# Area-Dependent Quantum Field Theories


Lorant Szegedy

Born 31 December 1990 in Szentes, Hungary



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*Supervisor:* Prof. Dr. Ingo Runkel

*Referee:* Prof. Dr. Jörg Teschner

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Department of Mathematics

Faculty of Mathematics, Informatics and Natural Sciences

University of Hamburg

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Hamburg, 28. September 2015 Lorant Szegedy

# Abstract

In this thesis we study area-dependent quantum field theories. We classify them in terms of regularized $*$-Frobenius algebras, which generalize the notion of $*$-Frobenius algebras. We then give a way of building examples, the so-called lattice construction. Finally we describe the example motivating the study of these theories, the 2d Yang–Mills theory, with our newly developed techniques.

# Contents

# Introduction

Quantum field theories in general depend on the metric on the manifold they are defined on, therefore these theories are complicated to describe and analyze mathematically. However one can consider simpler theories, where the correlation functions depend for example only on the conformal structure of the manifold, these are called conformal field theories. If the correlation functions do not depend on the metric at all, only on the topology of the manifold, we call the quantum field theory topological. Both types of theories have been extensively discussed in the literature and an explicit mathematical description is available for topological field theories (or TFTs) in any dimensions [1]. The most widely known TFTs are however in 2 [2] and 3 dimensions [3, 4]. One can also consider a theory, where the correlation functions depend on the metric only through the volume of the manifold. Unlike the former two theories, the latter has not been treated so thoroughly in the literature.

One can find a description of such 2-dimensional theories, called area-dependent quantum field theories (or AQFTs), in [5, 6] and [7], but none of them discusses the general theory in detail, they only describe an example. This example of AQFTs, which motivates the detailed analysis of the general theory, is the 2-dimensional Yang–Mills theory, explained in e.g. [8]. There the area works as the regulator: a general correlation function is well-defined only for non-zero area. While it is a well-known fact that the state spaces of 2d TFTs are finite dimensional [2], this minor modification of allowing area-dependence enables one to consider theories with infinite dimensional state spaces. This can be seen in the Yang–Mills theory, where the state space is the Hilbert space of square integrable class functions on a compact Lie group.

The thesis is structured as follows. Chapter 1 contains the algebraic tools needed for the treatment of AQFTs. We start by summarizing the theory of Frobenius algebras, then introduce the notion of $*$-categories and $*$-Frobenius algebras. Then we give an overview of compact operators focusing on their spectral theory. We give the definition of regularized $*$-Frobenius algebras (or RFAs), generalizing the notion of $*$-Frobenius algebras and show that they are governed by a family of compact operators. Then we show how under certain continuity condition RFAs decompose into a family of $*$-Frobenius algebras and that given a family of $*$-Frobenius algebras obeying certain non-trivial convergence conditions how we can recover the notion of an RFA. We finally show that these two constructions are inverse to each other.

In chapter 2 after discussing the category of smooth bordisms and topological field theories, we introduce the category of bordisms with areas (which are strictly positive). Using this notion we define area-dependent quantum field theories as symmetric monoidal functors from the latter category to the category of Hilbert spaces. We introduce the notion of strongly continuous AQFTs, which are described by strongly continuous RFAs, and which are therefore classified. Finally we consider AQFTs, which allow zero areas.

In chapter 3 we introduce a cell decomposition of bordisms, which we use to describe the lattice construction of TFTs. Then we show how the data needed for the lattice construction is encoded in a $\Delta$-separable symmetric Frobenius algebra, and that the state space of a lattice TFT is the center of this algebra. We generalize this construction to the area-dependent case, show how a $\Delta$-separable symmetric RFA describes the lattice construction of an AQFT and that its state space is given by a generalized version of the center.

Finally in chapter 4 we describe an example of AQFTs, the 2d Yang–Mills theory using our new formalism. We show that the Boltzmann weights are well-defined only for a non-zero area. We compute the state space of the theory and calculate the RFA, which describes the lattice construction and compute its center in order to make the connection of the general theory to this example explicit. Finally we calculate some zero area limits and show explicitly that the convolution product is an unbounded operator on $L^2(G)$.

In appendix A we describe the category of Hilbert spaces and show how one can endow it with a symmetric monoidal structure, usually discussed only for the full subcategory of finite dimensional Hilbert spaces.

The following results can to my knowledge not be found in the literature and are new contributions:

- the definition of regularized Frobenius algebras, in particular the careful treatment of their analytic properties,
- the classification of strongly continuous regularized $*$-Frobenius algebras,
- the classification of strongly continuous area-dependent quantum field theories in terms of regularized Frobenius algebras,
- the lattice construction of area-dependent quantum field theories, however the general ideas for TFTs are taken from the literature [9],
- the convergence results in the 2d Yang–Mills theory, however the key estimate is taken from from the literature [10].

This new formalism could allow one to systematically study 1-dimensional defects in area dependent quantum field theories. We would like to describe these as "bimodules" over regularized $*$-Frobenius algebras, which would generalize the results for TFTs [9]. Then we could apply this formalism to the 2d Yang–Mills theory and we hope that we can identify Wilson loop observables as examples of such defects.

Another goal is to fit the so-called q-deformed 2d Yang–Mills theory, which is a gauge theory based on a quantum group [11], into this framework. Here we would like to show that the observables - the gauge invariants - correspond to 1d defects. Then we will try to use this theory to calculate different quantities in some higher dimensional supersymmetric gauge theories [12].

I am indebted to André Henriques for explaining the decomposition of AQFTs into possibly infinite direct sums of TFTs combined with a simple area law. This observation was the basis of the classification result of strongly continuous AQFTs.

