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Optimization of Complex Systems – Additonal Material.

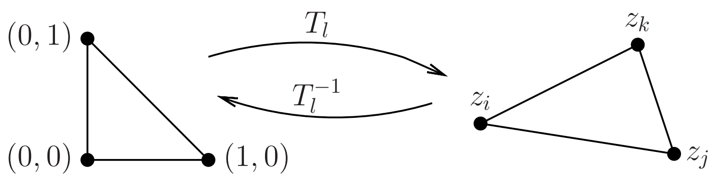
How do we assemble the Galerkin matrix (stiffness matrix) efficiently? Recall that for the Poisson equation we have

$$A_n = [a(\varphi_j, \varphi_i)]_{i,j=1}^n .$$

The actual computation of the coefficients $a_{ij} = a(\varphi_j, \varphi_i)$ is usually done as follows. We have

$$a_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}x = \sum_{l=1}^N \int_{\Omega_l} \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}x.$$

Thus it is enough to compute $a_{ij}^l := \int_{\Omega_l} \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}x$.
By construction, a_{ij}^l is zero if $z_i \notin \overline{\Omega}_l$ or $z_j \notin \overline{\Omega}_l$. For the remaining cases we go back to a reference triangle



and T_l is an affine linear mapping on triangles with

$$T_l\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = z_i, \quad T_l\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = z_k, \quad T_l\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = z_j.$$

We want to map a point $y = \left[\begin{smallmatrix}y_1\\y_2\end{smallmatrix}\right] \in \overline{\widehat{\Omega}}_l$ on the *reference triangle* (bijectively) to a point $x = \left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right] \in \overline{\Omega}$. It can be easily checked that $\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right] = T_l\left(\left[\begin{smallmatrix}y_1\\y_2\end{smallmatrix}\right]\right) = z_i + (z_j - z_i)y_1 + (z_k - z_i)y_2$ realizes such a mapping by an affine linear transformation. In the end, we use the changed coordinates to compute a_{ij}^l on the reference triangle as follows: With

$$z_i = \begin{bmatrix}z_{i1}\\z_{i2}\end{bmatrix}, \quad z_j = \begin{bmatrix}z_{j1}\\z_{j2}\end{bmatrix}, \quad z_k = \begin{bmatrix}z_{k1}\\z_{k2}\end{bmatrix},$$

we get

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = T_l(y) = \underbrace{\begin{bmatrix}z_{j1} - z_{i1} & z_{k1} - z_{i1}\\ z_{j2} - z_{i2} & z_{k2} - z_{i2}\end{bmatrix}}_{=:A_l} \begin{bmatrix}y_1\\y_2\end{bmatrix} + \begin{bmatrix}z_{i1}\\z_{i2}\end{bmatrix}. \tag{1}$$

Assume that there are two piecewise linear ansatz functions $\widehat{\varphi}_i$ and $\widehat{\varphi}_j$ defined on the reference triangle $\widehat{\Omega}_l$. Then we get

$$\varphi_i(x) = \varphi_i(T_l(y)) = \widehat{\varphi}_i(y), \quad \varphi_j(x) = \varphi_j(T_l(y)) = \widehat{\varphi}_j(y).$$

Now we want to do the change of coordinates in the integrals, in particular, we must change the coordinates when forming the gradients. According to the chain rule in multivariate differentiation, we obtain

$$\begin{aligned}\nabla_x\varphi(x) := \begin{bmatrix}\frac{\partial\varphi(x)}{\partial x_1}\\ \frac{\partial\varphi(x)}{\partial x_2}\end{bmatrix} &= \begin{bmatrix}\frac{\partial\varphi(T_l(y))}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial\varphi(T_l(y))}{\partial y_2}\frac{\partial y_2}{\partial x_1}\\ \frac{\partial\varphi(T_l(y))}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial\varphi(T_l(y))}{\partial y_2}\frac{\partial y_2}{\partial x_2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\partial\widehat{\varphi}(y)}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial\widehat{\varphi}(y)}{\partial y_2}\frac{\partial y_2}{\partial x_1}\\ \frac{\partial\widehat{\varphi}(y)}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial\widehat{\varphi}(y)}{\partial y_2}\frac{\partial y_2}{\partial x_2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1}\\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2}\end{bmatrix}\underbrace{\begin{bmatrix}\frac{\partial\widehat{\varphi}(y)}{\partial y_1}\\ \frac{\partial\widehat{\varphi}(y)}{\partial y_2}\end{bmatrix}}_{=:\nabla_y\widehat{\varphi}(y)}.\end{aligned}$$

