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Winter Term 2019/20

Optimization of Complex Systems – 9th Exercise Sheet.

Discussion of the solutions in the exercise on January 20, 2020.

Problem 1 (discretized box constraints): Let $\Omega \subset \mathbb{R}^N$ be a polygonally bounded domain and $U_h = U_h^{(0)}$ or $U_h = U_h^{(1)}$ be the spaces of piecewise constant or piecewise linear finite element ansatz functions with basis $\{\xi_1, \ldots, \xi_m\}$, respectively. Let

$$u_h = \sum_{i=1}^{m} u_i \xi_i.$$

Show that

$$u_a \leq u_h(x) \leq u_b \text{ a. e. in } \Omega \quad \Leftrightarrow \quad u_a \leq u_i \leq u_b \quad \forall\, i \in \{1, \ldots, m\}.$$

Problem 2 (energy norm): Let $\Omega \subset \mathbb{R}^N$ for $N = 1, 2$ be a bounded domain. A norm on $H_0^1(\Omega)$ is the *energy norm*, defined by

$$\|y\|_{H_0^1(\Omega)} = \|\nabla y\|_{L^2(\Omega)^N}.$$

a) Show that the energy norm is indeed a norm on $H_0^1(\Omega)$.

b) Consider now the Poisson equation

$$\begin{aligned} \Delta y &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma = \partial\Omega. \end{aligned}$$

Let y_h be the approximate solution obtain from a finite element discretization and $\widehat{y}_h$ be the computed solution (containing the numerical errors from solving the linear system). We obtain

- the *approximation error* $\|y - y_h\|_{H_0^1(\Omega)}$,
- the *algebraic error* $\left\|y_h - \widehat{y}_h\right\|_{H_0^1(\Omega)}$.

Show that the *total error* satisfies

$$\left\|y - \widehat{y}_h\right\|^2_{H_0^1(\Omega)} = \|y - y_h\|^2_{H_0^1(\Omega)} + \left\|y_h - \widehat{y}_h\right\|^2_{H_0^1(\Omega)}.$$

Problem 3 (optimization in MATLAB): Let $\Omega = (0, 1)^2$. Consider the minimization problem

$$\min J(y, u) = \frac{1}{2}\|y - y_{\mathrm{d}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$$

subject to

$$\begin{aligned} \Delta y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \Gamma = \partial\Omega, \\ u &\in \left\{u \in L^2(\Omega) \,:\, u_a \leq u(x) \leq u_b\right\}. \end{aligned}$$

Here, $y_{\mathrm{d}}(x_1, x_2) = 1 + \cos(x_1)\cos(x_2)$, $u_a = -1$, and $u_b = 1$.

a) Write a function to perform a full discretization of the problem in MATLAB using the discretization for the solution y from homework sheet 8 and piecewise constant functions to discretize u. You can make use of the code provided on the course website. As input, the function should take the number `n` of inner nodes per dimension and return the matrices `M`, `N`, `A`, `R`, `Q`, `G`, and `H` (see lecture) defining the finite-dimensional QP.

b) Write a function to solve the problem using the MATLAB command `fmincon`. This function should take `n` and should return the coordinate vectors `uopt` and `yopt` of the discrete optimal control $\overline{u}_h$ and the the corresponding optimal state $\overline{y}_h$. Visualize the optimal control and the optimal state for various mesh sizes.