# Chapter 1

# Algebraic Considerations

## 1.1 Frobenius Algebras

Let $\mathcal{C}$ be a strict monoidal category with tensor product $\otimes$ and tensor unit $I$. An algebra in $\mathcal{C}$ is an object $A$ together with morphisms $\mu_A : A \otimes A \to A$, $\eta_A : I \to A$ (called product and unit respectively) such that

- $\mu_A \circ (id_A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes id_A)$ (associativity) and
- $\mu_A \circ (id_A \otimes \eta_A) = \mu_A \circ (\eta_A \otimes id_A) = id_A$ (unitality).

A morphisms of algebras $A, B \in \mathcal{C}$ is a morphism $A \xrightarrow{\phi} B$ in $\mathcal{C}$, such that

- $\phi \circ \mu_A = \mu_B \circ (\phi \circ \phi)$ and $\phi \circ \eta_A = \eta_B$.

A left module over an algebra $A \in \mathcal{C}$ (or an $A$-module in short) is an object $M \in \mathcal{C}$ together with a morphism $\mu_M : A \otimes M \to M$ in $\mathcal{C}$ such that

- $\mu_M \circ (id_A \otimes \mu_M) = \mu_M \circ (\mu_A \otimes id_M)$ and
- $\mu_M \circ (\eta_A \otimes id_M) = id_M$.

One can define right modules analogously. We call a morphism $\phi : M \to N$ in $\mathcal{C}$ between two $A$-modules an $A$-module morphism, if $\phi \circ \mu_M = \mu_N \circ (id_A \otimes M)$.

A Frobenius algebra in $\mathcal{C}$ is an algebra $A$ in $\mathcal{C}$ together with a morphism $\varepsilon_A : A \to I$, such that the morphism $\beta_A := \varepsilon_A \circ \mu_A : A \otimes A \to I$ (the pairing) is non-degenerate. By this we mean that there is a morphism $\gamma_A : I \to A \otimes A$ (the copairing), such that

$$(id_A \otimes \beta_A) \circ (\gamma_A \otimes id_A) = (\beta_A \otimes id_A) \circ (id_A \otimes \gamma_A) = id_A. \tag{1.1}$$

Note that this equation determines the copairing uniquely. We will not write the indices of the morphisms, when it is understood, which Frobenius algebra we are talking about. We have an equivalent definition of Frobenius algebras in any monoidal category $\mathcal{C}$ [2]:

**Proposition 1.** *Let $F$ be an object in $\mathcal{C}$ together with morphisms $\mu_F : F \otimes F \to F$, $\eta_F : I \to F$ , $\Delta_F : F \to F \otimes F$ and $\varepsilon_F : F \to I$ (the product, unit, coproduct and counit respectively), such that the following relations hold:*

1. $\mu_F \circ (id_F \otimes \eta_F) = \mu_F \circ (\eta_F \otimes id_F) = id_F$ *(unitality),*

2. $(id_F \otimes \varepsilon_F) \circ \Delta_F = (\varepsilon_F \otimes id_F) \circ \Delta_F = id_F$ *(counitality) and*

3. $\mu_F \circ \Delta_F = (\mu_F \otimes id_F) \circ (id_F \otimes \Delta_F) = (id_F \otimes \mu_F) \circ (\Delta_F \otimes id_F)$ *(Frobenius relation).*

*Then $F$ together with $\mu_F$, $\eta_F$ and $\varepsilon_F$ is a Frobenius algebra.*

*Conversely let $F$ together with $\mu_F$, $\eta_F$ and $\varepsilon_F$ be a Frobenius algebra in $\mathcal{C}$, with copairing $\gamma_F$. Set*

$$\Delta_F := (\mu_F \otimes id_F) \circ (id_F \otimes \gamma_F) : F \to F \otimes F. \tag{1.2}$$

*Then $\mu_F$, $\eta_F$, $\Delta_F$ and $\varepsilon_F$ satisfy the relations 1-3.*

The proof is essentially the same when one considers $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$ (defined below), in which case the proof can be found in [2] section 2.3. Note that the Frobenius relation can be used to show that associativity holds. Furthermore it can be used to show coassociativity [2]:

$$(\Delta_F \otimes id_F) \circ \Delta_F = (id_F \otimes \Delta_F) \circ \Delta_F. \tag{1.3}$$

There is a useful tool in strict monoidal categories for doing calculations. We draw string diagrams for morphisms, an example is shown on figure 1.1. We draw the tensor product of morphisms by placing the two diagrams next to each other, and we draw the composition of two morphisms by placing the two diagrams on top of each other. Some examples are given on figure 1.2. For more details on this graphical calculus see e.g. [13] chapter XIV.1.

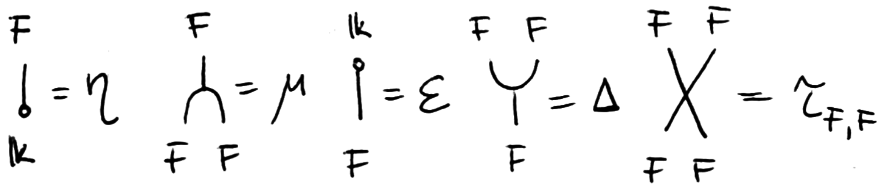

Figure 1.1: Graphical notation of the Frobenius algebra maps and the braiding

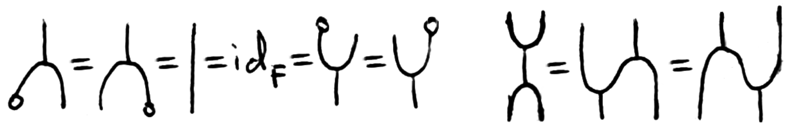

Figure 1.2: Graphical notation of some relations in a Frobenius algebra

Let $\mathbf{Frob}(\mathcal{C}) \subset \mathcal{C}$ denote the category of Frobenius algebras in $\mathcal{C}$. Its objects are Frobenius algebras in $\mathcal{C}$ and its morphisms are morphisms $A \xrightarrow{\phi} B$ in $\mathcal{C}$ for $A$, $B$ Frobenius algebras, which satisfy,

- $\phi \circ \mu_A = \mu_B \circ (\phi \circ \phi)$, $\phi \circ \eta_A = \eta_B$,
- $(\phi \circ \phi) \circ \Delta_A = \Delta_B \circ \phi$, $\varepsilon_A = \varepsilon_B \circ \phi$.

