Scattering theory for Dirac and Klein-Gordon fields on the (De Sitter) Kerr metric and the Hawking effect

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Field equations on lorentzian space-times,
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Scattering theory on black hole type spacetimes and related subjects

- 1975 : Hawking effect. Contributions by many others including Gibbons, Unruh, Wald,...
- 1980's program by Dimock, Kay on scattering theory on the Schwarzschild metric. Related work by Fredenhagen, Haag. "Conformal scattering" by Friedlander.
- 1990's Further developed by Alain Bachelot giving a mathematically rigorous description of the Hawking effect in the spherically symmetric setting in 1999. Further contributions by Nicolas, Melnyk, Daudé,...
- 2000's Kerr
  - Scattering theory on Kerr H, H-Nicolas, rigorous description of the Hawking effect for fermions (H '09).
  - Decay of the local energy for field equations. Andersson-Blue, Dafermos-Rodnianski, Shlapentokh-Rothman, Dyatlov, Finster-Kamran-Smoller-Yau, Tataru-Tohaneanu, Vasy,...
- 2010's Nonlinear stability of the De Sitter Kerr metric : Hintz, Vasy (2016). Nonlinear stability of Schwarzschild for axial symmetric polarized perturbations : Klainerman, Szeftel (2017).

  Scattering theory for Klein-Gordon equations without positive conserved energy (Kako, Gérard, Bachelot, Georgescu-Gérard-H.), on (De Sitter) Kerr (Georgescu-Gérard-H., Dafermos-Rodnianski-Shlapentokh-Rothman), scattering theory via vector field methods (Mason, Nicolas, Joudioux, Dafermos-Rodnianski-Shlapentokh-Rothman).

De Sitter Kerr metric in Boyer-Lindquist coordinates

$\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, with spacetime metric

$$\begin{aligned} g &= \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\lambda^2\rho^2} dt^2 + \frac{2a \sin^2\theta((r^2+a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta_r)}{\lambda^2\rho^2} dt d\varphi \\ &- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2\theta\sigma^2}{\lambda^2\rho^2} d\varphi^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right)(r^2+a^2) - 2Mr, \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta, \ \sigma^2 = (r^2+a^2)^2\Delta_\theta - a^2\Delta_r \sin^2\theta, \ \lambda = 1 + \frac{1}{3}\Lambda a^2. \end{aligned}$$

$\Lambda \geq 0$: cosmological constant ($\Lambda = 0$: Kerr), $M > 0$: masse, a : angular momentum per unit masse ($|a| < M$).

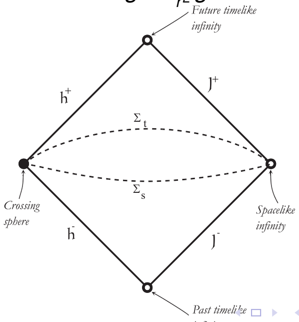
- $\rho^2 = 0$ is a curvature singularity, $\Delta_r = 0$ are coordinate singularities. $\Delta_r > 0$ on some open interval $r_- < r < r_+$. $r = r_-$: black hole horizon, $r = r_+$ cosmological horizon.
- ∂_φ and ∂_t are Killing. There exist $r_1(\theta)$, $r_2(\theta)$ s. t. ∂_t is
  - timelike on $\{(t, r, \theta, \varphi) : \ r_1(\theta) < r < r_2(\theta)\}$,
  - spacelike on $\{(t, r, \theta, \varphi) : \ r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi : \ r_2(\theta) < r < r_+\} =: \mathcal{E}_- \cup \mathcal{E}_+$. The regions $\mathcal{E}_-$, $\mathcal{E}_+$ are called ergospheres.

The Penrose diagram ($\Lambda = 0$)

- Kerr-star coordinates :

$$t^* = t + r_*, r, \theta, \varphi^* = \varphi + \Lambda(r), \ \frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}, \ \frac{d\Lambda(r)}{dr} = \frac{a}{\Delta}.$$

  .
  Along incoming principal null geodesics : $\dot{t}^* = \dot{\theta} = \dot{\varphi}^* = 0, \ \dot{r} = -1$.
- Form of the metric in Kerr-star coordinates :
  $g = g_{tt} {dt^*}^2 + 2 g_{t\varphi} dt^* d\varphi^* + g_{\varphi\varphi} {d\varphi^*}^2 + g_{\theta\theta} d\theta^2 - 2 dt^* dr + 2a \sin^2 d\varphi^* dr$.
- Future event horizon : $\mathfrak{H}^+ := \mathbb{R}_{t^*} \times \{r = r_-\} \times S^2_{\theta,\varphi^*}$.
- The construction of the past event horizon $\mathfrak{H}^-$ is based on outgoing principal null geodesics (star-Kerr coordinates). Similar constructions for future and past null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$ using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$.



Part 1 :

Scattering theory for massless Dirac fields on the Kerr metric

D.H., J.-P. Nicolas, Rev. Math. Phys. 16(1) : 29-123, 2004.

Weyl equation :

$$\nabla^{A}_{A'}\phi_A = 0.$$

Conserved current :

$$V^a = \phi^A \overline{\phi}^{A'},\ C(t) = \frac{1}{\sqrt{2}} \int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} = const.$$

T^a : normal to Σ_t.

- Newman-Penrose tetrad $l^a, n^a, m^a, \overline{m}^a$:
  $l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0.$
  - Normalization $l_a n^a = 1$, $m_a \overline{m}^a = -1$
  - l^a, n^a : Scattering directions.
- Spin frame $o^A \overline{o}^{A'} = l^a$, $\iota^A \overline{\iota}^{A'} = n^a$, $o^A \overline{\iota}^{A'} = m^a$
  $\iota^A \overline{o}^{A'} = \overline{m}^a$, $o_A \iota^A = 1$
- Components in the spin frame : $\phi_0 = \phi_A o^A, \phi_1 = \phi_A \iota^A$
- Weyl equation :

$$\begin{cases} n^a \partial_a \phi_0 - m^a \partial_a \phi_1 + (\mu - \gamma)\phi_0 + (\tau - \beta)\phi_1 = 0, \\ l^a \partial_a \phi_1 - \overline{m}^a \partial_a \phi_0 + (\alpha - \pi)\phi_0 + (\epsilon - \tilde{\rho})\phi_1 = 0. \end{cases}$$

A new Newman Penrose tetrad

Problem : The Kerr metric is at infinity a long range perturbation of the Minkowski metric. In the long range situation asymptotic completeness is generically false without modification of the wave operators.

