Trees of tangles in infinite separation systems

Part I.

with Christian and Jakob

2020-06-09

Reviewing the Splinter Theorem

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Let U be a universe of separations and $(\mathcal{A}_i)_{i \leq n}$ a family of subsets of U. If $(\mathcal{A}_i)_{i \leq n}$ splinters then we can pick an element a_i from each $\mathcal{A}_i$ so that $\{a_1, \ldots, a_n\}$ is nested.

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How can we extend this to the infinite?

The profinite approach

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Apply the splinter theorem to finite restrictions.

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First Observation: If $(\mathcal{A}_i \mid i \in I)$ in $\vec{U}$ splinters, then so does every projection to a $\vec{U}_p$. So, apply the finite Splinter Theorem!

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Second Observation: If we apply the Splinter Theorem to $\vec{U}_p$ and map the nested set to $\vec{U}_q$ we get a splinter solution for $\vec{U}_q$.

For every $\vec{U}_p$ let $\mathcal{N}_p$ be the set of all nested sets which meet every $\mathcal{A}_i \restriction p$.

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For every $\vec{U}_p$ let $\mathcal{N}_p$ be the set of all nested sets which meet every $\mathcal{A}_i \restriction p$. Splinter Theorem says that these are non-empty.

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Consider $N \in \varprojlim (\mathcal{N}_p \mid p \in P)$. We can turn N into a nested set in $\vec{U}$.

If the $\mathcal{A}_i$ are closed, N meets all of them.

Theorem (Profinite Splinter Theorem)
Let $\vec{U} = \varprojlim (\vec{U}_p \mid p \in P)$ be a profinite universe and $(\mathcal{A}_i \mid i \in I)$ a family of non-empty closed subsets of $\vec{U}$. If $(\mathcal{A}_i \mid i \in I)$ splinters then there is a closed nested set $N \subseteq \vec{U}$ containing at least one element from each $\mathcal{A}_i$.

Application to graphs

Let $\mathcal{A}_{P,P'}$ be the set of all efficient P–P'-distinguishers.

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Lemma
$\mathcal{A}_{P,P'}$ is closed for bounded P, P'.

Application to graphs

Lemma
$(\mathcal{A}_{P,P'} \mid P, P'$ bounded, robust and distinguishable profiles in $G)$ splinters.

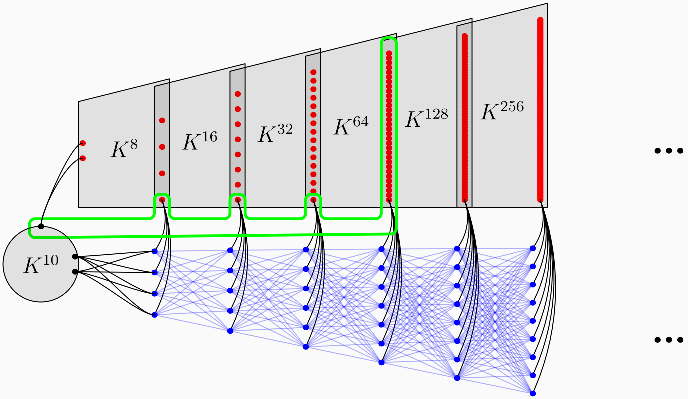
Luckily, the splinter condition was designed for this.

We only need robust and distinguishable.

Done!

Can we build a tree-decomposition from this?

No.



Can we do something without inverse limits?

Relations, Crossing Numbers and Canonicity

Finite crossing number

In graph separations: Crossing number is *strongly submodular*.

Encode this in our splinter-condition.

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Let **nested** be a reflexive and symmetric relation.

Crossing means 'not nested'.

A **corner of** a **and** b is an element c of $\mathcal{A}$, such that anything that crosses c also crosses a or b.

Finite crossing number

Split $(\mathcal{A}_i \mid i \in I)$ into 'levels':

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We take care here not to count multiplicities.

The thin splinter theorem

$(\mathcal{A}_i \mid i \in I)$ **thinly splinters** if:

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3. If $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ cross with $|i| = |j| = k$, then
    - either $\mathcal{A}_i$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_i,
    - or else $\mathcal{A}_j$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_j.

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Theorem (Thin Splinter Theorem)
If $(\mathcal{A}_i \mid i \in I)$ thinly splinters, then there is a canonical set $N \subseteq \mathcal{A}$ which meets every $\mathcal{A}_i$ and is pairwise nested.

Proof

Construct $N_0 \subseteq N_1 \subseteq N_2 \ldots$,
s.t. N_k takes care of all $\mathcal{A}_i$ with $|i| \leq k$.

Set $N_{-1} \coloneqq \emptyset$. In step k:

Let N_k^+ consist of
from each $\mathcal{A}_i$ with $|i| = k$
among those elements nested with N_{k-1}
all of minimum k-crossing number.

Set $N_k \coloneqq N_{k-1} \cup N_k^+$. Need to show:

- For each $\mathcal{A}_i$ we had elements to choose from.
- N_k is nested.

Proof

- For each $\mathcal{A}_i$, $|i| = k$, we had elements to choose from.

That is, $\mathcal{A}_i$ has an element that is nested with N_{k-1}.

By (1) every element of $\mathcal{A}_i$ crosses only finitely many elements of N_{k-1}.
Let a_i be one that crosses as few as possible.

Suppose it crosses some $a_j \in N_{k-1}$, then $a_j \in \mathcal{A}_j$ with $|j| < k$.
By (2), a_i and a_j have a corner in $\mathcal{A}_i$ that is nested with a_j.
$\lightning$ This corner was a better choice for a_i.

Proof

- N_k is nested.

Every element of N_k^+ is nested with N_{k-1} by construction.
Only need to show that N_k^+ is nested.

Suppose a_i and a_j in N_k^+ cross.
By (3) there is a corner of a_i and a_j in $\mathcal{A}_i$ or $\mathcal{A}_j$,
with a strictly lower k-crossing number than
the corresponding a_i or a_j. ϟ

$\square$