Universität Hamburg
Fachbereich Mathematik
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Model Reduction Homework Sheet 1.

The problems will be discussed in the exercise on Thursday, April 18.

Problem 1: Let $A \in \mathbb{R}^{n \times n}$ be given. Show the following statements:

a) The ODE $\dot{x}(t) = Ax(t)$ is asymptotically stable, if and only if $\Lambda(A) \subset \mathbb{C}^-$.

b) The ODE $\dot{x}(t) = Ax(t)$ is (Lyapunov) stable (i. e., $x(\cdot)$ remains bounded for all initial conditions), if and only if $\Lambda(A) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R}$ and the eigenvalues on the imaginary axis are semi-simple (i. e., they only have Jordan blocks of size at most 1×1).

Hint: Transform A to Jordan canonical form and consider the matrix exponential.

Problem 2: Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be given. Show that the following statements are equivalent:

a) The pair (A, B) is controllable.

b) It holds that $\operatorname{rank} \begin{bmatrix} \lambda I_n - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$.

c) It holds that $\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = n$.

d) It holds that $v^\mathsf{T} B \neq 0$ for all (right) eigenvectors $v \in \mathbb{C}^n \setminus \{0\}$ of A^T.

Hint: An input connecting two vectors $x(t_0) = x_0$ and $x(t_\mathrm{f}) = x_\mathrm{f}$ in state space is given by

$$u(t) = B^\mathsf{T} \mathrm{e}^{A^\mathsf{T}(t-t_0)} P(t_0, t_\mathrm{f})^{-1} \left(x_\mathrm{f} - \mathrm{e}^{A(t_\mathrm{f}-t_0)} x_0 \right),$$

where

$$P(t_0, t_\mathrm{f}) = \int_{t_0}^{t_\mathrm{f}} \mathrm{e}^{A(t-t_0)} BB^\mathsf{T} \mathrm{e}^{A^\mathsf{T}(t-t_0)} \mathrm{d}t,$$

and it can be shown that controllability is equivalent to invertibility of $P(t_0, t_\mathrm{f})$.

Problem 3: For asymptotically stable A ($\Lambda(A) \subset \mathbb{C}^-$) define the infinite controllability Gramian

$$P = \int_0^\infty \mathrm{e}^{At} BB^\mathsf{T} \mathrm{e}^{A^\mathsf{T} t} \mathrm{d}t.$$

Show that:

a) The matrix P is the solution of the Lyapunov equation

$$AP + PA^\mathsf{T} = -BB^\mathsf{T}. \tag{1}$$

b) The matrix P solving (1) is symmetric and positive semi-definite.

c) The pair (A, B) is controllable, if and only if $P > 0$.

Problem 4: Check the following systems for controllability:

a) $\dot{x}(t) = x(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t);$

b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \dots & \dots & -\alpha_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t).$