Dirac equation on Schwarzschild :

$$i\partial_t \Psi = \not{D}_S \Psi, \not{D}_S = \Gamma^1 D_{r_*} + \frac{(1 - \frac{2M}{r})^{1/2}}{r} \not{D}_{S^2} + V.$$

ok because of spherical symmetry.

Tetrad adapted to the foliation : $l^a + n^a = T^a$. Conserved quantity :

$$\frac{1}{\sqrt{2}} \int_{\Sigma_t} (|\phi_0|^2 + |\phi_1|^2) d\sigma_{\Sigma_t}.$$

$l^a, n^a \in \text{span}\{T^a, \partial_r\}$. Ψ spinor multiplied by a certain weight :

$$i\partial_t \Psi = \not{D}_K \Psi, \quad \not{D}_K = h \not{D}_{sym} h + V_\varphi D_\varphi + V.$$

Well adapted to time dependent scattering : $h^2 - 1,\ V_\varphi,\ V$ short range.

1.2 Principal results

Comparison dynamics

$$\mathcal{H} = L^2((\mathbb{R}\times S^2); dr_* d\omega); \mathbb{C}^2), \ \mathbb{D}_H = \gamma D_{r_*} - \frac{a}{r_+^2 + a^2} D_\varphi, \mathbb{D}_\infty = \gamma D_{r_*},$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathcal{H}^- = \{(\psi_0, 0) \in \mathcal{H}\} \text{ (resp. } \mathcal{H}^+ = \{(0,\psi_1) \in \mathcal{H}\}).$$

Theorem (Asymptotic velocity)

There exist bounded selfadjoint operators s.t. for all $J \in \mathcal{C}_\infty(\mathbb{R})$ *:*

$$\begin{aligned} J(P^\pm) &= s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} J\left(\frac{r_*}{t}\right) e^{it\not{D}_K}, \\ J(\mp\gamma) &= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_H} \\ &= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{r_*}{t}\right) e^{it\mathbb{D}_\infty}. \end{aligned}$$

In addition we have :

$$\sigma(P^+) = \{-1, 1\}.$$

Theorem (Asymptotic completeness)

The classical wave operators defined by the limits

$$
\begin{aligned}
W_H^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_H} P_{\mathcal{H}^{\mp}} \,, \\
W_\infty^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_\infty} P_{\mathcal{H}^{\pm}} \,, \\
\Omega_H^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\mathbb{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm}) \,, \\
\Omega_\infty^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm})
\end{aligned}
$$

exist.

Remark

1. *Proof based on Mourre theory.*
2. *The same theorem holds with more geometric comparison dynamics.*
3. *Generalized by Daudé to the massive charged case.*
4. *Results valid for quite general perturbations of Kerr.*
5. *Schwarzschild : Nicolas (95), Melnyk (02), Daudé (04).*

1.3 Geometric interpretation

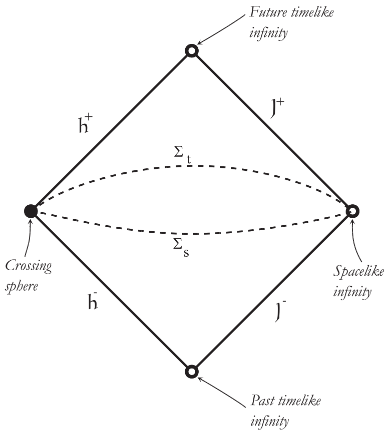


FIGURE – Penrose compactification of block *I*

- $\mathcal{J}^{\pm}$ are constructed using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$.
- The Weyl equation is conformally invariant : $\hat{\nabla}^{AA'} \hat{\phi}_A = 0$, where $\hat{\phi}_A = r\phi_A$.

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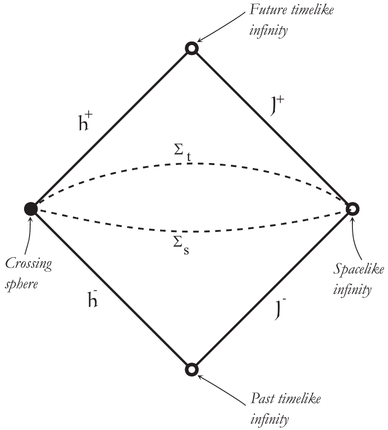

FIGURE – Penrose compactification of block I

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- The Weyl equation is conformally invariant :
  $\hat{\nabla}^{AA'} \hat{\phi}_A = 0$, where $\hat{\phi}_A = r\phi_A$.

- $\lim_{r\to r_+} \Psi_0(\gamma^-_{V,\theta,\varphi^\sharp}(r)) =: \Psi_0|_{\mathfrak{H}^+}(0, V, \theta, \varphi^\sharp)$,
  $\lim_{r\to r_+} \Psi_1(\gamma^-_{V,\theta,\varphi^\sharp}(r)) = 0$.
  Ψ is solution of the Dirac equation. $\gamma^-_{V,\theta,\varphi^\sharp}$ is the principal incoming null geodesic meeting $\mathfrak{H}^+$ at $(0, V, \theta, \varphi^\sharp)$.
- Trace operators :

$$\mathcal{T}^+_{\mathfrak{H}} \quad : \quad \begin{array}{ccc} C_0^\infty(\Sigma_0, \mathbb{C}^2) & \to & C^\infty(\mathfrak{H}^+, \mathbb{C}) \\ \Psi_{\Sigma_0} & \mapsto & \Psi_0|_{\mathfrak{H}^+}. \end{array}$$

- $\mathcal{H}$: Hilbert space associated to Σ_0, $\mathcal{H}_{\mathfrak{H}^\pm}$ Hilbert spaces associated to $\mathfrak{H}^\pm$.