There is an interesting feature of Frobenius algebras ([2] lemma 2.4.5).

**Proposition 2.** $\mathbf{Frob}(\mathcal{C})$ *is a groupoid.*

*Proof.* Let $A \xrightarrow{\phi} B$ be a morphism in $\mathbf{Frob}(\mathcal{C})$ and set

$$\overline{\phi} := (\varepsilon_B \circ \mu_B \otimes id_A) \circ (id_B \otimes \phi \otimes id_A) \circ (id_B \otimes \Delta_A \circ \eta_A)\,. \tag{1.4}$$

Then we have $\phi \circ \overline{\phi} = id_B$, calculated on figure 1.3 and $\overline{\phi} \circ \phi = id_A$, which can be calculated similarly, so $\phi$ is an isomorphism. $\square$

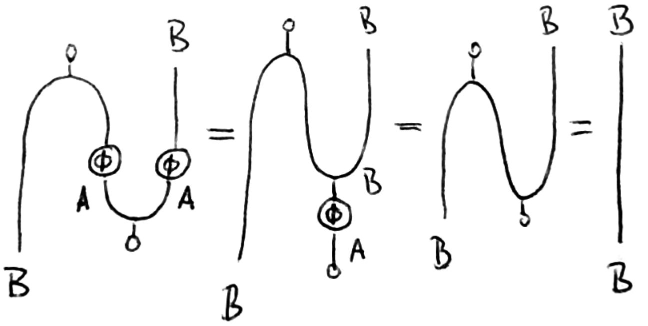

Figure 1.3: First use that $\phi \circ \mu_A = \mu_B \circ (\phi \circ \phi)$, then that $\phi \circ \eta_A = \eta_B$. In the last step use the Frobenius relation, then unitality and counitality.

If $\mathcal{C}$ is furthermore endowed with a braiding $\tau$, then we call an algebra $A$ commutative, if $\mu_A \circ \tau_{A,A} = \mu_A$ and write $\mathbf{cFrob}(\mathcal{C})$ for the subcategory of commutative Frobenius algebras in $\mathcal{C}$. Note that a Frobenius algebra is commutative, if and only if it is cocommutative, i.e. $\tau_{A,A} \circ \Delta_A = \Delta_A$. Now recall the definition of the pairing: $\beta_A = \varepsilon_A \circ \mu_A$. We call a Frobenius algebra $A$ symmetric, if the pairing is symmetric: $\beta_A = \beta_A \circ \tau_{A,A}$. Then the uniqueness of the copairing implies that $\gamma_A = \tau_{A,A} \circ \gamma_A$.

Let $\Bbbk$ be a field. We denote by $\mathbf{Vect}_\Bbbk$ the category of vector spaces over $\Bbbk$. It has objects vector spaces over $\Bbbk$ and morphisms $\Bbbk$-linear maps. $\mathbf{Vect}_\Bbbk$ becomes a monoidal category via the tensor product of vector spaces ($\otimes$) and tensor unit $\Bbbk$ and the flip map $v \otimes w \mapsto w \otimes v$ endows it with a symmetric structure. This is not a strict monoidal category, but as usual[1] we will work with an equivalent strict monoidal category instead and use the same notation.

If $\mathcal{C} = \mathbf{Vect}_\Bbbk$ with the above braiding then we call $(\mathbf{c})\mathbf{Frob}(\mathbf{Vect}_\Bbbk)$ the category of (commutative) Frobenius algebras over $\Bbbk$ and write $(\mathbf{c})\mathbf{Frob}_\Bbbk$ for simplicity. Since the copairing exists for every $A \in \mathbf{Frob}_\Bbbk$, $A$ is finite dimensional. For the proof see [2] proposition 2.3.24.

**Proposition 3.** *Let $A$ be an algebra in $\mathbf{Vect}_\Bbbk$. Then the following are equivalent:*

1. *$A$ is semi-simple as a left module over itself,*
2. *every left $A$-module is semi-simple,*
3. *every left $A$-module is projective.*

[1]See e.g. [13] chapter XI.5.

For the proof see e.g. [14] proposition 13.9 and 17.2. We call an algebra $A$ semi-simple, if it satisfies the equivalent conditions of proposition 3. We call a Frobenius algebra semi-simple, if it is semi-simple as an algebra. An algebra $A \in \mathbf{Vect}_{\Bbbk}$ is called separable, if there exists an $A$-module morphism $\iota : A \to A \otimes A$ such that $\mu_A \circ \iota = id_A$. A Frobenius algebra $A$ is $\Delta$-separable, if $\mu_A \circ \Delta_A = id_A$.

**Lemma 4.** *Let $A \in \mathbf{Vect}_{\Bbbk}$ be a separable algebra. Then $A$ is semi-simple.*

*Proof.* Let ${}_AM$ be a left $A$-module with action $\mu_M$ and consider the left $A$-module ${}_AA \otimes M$ where $A$ acts only on the $A$ factor. Then ${}_AA \otimes M$ is a free $A$-module.