Since $y = A_l^{-1}(x - z_i)$, we see that

$$\begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1}\\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2}\end{bmatrix} = A_l^{-\mathsf{T}},$$

and therefore,

$$\nabla_x\varphi(x) = A_l^{-\mathsf{T}}\nabla_y\widehat{\varphi}(y).$$

This and the the transformation formula for coordinate changes in integration gives us

$$\begin{aligned}a_{ij}^l &= \int_{\Omega_l}\nabla_x\varphi_i(x)\cdot\nabla_x\varphi_j(x)\mathrm{d}x\\ &= \int_{\widehat{\Omega}_l}\left(A_l^{-\mathsf{T}}\nabla_y\widehat{\varphi}_i(y)\right)\cdot\left(A_l^{-\mathsf{T}}\nabla_y\widehat{\varphi}_j(y)\right)\cdot\left|\det\left(\begin{bmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{bmatrix}\right)\right|\mathrm{d}y\\ &= \int_{\widehat{\Omega}_l}\left(\nabla_y\widehat{\varphi}_i(y)\right)^{\mathsf{T}}A_l^{-1}A_l^{-\mathsf{T}}\left(\nabla_y\widehat{\varphi}_j(y)\right)\cdot\left|\det\left(\begin{bmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{bmatrix}\right)\right|\mathrm{d}y.\end{aligned}$$

With (1) we get

$$\begin{bmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{bmatrix} = \begin{bmatrix}z_{j1} - z_{i1} & z_{k1} - z_{i1}\\ z_{j2} - z_{i2} & z_{k2} - z_{i2}\end{bmatrix} = A_l.$$

Therefore, with

$$B_l = \begin{bmatrix}b_{11}^l & b_{12}^l\\ b_{12}^l & b_{22}^l\end{bmatrix} := A_l^{-1}A_l^{-\mathsf{T}}$$

we finally get

$$\begin{aligned}a_{ij}^l &= \underbrace{|(z_{j1} - z_{i1})(z_{k2} - z_{i2}) - (z_{k1} - z_{i1})(z_{j2} - z_{i2})|}_{=:d_l}\int_{\widehat{\Omega}_l}\left(\nabla_y\widehat{\varphi}_i(y)\right)^{\mathsf{T}}A_l^{-1}A_l^{-\mathsf{T}}\left(\nabla_y\widehat{\varphi}_j(y)\right)\mathrm{d}y\\ &= d_l\cdot\Bigg(b_{11}^l\cdot\int_{\widehat{\Omega}_l}\frac{\partial\widehat{\varphi}_i(y)}{\partial y_1}\cdot\frac{\partial\widehat{\varphi}_j(y)}{\partial y_1}\mathrm{d}y + b_{12}^l\cdot\int_{\widehat{\Omega}_l}\frac{\partial\widehat{\varphi}_i(y)}{\partial y_1}\cdot\frac{\partial\widehat{\varphi}_j(y)}{\partial y_2} + \frac{\partial\widehat{\varphi}_i(y)}{\partial y_2}\cdot\frac{\partial\widehat{\varphi}_j(y)}{\partial y_1}\mathrm{d}y\\ &\quad + b_{22}^l\cdot\int_{\widehat{\Omega}_l}\frac{\partial\widehat{\varphi}_i(y)}{\partial y_2}\cdot\frac{\partial\widehat{\varphi}_j(y)}{\partial y_2}\mathrm{d}y\Bigg).\end{aligned}$$

So in fact, each of the integrals above on the reference triangle have to be computed only once in total! (and not once for each element). Therefore, the only values we have to compute for each element again are d_l, b_{11}^l, b_{12}^l, and b_{22}^l. But this is easy, since it only requires the knowledge of the vertices of the elements $\overline{\Omega}_l$.