Universität Hamburg
Fachbereich Mathematik
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Model Reduction Homework Sheet 2.

The problems will be discussed in the exercise on Thursday, May 02.

Problem 1: (Calculating with transfer functions). Consider two transfer functions $G_1(s)$, $G_2(s) \in \mathbb{R}(s)^{p\times m}$ with realizations $[A_1, B_1, C_1, D_1] \in \Sigma_{n_1,m,p}$ and $[A_2, B_2, C_2, D_2] \in \Sigma_{n_2,m,p}$. Show the following statements:

a) A realization of $G_1(s) + G_2(s)$ is given by

$$\left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_1 + D_2\right] \in \Sigma_{n_1+n_2,m,p}.$$

b) For $p = m$, two realizations of $G_1(s)G_2(s)$ are given by

$$\left[\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}, D_1D_2\right] \in \Sigma_{n_1+n_2,m,p},$$
$$\left[\begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix}, \begin{bmatrix} D_1C_2 & C_1 \end{bmatrix}, D_1D_2\right] \in \Sigma_{n_1+n_2,m,p}.$$

c) Let $G(s)$ with a realization $[A, B, C, D] \in \Sigma_{n,m,m}$ with invertible D be given. Assume there exists an inverse $G^{-1}(s)$ with $G(s)G^{-1}(s) = G^{-1}(s)G(s) = I_m$. Then a realization of $G^{-1}(s)$ is given by

$$\left[A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1}\right] \in \Sigma_{n,m,m}.$$

Problem 2: (Mechanical systems). Consider the *second-order* LTI system

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bu(t), \quad x(0) = x_0,\ \dot{x}(0) = x_1, \\ y(t) &= C_1x(t) + C_2\dot{x}(t), \end{aligned}$$

where

- the mass and stiffness matrices $M \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ are symmetric and positive definite;
- the damping matrix $D \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite;
- $B \in \mathbb{R}^{n\times m}$, and C_1, $C_2 \in \mathbb{R}^{p\times n}$.

a) Derive the transfer function of this system (including the initial conditions).

b) Transform the system into an LTI system $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}] \in \Sigma_{\widetilde{n},\widetilde{m},\widetilde{p}}$ of first order.

c) Show that the system is asymptotically stable, if the matrix D is positive definite.
Hint: Find a first-order realization in which

$$\mathcal{A} = \begin{bmatrix} 0 & \widetilde{K} \\ -\widetilde{K}^\mathsf{T} & -\widetilde{D} \end{bmatrix}$$

with symmetric positive definite $\widetilde{D}$.

Problem 3: Construct minimal realizations of the following rational functions:

- $G_1(s) = \begin{bmatrix} \frac{1}{s-2} & \frac{2}{s-2} \\ \frac{3}{s-2} & \frac{4}{s-2} \end{bmatrix}$,
- $G_2(s) = \begin{bmatrix} \frac{1}{s-2} & \frac{2}{s-2} \\ \frac{3}{s-2} & \frac{6}{s-2} \end{bmatrix}$.

What are the poles, zeros, and the rank of these matrices?