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Winter Term 2019/20

Mathematical Systems and Control Theory – 6th Exercise Sheet.

Discussion of the solutions in the exercise on January 29, 2020.

Problem 1 (Hamiltonian matrices): Let $\mathcal{H} \in \mathbb{R}^{2n \times 2n}$ and $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$.

a) Show that the following statements are equivalent:
   i) The matrix $\mathcal{H}$ is Hamiltonian, i. e., $\mathcal{H}^\mathsf{T} J + J\mathcal{H} = 0$.
   ii) The matrix $J\mathcal{H}$ is symmetric.
   iii) There exist F, G, $H \in \mathbb{R}^{n \times n}$ with $G = G^\mathsf{T}$ and $H = H^\mathsf{T}$ such that

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^\mathsf{T} \end{bmatrix}.$$

   iv) The matrix $\mathcal{H}$ is skew-adjoint with respect to the J-inner product

$$\langle x, y \rangle_J := y^\mathsf{T} J x,$$

   i. e., for all x, $y \in \mathbb{R}^{2n}$ it holds that $\langle \mathcal{H}x, y \rangle_J = -\langle x, \mathcal{H}y \rangle_J$.

b) Show that

$$\lambda \in \Lambda(\mathcal{H}) \quad \Rightarrow \quad \overline{\lambda},\ -\lambda,\ -\overline{\lambda} \in \Lambda(\mathcal{H}).$$

Problem 2 (Isotropic invariant subspaces): Let $\mathcal{H} \in \mathbb{R}^{2n \times 2n}$ be Hamiltonian and let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{1}$$

with X_1, $X_2 \in \mathbb{R}^{n \times r}$ such that the columns of X span an $\mathcal{H}$-invariant subspace $\mathcal{X}$, i. e.,

$$\mathcal{H}X = XZ$$

for some $Z \in \mathbb{R}^{r \times r}$. Show that if $\Lambda(Z) \subset \mathbb{C}^-$, then $\mathcal{X}$ is *isotropic*, meaning that $\langle x, y \rangle_J = 0$ for all x, $y \in \mathcal{X}$.

Problem 3 (Hamiltonian and symplectic matrices): Let $\mathbb{H}_{2n}$ and $\mathbb{S}_{2n}$ denote the sets of Hamiltonian and symplectic matrices in $\mathbb{R}^{2n \times 2n}$, respectively. A matrix $\mathcal{S} \in \mathbb{R}^{2n \times 2n}$ is called *symplectic*, if $\mathcal{S}$ is orthogonal with respect to the J-inner product, i. e., $\langle \mathcal{S}x, \mathcal{S}y \rangle_J = \langle x, y \rangle_J$ for all x, $y \in \mathbb{R}^{2n}$. Show the following statements:

a) The matrix $\mathcal{S}$ is symplectic, if and only if $\mathcal{S}^\mathsf{T} J \mathcal{S} = J$.

b) It holds that $\det(\mathcal{S}) \in \{-1, 1\}$.

c) The set $\mathbb{S}_{2n}$ equipped with the matrix multiplication forms a group.

d) Each $\mathcal{H} \in \mathbb{H}_{2n}$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle (I_{2n} + t\mathcal{H})x, (I_{2n} + t\mathcal{H})y\rangle_J\right)\big|_{t=0} = 0 \quad \text{for all } x,\, y \in \mathbb{R}^{2n}.$$

e) The set $\mathbb{H}_{2n}$ is a Lie algebra with respect to the *Lie bracket* $[\cdot,\cdot] : \mathbb{H}_{2n} \times \mathbb{H}_{2n} \to \mathbb{R}^{2n\times 2n}$ with

$$[\mathcal{H}_1, \mathcal{H}_2] := \mathcal{H}_1\mathcal{H}_2 - \mathcal{H}_2\mathcal{H}_1,$$

that is, the following properties are satsified:

i) $[\mathcal{H}_1, \mathcal{H}_2] \in \mathbb{H}_{2n}$ for all $\mathcal{H}_1,\, \mathcal{H}_2 \in \mathbb{H}_{2n}$;

ii) $[\mathcal{H}, \mathcal{H}] = 0$ for all $\mathcal{H} \in \mathbb{H}_{2n}$;

iii) $[\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3] = [\mathcal{H}_1, \mathcal{H}_3] + [\mathcal{H}_2, \mathcal{H}_3]$ for all $\mathcal{H}_1,\, \mathcal{H}_2,\, \mathcal{H}_3 \in \mathbb{H}_{2n}$;

iv) $[[\mathcal{H}_1, \mathcal{H}_2], \mathcal{H}_3] + [[\mathcal{H}_2, \mathcal{H}_3], \mathcal{H}_1] + [[\mathcal{H}_3, \mathcal{H}_1], \mathcal{H}_2] = 0$ for all $\mathcal{H}_1,\, \mathcal{H}_2,\, \mathcal{H}_3 \in \mathbb{H}_{2n}$ (Jacobi identity).

f) It holds that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S} \in \mathbb{H}_{2n}$ for all $\mathcal{H} \in \mathbb{H}_{2n}$ and $\mathcal{S} \in \mathbb{S}_{2n}$.