Let

$$\iota_M := (id_A \otimes \mu_M) \circ (\iota \circ \eta_A \otimes id_M) : {}_AM \to {}_AA \otimes M.$$

Since $A$ is separable we have

$$\mu_M \circ \iota_M = \mu_M \circ ((\mu_M \circ \iota \circ \eta_A) \otimes id_M)) = \mu_M \circ (\eta_A \otimes id_M) = id_M,$$

so ${}_AM$ is the direct summand of a free module, therefore it is a projective module. Therefore $A$ is a semi-simple algebra by proposition 3. $\square$

**Corollary 5.** *If for $A \in \mathbf{Frob}_{\Bbbk}$ we have*

$$\mu_A \circ \Delta_A = \xi id_A, \tag{1.5}$$

*with $\xi \in \Bbbk \setminus \{0\}$, then $A$ is semi-simple.*

## 1.2 $*$-Frobenius Algebras

In order to define $*$-Frobenius algebras we need two new notions. A $*$-structure or (dagger structure) on a category $\mathcal{C}$ is a contravariant functor $*_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, which is the identity on objects and an involution on morphisms, i.e. $*_{\mathcal{C}}^2 = Id_{\mathcal{C}}$. We call $\mathcal{C}$ a $*$-category, if it is equipped with a $*$-structure [15]. If the category $\mathcal{C}$ is monoidal with associator $\alpha$, left and right unit constraints $r$ and $l$, then we require that $\alpha^* = \alpha^{-1}$, $r^* = r^{-1}$ and $l^* = l^{-1}$, and furthermore that $(f \otimes g)^* = f^* \otimes g^*$ for all morphisms $f$ and $g$. If the category $\mathcal{C}$ is braided with braiding $c$, then we require that $c^* = c^{-1}$.

Let $\mathcal{C}$, $\mathcal{D}$ be categories with $*$-structures $*_{\mathcal{C}}$ and $*_{\mathcal{D}}$ respectively. We call a functor $F : \mathcal{C} \to \mathcal{D}$ a $*$-functor, if it commutes with the $*$-structures, i.e.

$$F \circ *_{\mathcal{C}} = *_{\mathcal{D}} \circ F. \tag{1.6}$$

A natural transformation of $*$-functors is a natural transformation of the functors. If $\mathcal{C}$ and $\mathcal{D}$ were additionally monoidal and $F$ was a monoidal functor with natural isomorphisms $\phi_2$ and $\phi_0$, then we require that $\phi_2^* = \phi_2^{-1}$ and $\phi_0^* = \phi_0^{-1}$. Now let us see an example of a $*$-category.

**Definition 6.** *Let* **Hilb** *denote the category of Hilbert spaces, whose*

- *objects are Hilbert spaces over $\mathbb{C}$,*
- *morphisms are bounded linear maps between Hilbert spaces.*

We write $\mathcal{B}(\mathcal{H},\mathcal{K})$ for the set of bounded linear maps $\mathcal{H} \to \mathcal{K}$, for $\mathcal{H}$, $\mathcal{K}$ Hilbert spaces. Note that **Hilb** is a symmetric monoidal category with the completed algebraic tensor product ($\hat{\otimes}$), for details see appendix A. A $*$-structure on **Hilb** is to take the adjoint of a bounded linear map. Let us keep this $*$-structure on **Hilb** fixed from now on.

**Definition 7.** *A $*$-Frobenius algebra is a Frobenius algebra in a monoidal $*$-category such that*

$$\Delta^* = \mu,\ \varepsilon^* = \eta. \tag{1.7}$$

*A morphism of $*$-Frobenius algebras is a morphism of the Frobenius algebras.*

**Proposition 8.** *Every $*$-Frobenius algebra in* **Hilb** *is semi-simple.*

*Proof.* Let $E$ denote a $*$-Frobenius algebra in **Hilb** and let $t := \mu \circ \Delta = \Delta^* \circ \Delta$. It is a self-adjoint operator, so it can be diagonalized[2] and $E$ decomposes into Hilbert spaces as

$$E = \bigoplus_{\alpha=1}^{N} F_\alpha, \tag{1.8}$$

where $F_\alpha$ is the eigenspace of $t$ with eigenvalue $t_\alpha$. Without loss of generality we can assume that

$$t_\alpha \neq t_\beta,\ \text{if}\ \alpha \neq \beta. \tag{1.9}$$

Now we show that this is a direct sum of Frobenius algebras.

From the Frobenius relation and associativity we have

$$\begin{aligned} t \circ \mu =& \mu \circ (\Delta \circ \mu) = \mu \circ (\mu \otimes id) \circ (id \otimes \Delta) \\ =& \mu \circ (id \otimes \mu) \circ (id \otimes \Delta) = \mu \circ (id \otimes t), \end{aligned} \tag{1.10}$$

i.e. $t$ is a left module map and one similarly shows that it is a right module map as well. Let $\alpha \neq \beta$ and take $a \in F_\alpha$, $b \in F_\beta$. Then $t(a) = t_\alpha a$ and $t(b) = t_\beta b$ and we have

$$\begin{aligned} t(ab) =& at(b) = t_\beta ab \\ =& t(a)b = t_\alpha ab \end{aligned} \tag{1.11}$$

using (1.10). Then (1.11) and (1.9) show that $ab = 0$, so (1.8) is a decomposition as algebras.

Similarly one shows that equation 1.8 is a decomposition as coalgebras. We have

$$\begin{aligned} \Delta \circ t =& \Delta \circ \mu \circ \Delta = (\mu \otimes id) \circ (id \otimes \Delta) \circ \Delta \\ =& (\mu \otimes id) \circ (\Delta \otimes id) \circ \Delta = (t \otimes id) \circ \Delta \end{aligned}$$

using the Frobenius relation and coassociativity. One similarly shows that $\Delta \circ t = (id \otimes t) \circ \Delta$. Thus we have for $\forall a \in F_\alpha$, using Sweedler notation:

$$\begin{aligned} \Delta(t(a)) =& t(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes t(a_{(2)}) \\ =& t_\alpha \Delta(a) = t_\alpha a_{(1)} \otimes a_{(2)}, \end{aligned} \tag{1.12}$$

[2] It is also compact as $E$ is finite dimensional, as noted at the end of section 1.1, so proposition 12 applies.

which shows that the comultiplication restricted to $F_\alpha$ lands in $F_\alpha \otimes F_\alpha$.