Theorem

The trace operators $\mathcal{T}^\pm_{\mathfrak{H}}$ extend in a unique manner to bounded operators from $\mathcal{H}$ to $\mathcal{H}_{\mathfrak{H}^\pm}$.

Remark

Let $\mathfrak{F}^\pm_{\mathfrak{H}}$ be the $\mathcal{C}^\infty$ diffeomorphisms from $\mathfrak{H}^\pm$ onto Σ_0 defined by identifying points along incoming (resp. outgoing) principal null geodesics and $\Omega^\pm_{H,pn}$ inverse wave operators with comparison dynamics given by the principal null directions. Then $\mathcal{T}^\pm_{\mathfrak{H}} = (\mathfrak{F}^\pm_{\mathfrak{H}})^*\Omega^\pm_{H,pn}$. Comparison dynamics $P_N = \gamma D_{r_*} - \frac{a^2}{r^2+a^2}D_\varphi$.

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Same construction for $\mathcal{T}_{\mathscr{I}}^{\pm}$ and $\mathcal{H}_{\mathscr{I}^{\pm}}$. $\mathcal{T}_{\mathscr{I}}^{\pm}$ can be extended to bounded operators from $\mathcal{H}$ to $\mathcal{H}_{\mathscr{I}^{\pm}}$.

$$\Pi_F : \begin{array}{ccc} \mathcal{H} & \rightarrow & \mathcal{H}_{\mathfrak{H}^+} \oplus \mathcal{H}_{\mathscr{I}^+} =: \mathcal{H}_F \\ \Psi_{\Sigma_0} & \mapsto & (\mathcal{T}_{\mathfrak{H}}^{+}\Psi_{\Sigma_0}, \mathcal{T}_{\mathscr{I}}^{+}\Psi_{\Sigma_0}). \end{array}$$

Theorem (Goursat problem)

Π_F is an isometry. In particular for all $\Phi \in \mathcal{H}_F$, there exists a unique solution of the Dirac equation $\Psi \in C(\mathbb{R}_t, \mathcal{H})$ s.t. $\Phi = \Pi_F \Psi(0)$.

Remark

1) First constructions of this type : Friedlander (Minkowski, 80, 01), Bachelot (Schwarzschild, 91).
2) The inverse is possible : Mason, Nicolas (04), Joudioux (10) (asymptotically simple space-times), Dafermos-Rodnianski-Shlapentokh-Rothman (Kerr).

Part 2 : The Hawking effect as a scattering problem

D. H., Creation of fermions by rotating charged black holes, Mémoires de la SMF 117 (2009), 158 pp.

2.1 The collapse of the star

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col},\ \Sigma_t^{col} = \{(t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_\omega;\ \hat{r} \geq \hat{z}(t, \theta)\}.$$

Assumptions :

- For $\hat{r} > \hat{z}(t, \theta)$, the metric is the Kerr Newman metric.
- $\hat{z}(t, \theta)$ behaves asymptotically like certain timelike geodesics in the Kerr-Newman metric. We suppose for the conserved quantities L (angular momentum), $\mathcal{Q}$ (Carter constant) and $\tilde{E}$ (rotational energy) : $L = \mathcal{Q} = \tilde{E} = 0$. We also suppose an asymptotic condition on the surface of the star :

$$\hat{z}(t, \theta) = -t - \hat{A}(\theta) e^{-2\kappa_- t} + \mathcal{O}(e^{-4\kappa_- t}),\ t \to \infty.$$

$\kappa_- > 0$ is the surface gravity of the outer horizon, $\hat{A}(\theta) > 0$.

Remark

1. $\hat{r}$ is a coordinate adapted to simple null geodesics.
2. Dirac in $\mathcal{M}_{col}$: we add a boundary condition (MIT)
$\rightarrow\ \Psi(t) = U(t, 0)\Psi_0$.

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2.2 Dirac quantum fields

Dimock '82.

$$\mathcal{M}_{col} = \bigcup_{t\in\mathbb{R}} \Sigma_t^{col}, \quad \Sigma_t^{col} = \{(t, \hat{r}, \theta, \varphi); \hat{r} \geq \hat{z}(t, \theta)\}.$$

Dirac quantum field Ψ_0 and the CAR-algebra $\mathcal{U}(\mathcal{H}_0)$ constructed in the usual way. Fermi-Fock representation.

$$S_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & \mathcal{H}_0 \\ \Phi & \mapsto & S_{col}\Phi := \int_{\mathbb{R}} U(0,t)\Phi(t)dt \end{array}$$

Quantum spin field :

$$\Psi_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & \mathcal{L}(\mathcal{F}(\mathcal{H}_0)) \\ \Phi & \mapsto & \Psi_{col}(\Phi) := \Psi_0(S_{col}\Phi) \end{array}$$

$\mathcal{U}_{col}(\mathcal{O})$ = algebra generated by $\Psi_{col}^*(\Phi^1)\Psi_{col}(\Phi^2)$, $\mathrm{supp}\,\Phi^j \subset \mathcal{O}$.

$$\mathcal{U}_{col}(\mathcal{M}_{col}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O})}.$$

Same procedure on $\mathcal{M}_{BH}$:

$$S : \Phi \in (C_0^\infty(\mathcal{M}_{BH}))^4 \mapsto S\Phi := \int_{\mathbb{R}} e^{-itH}\Phi(t)dt.$$

States

- $\mathcal{U}_{col}(\mathcal{M}_{col})$
  Vacuum state :

$$\begin{aligned}\omega_{col}(\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)) &:= \omega_{vac}(\Psi^*_0(S_{col}\Phi_1)\Psi_0(S_{col}\Phi_2)) \\ &= \langle \mathbf{1}_{[0,\infty)}(H_0)S_{col}\Phi_1, S_{col}\Phi_2\rangle.\end{aligned}$$

- $\mathcal{U}_{BH}(\mathcal{M}_{BH})$
  - Vacuum state

$$\omega_{vac}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\phi_2)) = \langle \mathbf{1}_{[0,\infty)}(H)S\phi_1, S\phi_2\rangle.$$

  - Thermal Hawking state

$$\begin{aligned}\omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)) &= \langle \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}S\Phi_1, S\Phi_2\rangle_{\mathcal{H}} \\ &=: \omega^{\eta,\sigma}_{KMS}(\Psi^*(S\Phi_1)\Psi(S\Phi_2)), \\ T_{Haw} &= \sigma^{-1},\ \mu = e^{\sigma\eta},\ \sigma > 0.\end{aligned}$$

  T_{Haw} Hawking temperature, μ chemical potential.

