Universität Hamburg
Fachbereich Mathematik
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Summer Term 2019

Model Reduction Homework Sheet 5.

The problems will be discussed in the exercise on Thursday, June 27.

Problem 1: Let $S \in \mathbb{C}^{n \times n}$ be asymptotically stable, $B \in \mathbb{C}^{n \times m}$, and $\rho < 0$. Define $A := (S - \rho I_n)(S + \rho I_n)^{-1}$ and $Q := -2\rho(S + \rho I_n)^{-1} BB^{\mathsf{H}}(S + \rho I_n)^{-\mathsf{H}}$. Show that the discrete-time Lyapunov equation

$$AXA^{\mathsf{H}} - X = -Q$$

is equivalent to a continuous-time Lyapunov equation.

Problem 2: Let $S \in \mathbb{C}^{n \times n}$ and $-p$, $-q \in \mathbb{C} \setminus \Lambda(S)$.

a) Show that the matrices $(S + pI_n)^{\pm 1}$, $(S + qI_n)^{\pm 1}$ commute with each other.

b) Define the Cayley transform $\mathcal{C}(S, p, q) := (S - qI_n)(S + pI_n)^{-1}$. Show that

$$\mathcal{C}(S, p, q) = I_n - (q + p)(S + pI_n)^{-1}.$$

c) Show that

$$\Lambda(\mathcal{C}(S, p, q)) = \left\{ \frac{\lambda - q}{\lambda + p} \;\middle|\; \lambda \in \Lambda(S) \right\}.$$

Now let S be diagonalizable and $\Lambda(S) \subset \mathbb{C}^-$. Show that the spectral radius of $\mathcal{C}(S, p, q)$ is smaller than one if $q = \overline{p} \in \mathbb{C}^-$.

d) Now consider $M_j := \prod_{i=1}^{j} \mathcal{C}(S, p_i, q_i)$ with $q_i = \overline{p_i} \in \mathbb{C}^-$, $i = 1, \ldots, j$. What are the eigenvalues of M_j?

e) Now assume that the shifts are chosen cyclicly, that is, it holds $p_i = p_{i+k\cdot\ell}$ for some $\ell \in \mathbb{N}$ and $i = 1, \ldots, \ell$ and $k = 1, 2, \ldots$. Show that spectral radius of M_j converges to zero for $j \to \infty$. Does this also hold for $\|M_j\|_2$?

Problem 3: Prove Theorem 4.19 from the lecture notes: Assume $P = \{p_1, \ldots, p_k\}$ to be a set of proper shifts and assume w. l. o. g. that $p_{j+1} = \overline{p}_j \notin \mathbb{R}$. Then for V_j, V_{j+1} it holds that

$$\begin{aligned} V_{j+1} &= \overline{V_j} + 2\beta_j \operatorname{Im}(V_j), \\ W_{j+1} &= W_{j-1} - 4 \operatorname{Re}(p_j) \left(\operatorname{Re}(V_j) + \beta_j \operatorname{Im}(V_j) \right), \end{aligned}$$

with $\beta_j = \frac{\operatorname{Re}(p_j)}{\operatorname{Im}(p_j)}$.