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Winter Term 2019/20

Optimization of Complex Systems – 2nd Exercise Sheet.

Discussion of the solutions in the exercise on October 28, 2019.

Problem 1 (norms on $C(\overline{\Omega})$): Show that for a bounded domain Ω, $\|\cdot\|_{C(\overline{\Omega})}$ and $\|\cdot\|_{L^2(\Omega)}$ are norms on $C(\overline{\Omega})$. Show further that $\left(C(\overline{\Omega}), \|\cdot\|_{C(\overline{\Omega})}\right)$ is complete (i. e., a Banach space).

Problem 2 (norms in inner product spaces): Show that for any inner product space with inner product $\langle\cdot,\cdot\rangle$, a norm is induced by

$$\|u\| := \sqrt{\langle u, u\rangle}.$$

Problem 3 (bump functions): A typical example of a function in $C_0^\infty(\Omega)$ is a so-called *bump function.* Such a function may be given as follows. Consider $h : (-2,2) \to \mathbb{R}$ with

$$h(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{for } x \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Show that indeed, $h \in C_0^\infty(-2,2)$.

Problem 4 (counter-examples for function spaces): State a bounded domain $\Omega \in \mathbb{R}$ and a function $f : \Omega \to \mathbb{R}$ such that

a) $f \in L^1(\Omega)$, but $f \notin L^2(\Omega)$;

b) $f \in L^1_{\text{loc}}(\Omega)$, but $f \notin L^1(\Omega)$.

Further show that:

c) For $\Omega \in (0,1)$ and $f(x) = \cos\left(\frac{1}{x}\right) \cdot x^2$, it holds that $f \in C^1(\Omega) \cap C(\overline{\Omega})$, but $f \notin C^1(\overline{\Omega})$.

Problem 5 (Sobolev spaces): For a bounded domain $\Omega \subset \mathbb{R}^N$ and $m \in \mathbb{N}$ we define

$$H^m(\Omega) := \left\{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \le m\right\}.$$

Determine the maximal m such that the following functions are in $H^m(\Omega)$:

a) $\Omega \in (-1,1)$, $f(x) = \begin{cases} x^2 & \text{for } x < 0, \\ x^3 & \text{for } x \ge 0, \end{cases}$

b) $\Omega \in (0,1)$, $f(x) = x^{-\frac{1}{4}}$.