The Hawking effect

$$\Phi \in (C_0^\infty(\mathcal{M}_{col}))^4, \; \Phi^T(t,\hat{r},\omega) = \Phi(t-T,\hat{r},\omega).$$

Theorem (Hawking effect)

Let $\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4$, $j=1,2$. *We have*

$$\begin{aligned}
&\lim_{T\to\infty} \omega_{col}(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)) \\
&= \omega_{Haw}^{\eta,\sigma}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)) \\
&+ \omega_{vac}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)), \\
T_{Haw} &= 1/\sigma = \kappa_-/2\pi, \quad \mu = e^{\sigma\eta}, \; \eta = \frac{qQr_-}{r_-^2+a^2} + \frac{aD_\varphi}{r_-^2+a^2}.
\end{aligned}$$

2.3 Explanation

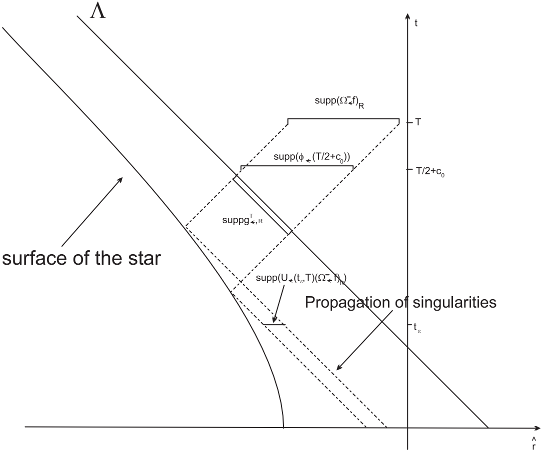
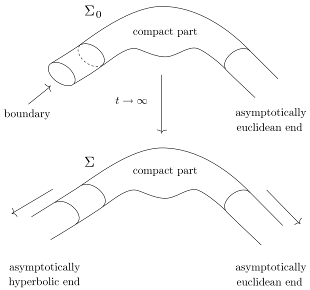


FIGURE – Collapse of the star

Change in frequencies : mixing of positive and negative frequencies.

2.4 The analytic problem



$$\begin{aligned} & \lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f||_0^2 \\ (1) \qquad & = \langle \mathbf{1}_{\mathbb{R}^+}(P^-)f, \mu e^{\sigma\not{D}}(1+\mu e^{\sigma\not{D}})^{-1}\mathbf{1}_{\mathbb{R}^+}(P^-)f\rangle \\ & + ||\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f||^2. \end{aligned}$$

Remark

1) **Hawking** *1975,*
2) **Bachelot** *(99), Melnyk (04).*
3) Schwarzschild : Moving mirror, equation with potential.

2.5 Toy model : The moving mirror

$$z(t) = -t - Ae^{-2\kappa t};\ A > 0,\ \kappa > 0,$$

$$\left\{ \begin{array}{rcl} \partial_t \psi & = & i\not{D}\psi, \\ \psi_1(t, z(t)) & = & \sqrt{\frac{1-\dot{z}}{1+\dot{z}}}\psi_2(t, z(t)) \\ \psi(t = s, .) & = & \psi_s(.) \end{array} \right. ,\ \not{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_x.$$

Solution given by a unitary propagator $U(t, s)$. Conserved L^2 norm :

$$||\psi||^2_{\mathcal{H}_t} = \int_{z(t)}^{\infty} |\psi|^2(t, x)dx.$$

Explicit calculation :

$$\begin{aligned} \lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0, T)f||_0^2 &= \langle e^{\frac{2\pi}{\kappa}\not{D}} \left(1 + e^{\frac{2\pi}{\kappa}\not{D}}\right)^{-1} P_2 f, P_2 f\rangle \\ &+ ||\mathbf{1}_{[0,\infty)}(\not{D})P_1 f||^2. \end{aligned}$$

Scattering problem : show that the real system behaves the same way.

2.6 Some remarks on the proof

- We compare to a dynamics for which the radiation can be explicitly computed.
- Can't compare dynamics on Cauchy surfaces $\rightarrow$ characteristic Cauchy problem.
- Three time intervals :
  - $[T/2 + c_0, T]$ no boundary involved $\rightarrow$ use asymptotic completeness+propagation estimates.
  - $[t_\epsilon, T/2 + c_0]$ use Duhamel formula + construction of tetrad and coordinates :
    - There exists a coordinate system $(t, \hat{r}, \omega)$ such that $\hat{r} = -t + c$ along incoming simple null geodesics ($L = Q = 0$).
    - There exists a Newman Penrose tetrad such that : $\not{D} = \Gamma D_{\hat{r}} + P_\omega + W$, $\Gamma = Diag(1, -1, -1, 1)$, P_ω is a differential operator with derivatives only in the angular directions and W is a potential.
  - $[0, t_\epsilon]$: $\|\mathbf{1}_{[0,\infty]}(\not{D}_0)U(0, t_\epsilon)U_H(t_\epsilon, T)\Omega_H^- f\| \sim \|\mathbf{1}_{[0,\infty)}(\mathbb{D}_{H,0})U_H(0, T)\Omega_H^- f\|$ if evolution is essentially given by the group (and not the evolution system). For this
    - $U_H(t_\epsilon, T)\Omega_H^- f \rightharpoonup 0$.
    - The hamiltonian flow stays outside the surface of the star for data in the given regime ($|\xi| >> |\Theta|$).
    - Propagation of singularities, compact Sobolev embeddings.

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Part 3 : Scattering theory for the Klein-Gordon equation on the De Sitter Kerr metric

V. Georgescu, C. Gérard, D. H., Asymptotic completeness for superradiant Klein-Gordon equations and applications to the De Sitter Kerr metric , J. Eur. Math. Soc. 19, 2171-2244.