Let $F_0$ denote the kernel of $t$, which is a Frobenius algebra. We have $t(x) = \Delta^* \circ \Delta(x) = 0$ for every $x \in F_0$, which is equivalent to $0 = \langle y, \Delta^* \circ \Delta(x)\rangle = \langle \Delta(y), \Delta(x)\rangle$ for every $y \in F_0$. So in particular $\langle \Delta(x), \Delta(x)\rangle = 0$, so $\Delta(x) = 0$ for every $x \in F_0$, which is absurd in a Frobenius algebra, as counitality cannot be satisfied. Therefore we conclude that $t$ is injective.

Now the only thing left to show is that each summand $F_\alpha$ is semi-simple but it follows from corollary 5. □

By the Wedderburn–Artin theorem[3] every semi-simple algebra over $\mathbb{C}$, in particular every $*$-Frobenius algebra, is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$. Note that there are different Frobenius algebra structures on $Mat_n(\mathbb{C})$, the standard one is when one takes the trace to be the counit $\varepsilon$.

Let us now see an example of $*$-Frobenius algebras. Let $\mathbb{C}_\epsilon$ denote $\mathbb{C}$ with its standard Hilbert space structure (and with basis vector 1), $\epsilon$ a non-zero complex number, and set

- $\varepsilon(1) := \epsilon$, $\eta(1) := \epsilon^*$,
- $\Delta(1) := \frac{1}{\epsilon} 1 \otimes 1$, $\mu(1 \otimes 1) := \frac{1}{\epsilon^*}$.

It is not hard to see that the above equations define a $*$-Frobenius algebra structure and that $\mathbb{C}_{\epsilon_1}$ and $\mathbb{C}_{\epsilon_2}$ are isomorphic as Frobenius algebras, if and only if $|\epsilon_1| = |\epsilon_2|$. The following corollary shows that actually all commutative $*$-Frobenius algebras are the sums of such.

**Corollary 9.** *Every commutative $*$-Frobenius algebra $E$ in* **Hilb** *is isomorphic to*

$$\bigoplus_{i=1}^{\dim E} \mathbb{C}_{\epsilon_i} \tag{1.13}$$

*as $*$-Frobenius algebras for some values of $\epsilon_i$.*

*Proof.* We know from proposition 8, that every $*$-Frobenius algebra is semi-simple. By the Wedderburn–Artin theorem, every semi-simple algebra is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$ and a commutative matrix algebra is 1-dimensional. So

$$E \cong \bigoplus_{i=1}^{\dim E} \mathbb{C}$$

as algebras, i.e. $E$ has a basis $\{e_i\}_{i=1}^{\dim E}$, in which the multiplication is diagonal:

$$\mu_E(e_i \hat{\otimes} e_j) = \delta_{ij} \mu_E(e_i \hat{\otimes} e_i) \propto e_i.$$

Define $\epsilon_i$, such that $\mu_E(e_i \hat{\otimes} e_i) = e_i / \epsilon_i^*$. Note that $\epsilon_i$ cannot be zero, because then the pairing $\varepsilon_E \circ \mu_E$ would be degenerate. This basis can also be chosen to be orthonormal w.r.t the inner product on $E$ and therefore we get an isometry.

The above decomposition is a decomposition of coalgebras:

$$\langle e_i \hat{\otimes} e_j, \Delta_E(e_k)\rangle = \langle \mu_E(e_i \hat{\otimes} e_j), e_k\rangle = \frac{\delta_{ij}}{\epsilon_i} \langle e_i, e_k\rangle = \frac{\delta_{ij}\delta_{ik}}{\epsilon_i},$$

where we used that $\Delta_E^* = \mu_E$. □

[3] See e.g. [14] theorem 13.7.

## 1.3 Compact Operators

It will be useful later on to collect some properties of compact operators. Let $\mathcal{H}, \mathcal{K} \in \mathbf{Hilb}$. We call $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ compact, if there exists a sequence of finite-rank operators $T_n \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ $(n = 1, 2, \dots)$, such that

$$\|T - T_n\| \xrightarrow{n\to\infty} 0. \tag{1.14}$$

In particular, finite-rank operators are compact.

**Proposition 10.** *Let $S \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ and $T \in \mathcal{B}(\mathcal{K}', \mathcal{K}'')$, one of them compact. Then $T \circ S$ is compact.*

*Proof.* See [16] theorem VI.12. □

**Proposition 11.** *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{K})$ be a compact operator on $\mathcal{K} \in \mathbf{Hilb}$. Then*

1. *the spectrum of $T$ is at most countably infinite,*
2. *if $\lambda \neq 0$ is in the spectrum of $T$ then it is an eigenvalue of $T$,*
3. *the eigenspace of $T$ for eigenvalue $\lambda \neq 0$ is finite dimensional,*
4. *the only possible accumulation point of the spectrum of $T$ is 0,*
5. *if $\mathcal{K}$ is infinite dimensional then 0 is in the spectrum of $T$.*

*Proof.* See corollary 6.34. and proposition 6.35 of [17]. □

A resolution of the identity on $\mathcal{K} \in \mathbf{Hilb}$ is a family $\{R_i\}_{i\in I}$ of orthogonal projections $(R_i R_j = \delta_{i,j} R_j)$ on $\mathcal{K}$ such that $\sum_{i\in I} R_i x = x$ for every $x \in \mathcal{K}$.[4]

An operator $T \in \mathcal{B}(\mathcal{K}, \mathcal{K})$ is called diagonalizable, if there exists a resolution of the identity $\{R_i\}_{i\in I}$ on $\mathcal{K}$ and a bounded family of complex numbers $\{\lambda_i\}_{i\in I}$ such that $Tx = \lambda_i x$, when $x$ is in the range of $R_i$. The operators $X, Y \in \mathcal{B}(\mathcal{K}, \mathcal{K})$ are called simultaneously diagonalizable, if there exists a resolution of the identity on $\mathcal{K}$, which diagonalizes both $X$ and $Y$.

