On coconvex polynomial approximation

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Let $Y_s = \{y_i\}_{i=1}^s$ be a set of points y_i, such that $-1 < y_s < \cdots < y_1 < 1$. We denote by $\Delta^2(Y_s)$, the collection of all functions $f \in C[-1,1]$ that change convexity at the set Y_s, and are convex in $[y_1, 1]$. If, say $f \in C^2[-1,1]$, then the above is equivalent to $f''(x)\prod_{i=1}^s (x - y_i) \geq 0$, in $[-1,1]$. For $f \in \Delta^2(Y_s)$ we denote by

$$E_n^{(2)}(f, Y_s) := \inf_{p_n \in \Delta^2(Y_s)} \max_{x \in [-1,1]} |f(x) - p_n(x)|,$$

the error of the best uniform coconvex approximation by algebraic polynomials p_n of degree $\leq n$.
Theorem 1.*If $f \in \Delta^2(Y_s)$, then*

$$E_n^{(2)}(f, Y_s) \leq c(s)\omega_3(f, 1/n), \qquad n > N(Y_s), \tag{1}$$

where $c(s)$ is a constant, depending only on s, $N(Y_s)$ is a constant, depending only on Y_s, $\omega_k(f,t)$ is the k-th modulus of smoothness of f.
It is well-known, that (1) cannot be had with ω_k, $k > 3$, instead of ω_3, even if one allow both constants c and N to depend on f.
Let W^r be Sobolev class of functions $f \in C[-1,1]$, that is $f \in W^r$, iff f has an absolutely continuous derivative and $|f^{(r)}(x)| \leq 1$ a.e. in $[-1,1]$.
Corollary 1.*Let $r = 1, 2$ or 3. If $f \in \Delta^2(Y_s) \cap W^r$, then*

$$E_n^{(2)}(f, Y_s) \leq c(s) n^{-r}, \qquad n > N(Y_s). \tag{2}$$

Theorem 2. If $s > 1$, then in (2) one cannot replace $N(Y_s)$ with a constant N, independent of Y_s; if $s = 1$, then it is possible.

References

K. Kopotun, D. Leviatan and I. A. Shevchuk, The degree of coconvex polynomial approximation, Proc. AMS, **127** (1999), 409-415.
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