescu-Gérard-H. ’14.
- Time dependent scattering should depend only on the behavior of the resolvent on the real axis.

Hawking effect

- Proof of a theorem about the Hawking effect for bosons should now work in the same way. Temperature depends on $n$.
- Highly idealized model.
Thank you for your attention!