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Winter Term 2019/20

Optimization of Complex Systems – 3rd Exercise Sheet.

Discussion of the solutions in the exercise on November 18, 2019.

Problem 1 (counter-example for embeddings in unbounded domains): Find an (unbounded) domain $\Omega \subseteq \mathbb{R}$ and a function $f : \Omega \to \mathbb{R}$ such that $f \in L^2(\Omega)$, but $f \notin L^1(\Omega)$.

Problem 2 (boundedness and continuity): Let $\mathcal{U}$ and $\mathcal{V}$ be two normed vector spaces and $A : \mathcal{U} \to \mathcal{V}$ be a linear operator.

a) Show that A is bounded if and only if it is continuous.

b) Let $A : C^1([0,1]) \to C([0,1])$ with $f \mapsto f'$. Show that A is linear and unbounded.

Problem 3 (operator norm): Determine the norm of the integral operator $A : C([0,1]) \to C([0,1])$ with

$$(Au)(t) = \int_0^1 \mathrm{e}^{t-s} u(s) \mathrm{d}s.$$

Problem 4 (Hölder inequality): Let Ω be a bounded domain and $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume further $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Prove the Hölder inequality

$$\int_\Omega |fg| \mathrm{d}x \leq \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^q(\Omega)} .$$

Hint: Make use of the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ which holds for $a, b \geq 0$ and $p, q > 1$.

Problem 5 (continuous embedding): Let Ω be a bounded domain and $1 \leq s \leq r \leq \infty$. Let $u \in L^r(\Omega)$. Show the estimate

$$\|u\|_{L^s(\Omega)} \leq |\Omega|^{\frac{r-s}{rs}} \|u\|_{L^r(\Omega)} ,$$

where for $r = \infty$, $(r-s)/rs = 1/s$ (for $s < \infty$) and $(r-s)/rs = 0$ (for $s = \infty$).
Hint: Use the Hölder inequality on $\int_\Omega |u|^s \cdot 1 \mathrm{d}x$ with $p = r/s$ and $q = r/(r-s)$.