3.1 The Klein-Gordon equation on the De Sitter Kerr metric

De Sitter Kerr metric in Boyer-Lindquist coordinates

$\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, with spacetime metric

$$\begin{aligned} g &= \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\lambda^2\rho^2} dt^2 + \frac{2a\sin^2\theta((r^2+a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta_r)}{\lambda^2\rho^2} dt d\varphi \\ &- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2\theta\sigma^2}{\lambda^2\rho^2} d\varphi^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right)(r^2+a^2) - 2Mr, \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta, \; \sigma^2 = (r^2+a^2)^2\Delta_\theta - a^2\Delta_r \sin^2\theta, \; \lambda = 1 + \frac{1}{3}\Lambda a^2. \end{aligned}$$

$\Lambda > 0$: cosmological constant, $M > 0$: masse, a : angular momentum per unit masse.

- $\rho^2 = 0$ is a curvature singularity, $\Delta_r = 0$ are coordinate singularities. $\Delta_r > 0$ on some open interval $r_- < r < r_+$. $r = r_-$: black hole horizon, $r = r_+$ cosmological horizon.
- ∂_φ and ∂_t are Killing. There exist $r_1(\theta)$, $r_2(\theta)$ s. t. ∂_t is
  - timelike on $\{(t, r, \theta, \varphi) : \; r_1(\theta) < r < r_2(\theta)\}$,
  - spacelike on $\{(t, r, \theta, \varphi) : \; r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi : \; r_2(\theta) < r < r_+\} =: \mathcal{E}_- \cup \mathcal{E}_+$.

  The regions $\mathcal{E}_-$, $\mathcal{E}_+$ are called ergospheres.

3.1 The Klein-Gordon equation on the De Sitter Kerr metric

We now consider the unitary transform

$$U: \begin{array}{ccc} L^2(\mathcal{M}; \frac{\sigma^2}{\Delta_r\Delta_\theta} dr d\omega) & \to & L^2(\mathcal{M}; dr d\omega) \\ \psi & \mapsto & \frac{\sigma}{\sqrt{\Delta_r\Delta_\theta}}\psi \end{array}$$

If ψ fulfills $(\Box_g + m^2)\psi = 0$, then $u = U\psi$ fulfills

(2) $$(\partial_t^2 - 2ik\partial_t + h)u = 0.$$

with

$$\begin{aligned} k &= \frac{a(\Delta_r - (r^2+a^2)\Delta_\theta)}{\sigma^2} D_\varphi, \\ h &= -\frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2. \end{aligned}$$

h is not positive inside the ergospheres. This entails that the natural conserved quantity

$$\tilde{\mathcal{E}}(u) = \|\partial_t u\|^2 + (hu|u)$$

is not positive.

3+1 decomposition, energies, Killing fields

Let $v = e^{-ikt}u$. Then u is solution of (2) if and only if v is solution of

$$(\partial_t^2 + h(t))v = 0, \quad h(t) = e^{-ikt}h_0e^{ikt}, \quad h_0 = h + k^2 \geq 0.$$

Natural energy :

$$\|\partial_t v\|^2 + (h(t)v|v).$$

Rewriting for u :

$$\dot{\mathcal{E}}(u) = \|(\partial_t - ik)u\|^2 + (h_0u|u).$$

This energy is positive, but may grow in time $\rightarrow$ superradiance.

Remark

- $\partial_t - ik = \frac{\nabla t}{(\nabla_b t \nabla^b t)}$.
- *$k = \Omega D_\varphi$ and Ω has finite limits $\Omega_{-/+}$ when $r \rightarrow r_\mp$. These limits are called angular velocities of the horizons. The Killing fields $\partial_t - \Omega_{-/+}\partial_\varphi$ on the De Sitter Kerr metric are timelike close to the black hole (-) resp. cosmological (+) horizon.*

3.2 The abstract equation

$\mathcal{H}$ Hilbert space. h, k selfadjoint, $k \in \mathcal{B}(\mathcal{H})$.

$$(3) \qquad \begin{cases} (\partial_t^2 - 2ik\partial_t + h)u &= 0, \\ u|_{t=0} &= u_0, \\ \partial_t u|_{t=0} &= u_1. \end{cases}$$

Hyperbolic equation

$$(\mathrm{A1}) \qquad h_0 := h + k^2 \geq 0.$$

Formally $u = e^{izt}v$ solution if and only if

$$p(z)v = 0$$

with $p(z) = h_0 - (k - z)^2 = h + z(2k - z)$, $z \in \mathbb{C}$. $p(z)$ is called the quadratic pencil.

Conserved quantities

$$\langle u|u\rangle_\ell := \|u_1 - \ell u_0\|^2 + (p(\ell)u_0|u_0),$$

where $p(\ell) = h_0 - (k - \ell)^2$. Conserved by the evolution, but in general not positive definite, because none of the operators $p(\ell)$ is in general positive.

Spaces and operators

$\mathcal{H}^i$: scale of Sobolev spaces associated to h_0.

(A2) $$0 \notin \sigma_{pp}(h_0);\ h_0^{1/2} k h_0^{-1/2} \in \mathcal{B}(\mathcal{H}).$$

Homogeneous energy spaces

$$\dot{\mathcal{E}} = \Phi(k) h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \Phi(k) = \begin{pmatrix} 1\!\!1 & 0 \\ k & 1\!\!1 \end{pmatrix}.$$

where $\dot{\mathcal{E}}$ is equipped with the norm $\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}} = \|u_1 - ku_0\|^2 + (h_0 u_0 | u_0)$.

Klein Gordon operator

$$\begin{aligned} \psi &= (u, \frac{1}{i}\partial_t u), \quad (\partial_t - iH)\psi = 0, \quad H = \begin{pmatrix} 0 & 1\!\!1 \\ h & 2k \end{pmatrix}, \\ (H - z)^{-1} &= p^{-1}(z) \begin{pmatrix} z - 2k & 1\!\!1 \\ h & z \end{pmatrix}. \end{aligned}$$

We note $\dot{H}$ the Klein-Gordon operator on the homogeneous energy space.

