University of Hamburg
Department of Mathematics
Dr. Matthias Voigt

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Optimization of Complex Systems – 5th Exercise Sheet.

Discussion of the solutions in the exercise on December 2, 2019.

Problem 1 (Gâteaux and Fréchet differentiability):

a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} 1, & \text{if } y = x^2 \text{ and } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Check whether this function is Gâteaux and Fréchet differentiable in $(x,y) = (0,0)$ and if so, state the derivatives.

b) Consider the mappting $g : C([0,1]) \to C([0,1])$ with

$$g(u)(t) = \int_0^t \cos\left(u(\tau)^2\right) \mathrm{d}\tau, \quad t \in [0,1].$$

Show that g is Fréchet differentiable for all $u \in C([0,1])$ and compute the Fréchet derivative for each $u \in C([0,1])$.

c) Prove the chain rule for Fréchet differentiable functions: Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Z}$ be Banach spaces and $F : \mathcal{U} \to \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{Z}$ be Fréchet differentiable in $u \in \mathcal{U}$ and $F(u) \in \mathcal{V}$, respectively. Then $E : \mathcal{U} \to \mathcal{Z}$ with $E(u) = (G \circ F)(u)$ is Fréchet differentiable in u with

$$E'(u) = G'(F(u)) \circ F'(u).$$

Problem 2 (adjoint operators): Let $A : L^2(0,1) \to L^2(0,1)$ be given as

$$(Au)(t) = \int_0^t \mathrm{e}^{t-s} u(s) \mathrm{d}s.$$

Determine the adjoint operator of A.

Problem 3 (Cones and optimality conditions): Let $U_{\mathrm{ad}} \subseteq \mathbb{R}^n$ be the set of admissable controls which is assumed to be convex. Further, let $f : U_{\mathrm{ad}} \to \mathbb{R}$ be continuously differentiable.

a) The "smallest" conic superset of U_{ad} at the point $\bar{u} \in U_{\mathrm{ad}}$, called *conic hull*, is defined by

$$\mathcal{K}(U_{\mathrm{ad}}, \bar{u}) := \{\alpha(u - \bar{u}) \;:\; u \in U_{\mathrm{ad}},\, \alpha > 0\}.$$

Show that $\mathcal{K}(U_{\mathrm{ad}}, \bar{u})$ is convex. (Further, $\mathcal{K}(U_{\mathrm{ad}}, \bar{u})$ is a cone, that is, $v \in \mathcal{K}(U_{\mathrm{ad}}, \bar{u}) \;\Rightarrow\; \alpha v \in \mathcal{K}(U_{\mathrm{ad}}, \bar{u})$ for $\alpha > 0$.)

b) For every convex cone $\mathcal{K}$, there exists the *dual cone*

$$\mathcal{K}^* := \{v \in \mathbb{R}^n \;:\; (v,u)_{\mathbb{R}^n} \leq 0 \quad \forall\, u \in \mathcal{K}\}.$$

Show that

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall\, u \in U_{\mathrm{ad}} \quad \Leftrightarrow \quad -\nabla f(\bar{u}) \in \mathcal{K}(U_{\mathrm{ad}}, \bar{u})^*.$$

c) Determine the dual cone $\mathcal{K}(U_{\text{ad}}, \bar{u})^*$ for the set

$$U_{\text{ad}} = \{u \in \mathbb{R}^n \ : \ u_a \le u \le u_b\}$$

with $u_a \le u_b$ (be careful, if $\bar{u} \in \partial U_{\text{ad}}$). Use this representation to derive the KKT optimality system (in particular, the complementarity conditions).