**Proposition 12.** *A compact operator $T \in \mathcal{B}(\mathcal{K}, \mathcal{K})$ is diagonalizable, if and only if it is normal (commutes with its adjoint).*

*Proof.* See corollary 6.45. of [17]. □

**Lemma 13.** *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{K})$ be a compact diagonalizable operator. Then there exists a family of Hilbert spaces $\{E_n\}_{n=0}^{\infty}$, such that*

- $\mathcal{K} = \overline{\bigoplus_{n=0}^{\infty} E_n}$,
- *$E_n$ is an eigenspace of $T$.*

*Proof.* Since $T$ is diagonalizable, there exists a resolution of identity $\{R_i\}_{i\in I}$ with some index set $I$. Let us suppose that the eigenvalues corresponding to the $R_i$'s are distinct. Then since $T$ is compact the number of distinct eigenvalues are at most countably infinite by proposition 11, so the index set $I$ can actually be taken to be the natural numbers.

[4]For details on sums indexed by an uncountable set see [17] section 5.7. Later we will see that it is sufficient to consider separable Hilbert spaces and sums over countable sets.

Let $E_n$ be the range of $R_n$. Then we have $\mathcal{K} = \overline{\bigoplus_{n=0}^{\infty} E_n}$ and closure is necessary, if $\mathcal{K}$ is infinite dimensional and $T$ has an infinite number of distinct eigenspaces. The last point of proposition 11 shows that even an injective compact operator has 0 in its spectrum and a diagonalizable (hence normal) operator does not have 0 in its residual spectrum (see corollary 6.18 b) in [17]), hence 0 has to be in its continuous spectrum. This means that the the image of $T$ is only dense in $\mathcal{H}$. $\square$

This lemma in particular shows that if $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is an injective compact diagonalizable operator, then $\mathcal{H}$ is separable, since $\mathcal{H}$ is the completed direct sum of countably many finite dimensional Hilbert spaces.

**Lemma 14.** *Let $\underline{T} = \{T_i \in \mathcal{B}(\mathcal{K}, \mathcal{K})\}_{i \in I}$ be a family of diagonalizable compact operators, such that either one of the following conditions is satisfied:*

1. *$I$ is a finite set, or*
2. *$\mathcal{K}$ is finite dimensional.*

*Then the operators in $\underline{T}$ are simultaneously diagonalizable, if and only if the operators are pairwise commuting ($T_iT_j = T_jT_i$)*

*Proof.* We will prove the statement for each case separately.

1. Assume that $I$ is a finite set. Then it is enough to show this for two operators $S$ and $T$ and then use induction on the number of operators.

   It is clear that if they are simultaneously diagonalizable, then they commute. Pick a common eigenvector $v$, then $[S, T]v = 0$ and any vector in $\mathcal{K}$ can be approximated by a linear combination of eigenvectors.

   Now suppose that they commute and let $\overline{\bigoplus_{n=0}^{\infty} E_n} = \mathcal{K}$ be the decomposition of $\mathcal{K}$ into eigenspaces of $S$. Then $E_n$ is a $T$ invariant subspace, as $STx = TSx = \lambda_n Tx$, with $x \in E_n$ and $\lambda_n$ the eigenvalue of $S$ on $E_n$. Hence $T$ can be diagonalized on $E_n$, giving the simultaneous diagonalization of $S$ and $T$.

2. Assume that $\mathcal{K}$ is finite dimensional. Then the subspace of $\mathcal{B}(\mathcal{K}, \mathcal{K})$ spanned by the operators $T_i$ is finite dimensional. It is enough to show the statement for the basis of this subspace and it is implied by the first part.

$\square$

**Proposition 15.** *Let $\{P_a \in \mathcal{B}(\mathcal{H}, \mathcal{H}), a > 0\}$ denote a semi-group of diagonalizable compact operators ($P_a \circ P_b = P_{a+b}$). Then there is a family of Hilbert spaces $\{E_n\}_{n=0}^{\infty}$, such that $\mathcal{K} = \overline{\bigoplus_{n=0}^{\infty} E_n}$ and for every $a$*

- *$E_0 = \mathrm{Ker} P_a$,*
- *$E_n$ is an eigenspace of $P_a$ and is finite dimensional for every $n \geq 1$.*

*Proof.* Let us fix $a_0$ and apply lemma 13 on $P_{a_0}$. Then $\mathcal{K} = \overline{\bigoplus_{n=0}^{\infty} F_n}$, where $F_0$ is the kernel of $P_{a_0}$. Consider $F_n$ for $n \neq 0$, it is finite dimensional by proposition 11, so we can simultaneously diagonalize on $F_n$ the operators $P_a$

for every $a$ by lemma 14. Let $x \in F_n$ be a common eigenvector for every $P_a$: $P_a x = c(a)x$, where $c : \mathbb{R}_+ \to \mathbb{C}$. Then we have

$$P_a \circ P_b x = c(a)c(b)x = c(a+b)x \implies c(a)c(b) = c(a+b) \tag{1.15}$$

as $P_a \circ P_b = P_{a+b}$. This equation implies that either $c(a) = 0$ for every $a$ or $c(a) \neq 0$ for every $a$ (for details see [18] page 28). Since $c(a_0) \neq 0$, none of the $F_n$ for $n \geq 1$ contain 0 eigenvalue eigenspaces of $P_a$ for any $a$. We have just shown that

$$\mathrm{Ker} P_a \subset \mathrm{Ker} P_{a_0} = F_0 \tag{1.16}$$

for every $a$. So if we exchange the role of $a$ and $a_0$, we get the converse inclusion, therefore the kernels coincide, so set $E_0 = F_0$. The previous diagonalization delivers the eigenspaces $E_n$ for $n \geq 1$. □

## 1.4 Regularized Frobenius Algebras

Motivated by area-dependent quantum field theories (defined in chapter 2), we would like to generalize the notion of $*$-Frobenius algebras, which will allow infinite dimensional spaces.

