Exercises on Supergeometry

1. Define the tensor product $A \otimes B$ of two superalgebras A and B. Prove that if A and B are commutative, then $A \otimes B$ is commutative too. Show that $\Lambda(n) \otimes \Lambda(m) \simeq \Lambda(n+m)$.

2. Find the canonical forms of non-degenerate even symmetric, even skew-symmetric, odd symmetric, odd skew-symmetric bilinear forms on the vector superspace $\mathbb{R}^{n|m}$. Find the matrix form of the Lie superalgebras preserving the corresponding structure (these Lie superalgebras are denoted by $\mathfrak{osp}(p,q|2k)$, $\mathfrak{osp}^{sk}(2k|p,q)$, $\mathfrak{pe}(n,\mathbb{R})$, $\mathfrak{pe}^{sk}(n,\mathbb{R})$, respectively). Prove that there exist isomorphisms $\mathfrak{osp}(p,q|2k) \simeq \mathfrak{osp}^{sk}(2k|p,q)$, $\mathfrak{pe}(n,\mathbb{R}) \simeq \mathfrak{pe}^{sk}(n,\mathbb{R})$.

3. Find the canonical form of an odd complex structure on $\mathbb{R}^{n|m}$. Find the matrix form of the Lie superalgebra $\mathfrak{q}(n,\mathbb{R})$ commuting with this structure.

4. Show that the representation of the simple Lie superalgebra $\mathfrak{vect}(0|2,\mathbb{R})$ on the vector superspace $\Lambda(2) \simeq \mathbb{R}^{2|2}$ is not irreducible and not totally reducible.

5. Construct the isomorphisms of the Lie superalgebras $\mathfrak{vect}(0|2,\mathbb{R}) \simeq \mathfrak{sl}(2|1,\mathbb{R})$, $\mathfrak{sl}(2|1,\mathbb{C}) \simeq \mathfrak{osp}(2|2,\mathbb{C})$. Which of the isomorphisms do exist: $\mathfrak{sl}(2|1,\mathbb{R}) \simeq \mathfrak{osp}(2|2,\mathbb{R})$, $\mathfrak{sl}(2|1,\mathbb{R}) \simeq \mathfrak{osp}(1,1|2,\mathbb{R})$?

6. Show that the real Lie superalgebra $\mathfrak{g}$ spanned by the vector fields ∂_x and $D = -\xi\partial_x + \partial_\xi$ on $\mathbb{R}^{1|1}$ is nilpotent. Prove that its representation on the space $\mathrm{span}_{\mathbb{R}}\{e^x, e^x\xi\}$ is irreducible.

7. Find all possible structures of the Lie superalgebra on $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$, $\mathfrak{g}_{\bar 0} = \mathfrak{so}(n,\mathbb{C})$, $\mathfrak{g}_{\bar 1} = \mathbb{C}^n$.

8. Find all possible structures of the Lie superalgebra on $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$, $\mathfrak{g}_{\bar 0} = \mathfrak{sp}(2m,\mathbb{C})$, $\mathfrak{g}_{\bar 1} = \mathbb{C}^{2m}$. In particular, find the structure of the Lie superalgebra $\mathfrak{osp}(1|2m,\mathbb{C})$.

9. Prove that $\mathrm{str}[K,L] = 0$, where $K, L \in \mathrm{Mat}(n|m, A)$.

10. Let A be a commutative superalgebra with a unite. Let $(A_{\bar 1}) \subset A$ be the ideal generated by $A_{\bar 1}$. Show that $(A_{\bar 1}) = (A_{\bar 1})^2 \oplus A_{\bar 1}$. Consider the projection $\pi : A \to \mathcal{A} = A/(A_{\bar 1}) = A_{\bar 0}/(A_{\bar 1})^2$. Prove that $a \in A$ is invertible if and only if $\pi(a) \in \mathcal{A}$ is invertible.

11. Let $\pi : \mathrm{Mat}(n|m, A) \to \mathrm{Mat}(n|m, \mathcal{A})$ be the extension of the map π from the previous exercise. Show that $L \in \mathrm{Mat}(n|m, A)$ is invertible if and only if $\pi(L) \in \mathrm{Mat}(n|m, \mathcal{A})$ is invertible.

12. Let $L = \begin{pmatrix} L_{\bar 0\bar 0} & L_{\bar 0\bar 1} \\ L_{\bar 1\bar 0} & L_{\bar 1\bar 1} \end{pmatrix} \in \mathrm{Mat}(n|m, A)$ be even. Show that L is invertible if and only if $L_{\bar 0\bar 0} \in \mathrm{Mat}(n, A)$ and $L_{\bar 1\bar 1} \in \mathrm{Mat}(m, A)$ are invertible.

13. Describe the morphisms $\mathbb{R}^{1|2} \to M$, where M is a smooth manifold.

14. Show that the Lie superbracket of two left-invariant vector fields on a Lie supergroup is a left-invariant vector field.

15. Show that the map $\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ given by

$$\mu^*(x) = x' + x'' + \xi'\xi'', \quad \mu^*(\xi) = \xi' + \xi''$$

defines the structure of a Lie supergroup on the supermanifold $\mathbb{R}^{1|1}$. Find the antipode map $i : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$.

16. Show that the Lie superalgebra corresponding to the Lie supergroup $\mathbb{R}^{1|1}$ from the previous exercise is spanned by the vector fields ∂_t and $-\xi\partial_t + \partial_\xi$.

17. Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds. Describe $\mathcal{M} \times \mathcal{N}$ using the functor of points.

18. Let $\mathcal{G}$ be a supermanifold. Prove that $\mathcal{G}$ is a Lie supergroup if and only if $\mathcal{G}(\mathcal{S})$ is a group for any supermanifold $\mathcal{S}$, and $\mathcal{G}(\alpha) : \mathcal{G}(\mathcal{S}) \to \mathcal{G}(\mathcal{T})$ is a group homomorphism for any morphism $\alpha : \mathcal{T} \to \mathcal{S}$.

19. Which supermanifold $\mathcal{M}$ defines the following functor of points: $\mathcal{M}(\mathcal{S}) = C^\infty_{\mathcal{S}}(S)$, $\mathcal{M}(\alpha) = \alpha^*$?

20. Which supermanifold $\mathcal{M}$ defines the following functor of points: $\mathcal{M}(\mathcal{S}) = C^\infty_{\mathcal{S}}(S) \otimes \mathbb{R}^{n|m}$, $\mathcal{M}(\alpha) = \alpha^* \times \mathrm{id}_{\mathbb{R}^{n|m}}$?

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