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Winter Term 2019/20

Optimization of Complex Systems – 6th Exercise Sheet.

Discussion of the solutions in the exercise on December 9, 2019.

Problem 1 (projection formula): Let u_a, u_b, $p \in \mathbb{R}$ and $\lambda > 0$. Solve the optimization problem

$$\min_{v \in [u_a, u_b]} \left(pv + \frac{\lambda}{2} v^2 \right)$$

by deriving a projection formula.

Problem 2 (discontinuous trace operator): Assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with boundary Γ. Show that there exists no constant $c > 0$ that only depends on Ω such that

$$\|u\|_{L^2(\Gamma)} \leq c \|u\|_{L^2(\Omega)}$$

for each $u \in C(\overline{\Omega})$.

Problem 3 (adjoint equation and necessary optimality conditions): Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with boundary Γ. Let functions a_Ω, $u \in L^2(\Omega)$, $a_\Gamma \in L^2(\Gamma)$, $\beta \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma)$ with $\alpha \geq 0$ almost everywhere be given. Let y and p be the weak solutions of the elliptic boundary value problems

$$\begin{aligned} -\Delta y &= \beta u \quad \text{in } \Omega, \\ \frac{\partial y}{\partial n} + \alpha y &= 0 \quad \text{on } \Gamma, \end{aligned}$$

and

$$\begin{aligned} -\Delta p &= a_\Omega \quad \text{in } \Omega, \\ \frac{\partial p}{\partial n} + \alpha p &= a_\Gamma \quad \text{on } \Gamma, \end{aligned}$$

respectively.

a) Show that

$$\int_\Omega a_\Omega y \,\mathrm{d}x + \int_\Gamma a_\Gamma y \,\mathrm{d}s = \int_\Omega \beta p u \,\mathrm{d}x.$$

b) Assume now further that λ, λ_Ω, $\lambda_\Gamma \geq 0$ and $y_\Omega \in L^2(\Omega)$, $y_\Gamma \in L^2(\Gamma)$. Consider the optimal control problem

$$\min J(y,u) := \frac{\lambda_\Omega}{2} \|y - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2} \|y - y_\Gamma\|^2_{L^2(\Gamma)} + \frac{\lambda}{2} \|u\|^2_{L^2(\Omega)}$$

subject to

$$\begin{aligned} -\Delta y &= \beta u \quad \text{in } \Omega, \\ \frac{\partial y}{\partial n} + \alpha y &= 0 \quad \text{on } \Gamma, \\ u_a(x) \leq u(x) &\leq u_b(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Show that with $U_{\mathrm{ad}} := \big\{ u \in L^2(\Omega) \;:\; u_a(x) \leq u(x) \leq u_b(x) \quad \text{a. e. in } \Omega \big\}$ and for an optimal triple $(\bar{u}, \bar{y}, p)$, the optimality conditions are given by

$$\int_\Omega (\beta p + \lambda \bar{u})(u - \bar{u}) \mathrm{d}x \geq 0 \quad \forall u \in U_{\mathrm{ad}},$$

where the adjoint state p is given by the weak solution of

$$\begin{aligned} -\Delta p &= \lambda_\Omega (\bar{y} - y_\Omega) \quad \text{in } \Omega, \\ \frac{\partial p}{\partial n} + \alpha p &= \lambda_\Gamma (\bar{y} - y_\Gamma) \quad \text{on } \Gamma. \end{aligned}$$