**Definition 16.** *A regularized Frobenius algebra $\mathcal{F}$ is an object in* **Hilb** *together with families of morphisms*

$$\begin{aligned} &\left\{\mu_a : \mathcal{F}^{\hat{\otimes}2} \to \mathcal{F}\right\}, \quad \left\{\eta_a : \mathbb{C} \to \mathcal{F}\right\}, \\ &\left\{\Delta_a : \mathcal{F} \to \mathcal{F}^{\hat{\otimes}2}\right\}, \quad \left\{\varepsilon_a : \mathcal{F} \to \mathbb{C}\right\}, \end{aligned}$$

*where $a \in \mathbb{R}_+$, such that the following relations hold. Let $a_i, b_i, c_i, d_i \in \mathbb{R}_+$ ($i = 1, 2$), such that $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = d_1 + d_2 = a$. Then we require that*

$$\begin{aligned} &\mu_{a_1} \circ \left(id_{\mathcal{F}} \hat{\otimes} \eta_{a_2}\right) = \mu_{b_1} \circ \left(\eta_{b_2} \hat{\otimes} id_{\mathcal{F}}\right) \\ &= \left(id_{\mathcal{F}} \hat{\otimes} \varepsilon_{c_1}\right) \circ \Delta_{c_2} = \left(\varepsilon_{d_1} \hat{\otimes} id_{\mathcal{F}}\right) \circ \Delta_{d_2} =: P_a, \end{aligned} \tag{1.17}$$

$$\Delta_{a_1} \circ \mu_{a_2} = \left(id_{\mathcal{F}} \hat{\otimes} \mu_{b_1}\right) \circ \left(\Delta_{b_2} \hat{\otimes} id_{\mathcal{F}}\right) = \left(\mu_{c_1} \hat{\otimes} id_{\mathcal{F}}\right) \circ \left(id_{\mathcal{F}} \hat{\otimes} \Delta_{c_2}\right), \tag{1.18}$$

$$P_{a_1} \circ \mu_{a_2} = \mu_a, \ \Delta_{a_1} \circ P_{a_2} = \Delta_a, \tag{1.19}$$

*and that $P_a$ is injective. We call $\mathcal{F}$ a regularized $*$-Frobenius algebra, if furthermore*

$$\mu_a^* = \Delta_a \text{ and } \eta_a^* = \varepsilon_a. \tag{1.20}$$

We the abbreviation RFA for regularized $*$-Frobenius algebras, as these will appear mostly in this work. Note that equations 1.17 and 1.18 generalize the properties of a Frobenius algebra, the two equations in (1.20) generalize $*$-Frobenius algebras. Injectivity of $P_a$ is also required in order to be able to recover the definition of a Frobenius algebra. Using equations 1.18 and 1.19, we also have that

$$\mu_{a_1} \circ \left(id_{\mathcal{F}} \hat{\otimes} P_{a_2}\right) = \mu_{b_1} \circ \left(P_{b_2} \hat{\otimes} id_{\mathcal{F}}\right) = \mu_a, \tag{1.21}$$

$$\left(id_{\mathcal{F}} \hat{\otimes} P_{a_1}\right) \circ \Delta_{a_2} = \left(P_{b_1} \hat{\otimes} id_{\mathcal{F}}\right) \circ \Delta_{b_2} = \Delta_a, \tag{1.22}$$

$$P_{a_1} \circ \eta_{a_2} = \eta_a, \tag{1.23}$$

$$\varepsilon_{b_1} \circ P_{b_2} = \varepsilon_a, \tag{1.24}$$

$$P_{a_1} P_{a_2} = P_a, \tag{1.25}$$

where $a_i, b_i$ $(i = 1, 2)$ and $a$ are as in definition 16. Some of the relations in a regularized Frobenius algebra are shown on figure 1.4, we write the parameters of the morphisms next to their diagram.

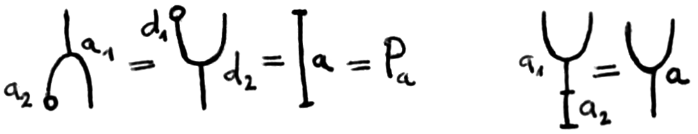


Figure 1.4: Some relations in a regularized Frobenius algebra

**Proposition 17.** *In a regularized $*$-Frobenius algebra the operators $P_a$ are self-adjoint and of trace class, hence compact.*

*Proof.* To show that $P_a$ are self-adjoint, just write using the definition of $P_a$ in equation 1.17 that

$$P_a = \left(id_{\mathcal{F}}\hat{\otimes}\varepsilon_{a/2}\right) \circ \Delta_{a/2} = \mu_{a/2} \circ \left(\eta_{a/2}\hat{\otimes}id_{\mathcal{F}}\right)$$

and use equation 1.20. The proof of $P_a$ being of trace-class is as follows: Set $\beta_a = \varepsilon_{a_1} \circ \mu_{a_2}$ and $\gamma_a = \Delta_{a_1} \circ \eta_{a_2}$. Then by equations 1.18 and 1.17 we have

$$P_a = \left(id_{\mathcal{F}}\hat{\otimes}\beta_{a_1}\right) \circ \left(\gamma_{a_2}\hat{\otimes}id_{\mathcal{F}}\right).$$

