Universität Hamburg
Fachbereich Mathematik
Dr. Matthias Voigt

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Model Reduction Homework Sheet 3.

The problems will be discussed in the exercise on Thursday, May 16.

Problem 1: (Stability-preserving model reduction). Consider the linear system $[A, B, C, D] \in \Sigma_{n,m,p}$. Show the following statements:

a) If A is *dissipative*, i. e., $\Lambda\big(A + A^\mathsf{T}\big) \subset \mathbb{C}^-$, then every reduced-order model $\big[V^\mathsf{T}AV, V^\mathsf{T}B, CV, D\big] \in \Sigma_{r,m,p}$ (with $V^\mathsf{T}V = I_r$) is asymptotically stable.

b) Assume that the reduced order model $\big[\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\big] := \big[W^\mathsf{T}AV, W^\mathsf{T}B, CV, D\big] \in \Sigma_{r,m,p}$ is constructed such that $W = PV\big(V^\mathsf{T}PV\big)^{-1}$, where P solves

$$A^\mathsf{T}P + PA^\mathsf{T} + 2\sigma P < 0 \quad (\text{“}<\text{” has to be understood in terms of definiteness}).$$

Then the reduced-order model is asymptotically stable with $\max_{\lambda\in\Lambda(\widetilde{A})} \mathrm{Re}(\lambda) < -\sigma$.

Problem 2: (Model reduction for unstable systems). Sometimes one would like to reduce linear systems that are unstable. Assume that $[A, B, C, D] \in \Sigma_{n,m,p}$ is an unstable linear system, but assume that A has a small number of eigenvalues $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Assume that an efficient algorithm is available that allows to compute the rightmost eigenvalues and their respective right and left eigenvectors. Further suppose that an algorithm for the reduction of asympotically stable systems is known (such as the dominant pole algorithm).
Derive an algorithm that allows the reduction of the system $[A, B, C, D] \in \Sigma_{n,m,p}$ using the above mentioned ingredients. Ensure that the output error is still bounded in the $\mathcal{L}_2$-norm for all $\mathcal{L}_2$ input functions u.

Problem 3: (Your first reduced-order model). Go to the course website and download the MATLAB code of the dominant pole algorithm (`samdp.m`) as well as the two benchmark examples (`iss.mat` and `CDplayer.mat`).

a) The quality of a reduced-order model is usually assessed by comparing the *sigma plot* of the full order model with the one of the reduced-order model. A sigma plot is created by plotting the 2-norm values of the system's transfer function $G(s)$ evaluated on the imaginary axis, i. e., $\|G(\mathrm{i}\omega)\|_2$ for $\omega \in [\omega_{\min}, \omega_{\max}]$ on a logarithmic scale. Write a MATLAB function

```
fig = sigmamax( A, B, C, D, wmin, wmax )
```

that generates a sigma plot for a given system $[A, B, C, D] \in \Sigma_{n,m,p}$ in the interval $\omega \in [\omega_{\min}, \omega_{\max}]$. Generate the sigma plots of the two benchmark systems.

b) Reduce the benchmark systems to order $r = 20$ using modal truncation. Sort the eigenvalues
   - according to the distance to the imaginary axis by computing a full diagonalization of A;
   - according to size of $\|R_k\|_2 / |\mathrm{Re}(\lambda_k)|$ by computing a full diagonalization of A;

- using the dominant pole algorithm implemented in `samdp.m`.

Generate the sigma plots of the reduced-order model with transfer function $\widetilde{G}(s)$ and the error model with transfer function $G(s) - \widetilde{G}(s)$. What do you observe?

c) Quantify the error of the reduced-order model. Compute the $\mathcal{H}_\infty$-norm and the $\mathcal{H}_2$-norm of each error transfer function. Use the commands

```
sys = ss( A, B, C, D );
hinfnorm = norm( sys, 'inf' );
h2norm = norm( sys, 2 );
```