3.3 Results in the De Sitter Kerr case

Uniform boundedness of the evolution

$$\mathcal{H}^n = \{u \in L^2(\mathbb{R} \times S^2) : (D_\varphi - n)u = 0\}, \; n \in \mathbb{Z}. \tag{4}$$

We construct the homogeneous energy space $\dot{\mathcal{E}}^n$ as well as the Klein-Gordon operator $\dot{H}^n$ as in Sect. 3.2.

Theorem

There exists $a_0 > 0$ such that for $|a| < a_0$ the following holds : for all $n \in \mathbb{Z}$, there exists $C_n > 0$ such that

$$\|e^{-it\dot{H}^n} u\|_{\dot{\mathcal{E}}^n} \leq C_n \|u\|_{\dot{\mathcal{E}}^n}, \; u \in \dot{\mathcal{E}}^n, \; t \in \mathbb{R}. \tag{5}$$

Remark

1. Note that for $n = 0$ the Hamiltonian $\dot{H}^n = \dot{H}^0$ is selfadjoint, therefore the only issue is $n \neq 0$.
2. Different from uniform boundedness on Cauchy surfaces crossing the horizon.

Asymptotic dynamics

Regge-Wheeler type coordinate $\frac{dx}{dr} = \frac{r^2+a^2}{\Delta_r}$.

$x \pm t = const.$ along principal null geodesics.

Unitary transform :

$$\mathcal{V} : \begin{array}{rcl} L^2(\mathbb{R}_{(r_-,r_+)} \times S^2) & \to & L^2(\mathbb{R} \times S^2, dxd\omega), \\ v(r,\omega) & \mapsto & \sqrt{\frac{\Delta_r}{r^2+a^2}} v(r(x),\omega). \end{array}$$

Asymptotic equations :

$$(\partial_t^2 - 2\Omega_{-/+}\partial_\varphi\partial_t + h_{-/+})u_{-/+} = 0, \quad h_{-/+} = \Omega_{-/+}^2\partial_\varphi^2 - \partial_x^2. \tag{6}$$

The conserved quantities :

$$\begin{aligned} & \|(\partial_t - i\Omega_{-/+}D_\varphi)u_{-/+}\|^2 + ((h_{-/+} - \Omega_{-/+}^2\partial_\varphi^2)u_{-/+}|u_{-/+}) \\ = \; & \|(\partial_t - i\Omega_{-/+}D_\varphi)u_{-/+}\|^2 + (-\partial_x^2 u_{-/+}|u_{-/+}) \end{aligned}$$

are positive.

Asymptotic profiles

Let $\ell_{-/+} = \Omega_{-/+} n$. Also let $i_{-/+} \in C^\infty(\mathbb{R})$, $i_- = 0$ in a neighborhood of ∞, $i_+ = 0$ in a neighborhood of $-\infty$ and $i_-^2 + i_+^2 = 1$. Let

$$h^n_{-/+} = -\partial_x^2 - \ell^2_{-/+}, \; k_{-/+} = \ell_{-/+}, \quad H^n_{-/+} = \begin{pmatrix} 0 & \mathbb{1} \\ h_{-/+} & 2k_{-/+} \end{pmatrix}$$

acting on $\mathcal{H}^n$ defined in (4).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{l/r}$. Let $\{q(q+1) : q \in \mathbb{N}\} = \sigma(-\Delta_{S^2})$ and $Z_q = \mathbb{1}_{\{q(q+1)\}}(-\Delta_{S^2})\mathcal{H}$. Let

$$\begin{aligned} W_q &:= (Z_q \otimes L^2(\mathbb{R})) \oplus (Z_q \otimes L^2(\mathbb{R})), \; \mathcal{E}^{q,n}_{-/+} := \mathcal{E}^n_{-/+} \cap W_q, \\ \mathcal{E}^{fin,n}_{-/+} &:= \left\{ u \in \mathcal{E}^n_{-/+} : \exists Q > 0, \; u \in \oplus_{q \le Q} \mathcal{E}^{q,n}_{-/+} \right\}. \end{aligned}$$

Theorem

There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z} \setminus \{0\}$ the following holds :

- *i) For all $u \in \mathcal{E}^{fin,n}_{-/+}$ the limits*

$$W_{-/+}u = \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{-/+} e^{-it\dot{H}^n_{-/+}} u$$

exist in $\dot{\mathcal{E}}^n$. The operators $W_{-/+}$ extend to bounded operators $W_{-/+} \in \mathcal{B}(\dot{\mathcal{E}}^n_{-/+}; \dot{\mathcal{E}}^n)$.

- *ii) The inverse wave operators*

$$\Omega_{-/+} = \mathrm{s}\text{-}\lim_{t\to\infty} e^{it\dot{H}^n_{-/+}} i^2_{-/+} e^{-it\dot{H}^n}$$

exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{-/+})$.

$i)$, $ii)$ also hold for $n = 0$ if $m > 0$.

Remark

1. We can also compare to comparison dynamics given by a product of transport equations along principal null geodesics. The appropriate energy space is the energy space of this comparison dynamics.
2. Results uniform in n recently obtained by Dafermos, Rodnianski, Shlapentokh-Rothman for the wave equation on Kerr.

3.4 Basic resolvent estimates and existence of the dynamics

Lemma (Basic resolvent estimates)

Let $\epsilon > 0$. *We have*

$$\begin{aligned} \|p^{-1}(z)u\| &\lesssim |z|^{-1}|\mathrm{Im} z|^{-1}\|u\|, \\ \|h_0^{1/2}p^{-1}(z)u\| &\lesssim |\mathrm{Im} z|^{-1}\|u\|. \end{aligned}$$

uniformly in $|z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}$, $|\mathrm{Im} z| > 0$.

Remark

i) Interpretation : superradiance does not occur for $|z| \geq (1+\epsilon)\|k\|$.
ii) Explanation : $p(z) = h_0 - (k-z)^2, \quad h_0 \geq 0$.

