1. Define the tensor product $A \otimes B$ of two superalgebras $A$ and $B$. Prove that if $A$ and $B$ are commutative, then $A \otimes B$ is commutative too. Show that $\Lambda(n) \otimes \Lambda(m) \simeq \Lambda(n + m)$.

2. Find the canonical forms of non-degenerate even symmetric, even skew-symmetric, odd skew-symmetric bilinear forms on the vector superspace $\mathbb{R}^{n|m}$. Find the matrix form of the Lie superalgebras preserving the corresponding structure (these Lie superalgebras are denoted by $\mathfrak{osp}(p,q|2k)$, $\mathfrak{osp}^k(2k|p,q)$, $\mathfrak{pe}(n,\mathbb{R})$, respectively). Prove that there exist isomorphisms $\mathfrak{osp}(p,q|2k) \simeq \mathfrak{osp}^k(2k|p,q)$, $\mathfrak{pe}(n,\mathbb{R}) \simeq \mathfrak{pe}^k(n,\mathbb{R})$.

3. Find the canonical form of an odd complex structure on $\mathbb{R}^{n|m}$. Find the matrix form of the Lie superalgebra $\mathfrak{q}(n,\mathbb{R})$ commuting with this structure.

4. Show that the representation of the simple Lie superalgebra $\mathfrak{vect}(0|2,\mathbb{R})$ on the vector superspace $\Lambda(2) \simeq \mathbb{R}^{2|2}$ is not irreducible and not totally reducible.

5. Construct the isomorphisms of the Lie superalgebras $\mathfrak{vect}(0|2,\mathbb{R}) \simeq \mathfrak{sl}(2|1,\mathbb{R})$, $\mathfrak{sl}(2|1,\mathbb{C}) \simeq \mathfrak{osp}(2|2,\mathbb{C})$. Which of the isomorphisms do exist: $\mathfrak{sl}(2|1,\mathbb{R}) \simeq \mathfrak{osp}(2|2,\mathbb{R})$, $\mathfrak{sl}(2|1,\mathbb{R}) \simeq \mathfrak{osp}(1,1|2,\mathbb{R})$?

6. Show that the real Lie superalgebra $\mathfrak{g}$ spanned by the vector fields $\partial_x$ and $D = -\xi \partial_x + \partial_\xi$ on $\mathbb{R}^{1|1}$ is nilpotent. Prove that its representation on the space $\text{span}_\mathbb{R}\{e^x,e^x\xi\}$ is irreducible.

7. Find all possible structures of the Lie superalgebra on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0 = \mathfrak{so}(n,\mathbb{C})$, $\mathfrak{g}_1 = \mathbb{C}^n$.

8. Find all possible structures of the Lie superalgebra on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0 = \mathfrak{sp}(2m,\mathbb{C})$, $\mathfrak{g}_1 = \mathbb{C}^{2m}$.

9. In particular, find the structure of the Lie superalgebra $\mathfrak{osp}(1|2m,\mathbb{C})$.

10. Prove that str$[K,L] = 0$, where $K,L \in \text{Mat}(n|m,\mathbb{R})$.

11. Let $A$ be a commutative superalgebra with a unite. Let $(A_1) \subset A$ be the ideal generated by $A_1$. Show that $(A_1) = (A_1)^2 \oplus A_1$. Consider the projection $\pi : A \to \mathcal{A} = A/(A_1) = A_0/(A_1)^2$. Prove that $a \in A$ is invertible if and only if $\pi(a) \in \mathcal{A}$ is invertible.

12. Let $L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix} \in \text{Mat}(n|m,\mathbb{R})$ be even. Show that $L$ is invertible if and only if $L_{00} \in \text{Mat}(n,\mathbb{R})$ and $L_{11} \in \text{Mat}(m,\mathbb{R})$ are invertible.

13. Describe the morphisms $\mathbb{R}^{1|2} \to M$, where $M$ is a smooth manifold.

14. Show that the Lie superbracket of two left-invariant vector fields on a Lie supergroup is a left-invariant vector field.

15. Show that the map $\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ given by

$$\mu^*(x) = x' + x'' + \xi'\xi'', \quad \mu^*(\xi) = \xi' + \xi''$$
defines the structure of a Lie supergroup on the supermanifold $\mathbb{R}^{1|1}$. Find the antipode map $i : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$.

16. Show that the Lie superalgebra corresponding to the Lie supergroup $\mathbb{R}^{1|1}$ from the previous exercise is spanned by the vector fields $\partial_t$ and $-\xi \partial_t + \partial_\xi$.

17. Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds. Describe $\mathcal{M} \times \mathcal{N}$ using the functor of points.

18. Let $\mathcal{G}$ be a supermanifold. Prove that $\mathcal{G}$ is a Lie supergroup if and only if $\mathcal{G}(\mathcal{S})$ is a group for any supermanifold $\mathcal{S}$, and $\mathcal{G}(\alpha) : \mathcal{G}(\mathcal{S}) \to \mathcal{G}(\mathcal{T})$ is a group homomorphism for any morphism $\alpha : \mathcal{T} \to \mathcal{S}$.

19. Which supermanifold $\mathcal{M}$ defines the following functor of points: $\mathcal{M}(\mathcal{S}) = C^\infty_S(\mathcal{S})$, $\mathcal{M}(\alpha) = \alpha^*$?

20. Which supermanifold $\mathcal{M}$ defines the following functor of points: $\mathcal{M}(\mathcal{S}) = C^\infty_S(\mathcal{S}) \otimes \mathbb{R}^{n|m}$, $\mathcal{M}(\alpha) = \alpha^* \times \text{id}_{\mathbb{R}^{n|m}}$?

References


