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Mathematical Systems and Control Theory – 1st Exercise Sheet.

Discussion of the solutions in the exercise on October 30, 2019.

Problem 1 (inverted pendulum): Consider the model of a controlled inverted pendulum which (after linearization) is given by a second-order differential equation

$$\ddot{\varphi}(t) - \varphi(t) = u(t).$$

Here, $\varphi(t) = \theta(t) - \pi$ is the angular deviation of the pendulum from the upright equilibrium at time $t \geq 0$ and $u(t)$ is the applied torque.

a) Show that for proportional feedback $u(t) = -\alpha\varphi(t)$ with $\alpha < 1$ it holds that: If the initial values satisfy $\dot{\varphi}(0) = -\varphi(0)\sqrt{1-\alpha}$, then $\lim_{t\to\infty} \varphi(t) = 0$.

b) Let $\alpha \in \mathbb{R}$ be fixed. Consider the energy function

$$V(x, y) := \cos x - 1 + \frac{1}{2}\big(\alpha x^2 + y^2\big).$$

Show that $V\big(\varphi(t), \dot{\varphi}(t)\big)$ is constant along the solutions of the nonlinear pendulum equation with proportional feedback, given by

$$\ddot{\varphi}(t) - \sin\varphi(t) + \alpha\varphi(t) = 0. \tag{1}$$

Conclude that there exist initial conditions $\varphi(0) = \varepsilon$, $\dot{\varphi}(0) = 0$ such that the solution of (1) for arbitrarily small ε does *not* satisfy $\lim_{t\to\infty} \varphi(t) = 0$, $\lim_{t\to\infty} \dot{\varphi}(t) = 0$.

Hint: Use that $x = 0$ is an isolated root of $V(\cdot, 0)$ which follows from analyticity of V (otherwise, $V \equiv 0$). You do *not* have to solve the differential equation at any point to do this task.

Problem 2 (stability of LTI systems): Let $A \in \mathbb{R}^{n\times n}$ be given. Show the following statements:

a) The ODE $\dot{x}(t) = Ax(t)$ is asymptotically stable, if and only if $\Lambda(A) \subset \mathbb{C}^-$.

b) The ODE $\dot{x}(t) = Ax(t)$ is (Lyapunov) stable (i. e., $x(\cdot)$ remains bounded for all initial conditions), if and only if $\Lambda(A) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R}$ and the eigenvalues on the imaginary axis are semi-simple (i. e., they only have Jordan blocks of size at most 1×1).

Hint: Transform A to Jordan canonical form and consider the matrix exponential, which for $A \in \mathbb{R}^{n\times n}$ is defined by

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Problem 3 (fundamental solution): Let $\Phi(\cdot, \cdot)$ be the fundamental solution of $\dot{x}(t) = A(t)x(t)$ with $A : \mathbb{R} \to \mathbb{R}^{n\times n}$. Show the following properties:

a) $\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)$ for all t, s, $\tau \in \mathbb{R}$;

b) $\Phi(t, s)$ is invertible for all t, $s \in \mathbb{R}$ and $\Phi(t, s)^{-1} = \Phi(s, t)$;

c) $\frac{\partial}{\partial s}\Phi(t, s) = -\Phi(t, s)A(s)$.