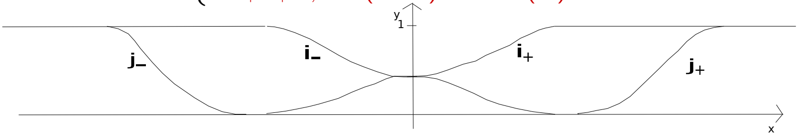
Lemma (Existence of the dynamics)

$(\dot{H}, D(\dot{H}))$ *is the generator of a* C_0- *group* $e^{-it\dot{H}}$ *on* $\dot{\mathcal{E}}$.

3.5 Klein-Gordon operators with "two ends"

$\mathcal{M} = \mathbb{R} \times S^2_\omega$, h second order differential operator, k bounded multiplication operator. We suppose

$$\begin{cases} w = w(x), \quad w \in C^\infty(\mathbb{R}), \\ wi_+ki_+w, \ wi_-(k-\ell)i_-w \in \mathcal{B}(\mathcal{H}). \end{cases}$$



$$\begin{cases} k_\pm &= k \mp \ell j_\mp^2, \\ h_\pm &= h_0 - k_\pm^2 \\ \tilde{h}_- &= h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2. \end{cases}$$

(TE) For $\epsilon > 0$ (h_+, k_+), $(\tilde{h}_-, k_- - \ell)$ satisfy

$$h_+ \geq 0, \quad \tilde{h}_- \geq 0, \quad w^{-\epsilon}(h_+ - z^2)^{-1}w^{-\epsilon}, \ w^{-\epsilon}(\tilde{h}_- - z^2)^{-1}w^{-\epsilon}$$

extend meromorphically to $\mathrm{Imz} > -\delta_\epsilon$.

Remark

In the De Sitter Kerr case the meromorphic extension follows from a result of Mazzeo-Melrose.

Construction of the resolvent

$$\dot{\mathcal{E}}_+ = h_+^{-1/2}\mathcal{H} \oplus \mathcal{H},\ \dot{\mathcal{E}}_- = \Phi(\ell)\tilde{h}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}.$$
$$\dot{H}_\pm = \begin{pmatrix} 0 & \mathbb{1} \\ h_\pm & 2k_\pm \end{pmatrix}.$$

are selfadjoint. We note $\dot{R}_\pm(z) := (\dot{H}_\pm - z)^{-1}$.

Proposition

Let $\epsilon > 0$. Then $w^{-\epsilon}\dot{R}_\pm(z)w^{-\epsilon}$ extends finite meromorphically to $\mathrm{Im} z > -\delta_{\epsilon/2}$ as an operator valued function with values in $\mathcal{B}(\dot{\mathcal{E}}_\pm)$.

Proposition

There exists a finite set $Z \subset \mathbb{C} \setminus \mathbb{R}$ with $\overline{Z} = Z$ such that the spectrum of $\dot{H}$ is included in $\mathbb{R} \cup Z$ and such that the resolvent $\dot{R}(z)$ is a finite meromorphic function on $\mathbb{C} \setminus \mathbb{R}$. Moreover the set Z consists of eigenvalues of finite multiplicity of $\dot{H}$.

Idea of the proof.

$$Q(z) := i_-(\dot{H}_- - z)^{-1}i_- + i_+(\dot{H}_+ - z)^{-1}i_+.$$

Then computation of $(H - z)Q(z)$+meromorphic Fredholm theory.

Smooth functional calculus

$$\|f\|_m := \sup_{\lambda\in\mathbb{R},\, \alpha\le m} |f^{(\alpha)}(\lambda)|.$$

Proposition

(i) Let $f \in C_0^\infty(\mathbb{R})$. Let $\tilde{f}$ be an almost analytic extension of f such that $\operatorname{supp}\tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$*. Then the integral*
$f(\dot{H}) := \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(z)\dot{R}(z)dz\wedge d\bar{z}$
is norm convergent in $\mathcal{B}(\dot{\mathcal{E}})$ and independent of the choice of the almost analytic extension of f.
(ii) The map $C_0^\infty(\mathbb{R}) \ni f \mapsto f(\dot{H}) \in \mathcal{B}(\dot{\mathcal{E}})$ is a homomorphism of algebras with
$f(\dot{H})^* = \bar{f}(\dot{H}^*), \quad \|f(\dot{H})\|_{B(\dot{\mathcal{E}})} \le \|f\|_m \quad$ *for some* $\quad m \in \mathbb{N}$.

Proposition

Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in a neighborhood of zero. Then
$s - \lim_{L\to\infty} \chi\left(\frac{\dot{H}}{L}\right) = \mathbb{1} - \mathbb{1}_{pp}^{\mathbb{C}}(\dot{H})$.

Lemma

$w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ can be extended meromorphically from the upper half plane to $\mathrm{Im} z > -\delta_\epsilon,\ \delta_\epsilon > 0$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. poles : resonances.

Proposition

Let $\epsilon > 0$. There exists a discrete closed set $\dot{\mathcal{T}}_H \subset \mathbb{R},\ \nu > 0$ such that for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$ we have

(7)

$$\sup_{\|u\|_{\dot{\mathcal{E}}}=1,\ \nu\ge\delta>0} \int_{\mathbb{R}} (\|w^{-\epsilon}\dot{R}(\lambda+i\delta)\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}}+\|w^{-\epsilon}\dot{R}(\lambda-i\delta)\chi(\dot{H})u\|^2_{\dot{\mathcal{E}}})d\lambda < \infty.$$

Definition

We call $\lambda \in \mathbb{R}$ a regular point of $\dot{H}$ if there exists $\chi \in C_0^\infty(\mathbb{R}),\ \chi(\lambda) = 1$ such that (7) holds. Otherwise we call it a singular point.

Remark

Note that in the selfadjoint case $\dot{\mathcal{T}}_H$ is the set of real resonances by Kato's theory of H-smoothness.

Propagation estimates

Proposition

Let $\epsilon > 0$. Then there exists a discrete closed set $\dot{\mathcal{T}} \subset \mathbb{R}$ such that for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}})$ and all $k \in \mathbb{N}$ we have

$$\|w^{-\epsilon} e^{-it\dot{H}} \chi(\dot{H}) w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \langle t \rangle^{-k}.$$

Proposition

Let $\epsilon > 0$. Then we have for all $\chi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$:

$$\int_{\mathbb{R}} \|w^{-\epsilon} e^{-it\dot{H}} \chi(\dot{H}) \varphi\|_{\dot{\mathcal{E}}}^2 dt \lesssim \|\varphi\|_{\dot{\mathcal{E}}}^2.$$

Theorem

Suppose that $\lambda_0 \in \mathbb{R}$ is neither a resonance of $w^{-\epsilon} \dot{R}(\lambda) w^{-\epsilon}$ nor of $w^{-\epsilon} Q(\lambda) w^{-\epsilon}$. Then λ_0 is a regular point of $\dot{H}$.

Proof.

$$w^{-\epsilon} \dot{R}(z) = w^{-\epsilon} Q(z) - w^{-\epsilon} \dot{R}(z) w^{-\epsilon} w^{\epsilon} \xi K(z).$$

$Q(z)$, $K(z)$ constructed using only resolvents of selfadjoint operators, $\xi \in C_0^\infty$.

3.7 Uniform boundedness of the evolution

For $\chi \in C^\infty(\mathbb{R})$ and $\mu > 0$ we put $\chi_\mu(.) = \chi\left(\frac{.}{\mu}\right)$.

Theorem

i) Let $\chi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi \subset \mathbb{R} \setminus [-1, 1]$, $\chi \equiv 1$ *on* $\mathbb{R} \setminus (-2, 2)$. *Then there exists* $\mu_0 > 0$, $C_1 > 0$ *such that we have for* $\mu \geq \mu_0$

$$\|e^{-it\dot{H}}\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_1\|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \quad \forall u \in \dot{\mathcal{E}}, \forall t \in \mathbb{R}.$$

ii) Let $\varphi \in C_0^\infty(\mathbb{R} \setminus \dot{\mathcal{T}}_H)$. *Then there exists* $C_2 > 0$ *such that for all* $u \in \dot{\mathcal{E}}$ *and* $t \in \mathbb{R}$ *we have*

$$\|e^{-it\dot{H}}\varphi(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_2\|\varphi(\dot{H})u\|_{\dot{\mathcal{E}}}.$$

Remark

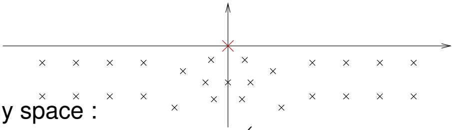
The general abstract framework, the work of Dyatlov and an hypoellipticity argument gives the uniform boundedness of the evolution and then the asymptotic completeness result.

Part 4 : Convergence rate for the Hawking radiation in the De Sitter Schwarzschild case

Alexis Drouot, A Quantitative version of Hawking's radiation, Annales Henri Poincaré 18 (2017), 757-806.

4.1 Local energy decay for the wave equation on the De Sitter Schwarzschild spacetime (a=0)

Dsitribution of resonances (Sa Barreto-Zworski '97) :



Modified energy space :

$$\|(u_0, u_1)\|^2_{\mathcal{E}^{(\mathrm{mod})}} = \|u_1\|^2 + \langle Pu_0, u_0\rangle + \left(\int_0^1 \int_{\mathbb{S}^2} |u_0(s,\omega)|^2 ds\, d\omega\right).$$

Theorem (Bony-Ha '08)

Let $\chi \in C_0^\infty(\mathcal{M})$. There exists $\varepsilon > 0$ such that $\chi e^{-itH}\chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2\rangle \\ 0 \end{pmatrix} + R_2(t)u, \quad \|R_2(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\varepsilon t}\| - \Delta_\omega u\|_{\mathcal{E}^{\mathrm{mod}}}.$

Remark

1. No resonance 0 *for Klein Gordon equation with positive masse of the field* $m > 0$.
2. Similar picture in much more general situations, see Vasy '13.

Consequence for asymptotic completeness

Theorem (Alexis Drouot '15)

Consider u solution in $\mathcal{M}$ of ($m > 0$)

$$(\Box + m^2)u = 0,\ u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1$$

with u_0, u_1 in C^1. There exists C^1 functions (called radiation fields of u) $u_\pm^* : \mathcal{M} \to \mathbb{R}$ and $C \in \mathbb{R}$ (depending only on $\operatorname{supp}(u_0; u_1)$) such that

$$u_\pm^*(x, \omega) = 0 \text{ for } x \leq C; \quad u_\pm^* = \mathcal{O}_{C^\infty}(e^{-\nu_0 \langle x \rangle}),$$

and

$$u(t, x, \omega) = u_+^*(-(t + x), \omega) + u_-^*(-t + x, \omega) + \mathcal{O}_{C^\infty(\mathcal{M}_-)}(e^{ct}),\ c > 0.$$

Proof uses results of Bony-H. '08 and Melrose-Sa-Barreto-Vasy '14.

Convergence rate for the Hawking effect

Theorem (Alexis Drouot '15)

There exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$ the following is true. Let

$$\mathbb{E}_T(u_0, u_1) = \mathbb{E}^{\mathbb{H}_0, T_0}(u(0), \partial_t u(0)),$$

where u solves for $m > 0$

$$\begin{cases} (\Box_g + m^2) & = & 0, \\ u|_{\mathcal{B}} & = & 0, \\ u(T) & = & u_0, \\ \partial_t u(T) & = & u_1 \end{cases}$$

Then

$$\mathbb{E}_T(u_0, u_1) = \mathbb{E}_+^{D_x^2, T_0}(u_+^*, D_x u_+^*) + \mathbb{E}_-^{D_x^2, T_{Haw}}(u_-^*, D_x u_-^*) + \mathcal{O}(e^{-cT}), \quad T \to \infty.$$

for some $c > 0$.

Comments

- Scattering theory
  - The fact that the mixed term has two different limits makes it more complicated than for the Klein-Gordon equation coupled to an electric field. Mourre theory on Krein spaces : Georgescu-Gérard-H. '14.
  - Time dependent scattering should depend only on the behavior of the resolvent on the real axis.
- Hawking effect
  - Proof of a theorem about the Hawking effect for bosons should now work in the same way. Temperature depends on n.
  - Highly idealized model.

Thank you for your attention !