Let $\{\phi_i\}_{i\in I}$ be a complete orthonormal set of $\mathcal{F}$, $\beta_a(\phi_i\hat{\otimes}\phi_j) = \beta_a^{ij}$ and $\gamma_a(1) = \sum_{i,j\in I} \gamma_a^{ij}\phi_i\hat{\otimes}\phi_j$. Using these we can calculate

$$\begin{aligned}
\mathrm{Tr}P_a &= \sum_{i\in I}\langle\phi_i, P_a\phi_i\rangle = \sum_{i\in I}\langle\phi_i, \left(id_{\mathcal{F}}\hat{\otimes}\beta_{a_1}\right) \circ \left(\gamma_{a_2}\hat{\otimes}\phi_i\right)\rangle \\
&= \sum_{i,j,k\in I} \langle\phi_i, \left(id_{\mathcal{F}}\hat{\otimes}\beta_{a_1}\right)\left(\gamma_{a_2}^{jk}\phi_j\hat{\otimes}\phi_k\hat{\otimes}\phi_i\right)\rangle \\
&= \sum_{i,j,k\in I} \langle\phi_i, \gamma_{a_2}^{jk}\phi_j\beta_{a_1}^{ki}\rangle = \sum_{j,k\in I} \beta_{a_1}^{kj}\gamma_{a_2}^{jk}.
\end{aligned}$$

On the other hand we have

$$\beta_{a_1} \circ \tau_{\mathcal{F},\mathcal{F}}\left(\gamma_{a_2}(1)\right) = \sum_{j,k\in I} \beta_{a_1} \circ \tau_{\mathcal{F},\mathcal{F}}\left(\gamma_{a_2}^{jk}\phi_j\hat{\otimes}\phi_k\right) = \sum_{j,k\in I} \beta_{a_1}^{kj}\gamma_{a_2}^{jk},$$

which is a convergent sum as $\beta_a$, $\tau_{\mathcal{F},\mathcal{F}}$ and $\gamma_a$ are bounded maps, and, which equals $\mathrm{Tr}P_a$.

To see that trace class implies compact, look at problem 5.66 c) and d) in [17]. □

Recall that in definition 16 we required that the operators $P_a$ are injective. Assume for a second that we have the set of data with the properties in definition 16, including equation 1.20, but the operators $P_a$ are not injective. They form a semigroup of normal compact (and hence diagonalizable) operators, hence by proposition 15 they have the same kernel, so restricting them to the closure of their image gives injective operators and by equations 1.19, 1.21 and 1.22 we do not lose anything by restricting to the image of $P_a$.

A regularized Frobenius algebra $\mathcal{F}$ is

- commutative, if $\mu_a \circ \tau_{\mathcal{F},\mathcal{F}} = \mu_a$,
- symmetric, if $\varepsilon_{a_1} \circ \mu_{a_2} = \varepsilon_{b_1} \circ \mu_{b_2} \circ \tau_{\mathcal{F},\mathcal{F}}$,
- $\Delta$-separable, if $\mu_{a_1} \circ \Delta_{a_2} = P_a$,
- strongly continuous, if the map $x \mapsto P_x$ is strongly continuous,

for any $a = a_1 + a_2 = b_1 + b_2$. Note that symmetric implies $\Delta_{a_1} \circ \eta_{a_2} = \tau_{\mathcal{F},\mathcal{F}} \circ \Delta_{b_1} \circ \eta_{b_2}$.[5] The center of $\mathcal{F}$ is the subspace

$$\left\{c \in \mathcal{F} | \mu_a(c \hat{\otimes} x) = \mu_a(x \hat{\otimes} c), \text{ for every } x \in \mathcal{F} \text{ and } a > 0\right\}. \tag{1.26}$$

**Lemma 18.** *The center of a regularized Frobenius algebra is closed.*

*Proof.* We can alternatively write the center $C$ as an intersection:

$$C = \bigcap_{\substack{x \in \mathcal{F} \\ a > 0}} \mathrm{Ker} \varphi_a^x,$$

where $\varphi_a^x(y) = \mu_a(x \hat{\otimes} y) - \mu_a(y \hat{\otimes} x)$. These are clearly bounded linear maps, hence their kernel is closed. The intersection of closed sets is closed. $\square$

A morphism of regularized Frobenius algebras is a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ in **Hilb** such that

$$\phi \circ \mu_a = \mu'_a \circ \left(\phi \hat{\otimes} \phi\right), \ \phi \circ \eta_a = \eta'_a, \tag{1.27}$$

$$\left(\phi \hat{\otimes} \phi\right) \circ \Delta_a = \Delta'_a \circ \phi, \ \varepsilon = \varepsilon'_a \circ \phi, \tag{1.28}$$

where the primed morphisms are the morphisms of $\mathcal{F}'$. We have the analogue of proposition 2.

**Proposition 19.** *Any morphism of regularized Frobenius algebras is injective. A morphism of regularized Frobenius algebras is an isomorphism, if its target is finite dimensional.*

*Proof.* Let $\mathcal{F} \xrightarrow{\phi} \mathcal{F}'$ be a morphism of regularized Frobenius algebras. Set

$$\overline{\phi_a} := \left(\varepsilon'_{a_1} \circ \mu'_{a_2} \hat{\otimes} id_{\mathcal{F}}\right) \circ \left(id_{\mathcal{F}'} \hat{\otimes} \phi \hat{\otimes} id_{\mathcal{F}}\right) \circ \left(id_{\mathcal{F}'} \hat{\otimes} \Delta_{a_3} \circ \eta_{a_4}\right), \tag{1.29}$$

with $\sum_{i=1}^4 a_i = a$. Then we have $\overline{\phi}_a \circ \phi = P_a$, by a same argument shown on figure 1.3. So $\phi$ is injective because $P_a$ is injective.

By a similar argument we have $\phi \circ \overline{\phi}_a = P'_a$. If $\mathcal{F}'$ is finite dimensional, then $P'_a$ is surjective, hence $\phi$ is surjective. $\square$

Note that the converse of the second statement is not true: the identity on an infinite dimensional RFA is bijective.

Let **RFrob** denote the category of regularized $*$-Frobenius algebras, **cRFrob** the category of commutative RFAs and $\mathbf{RFrob_{sc}}$ the category of strongly continuous RFAs. From now on we only treat regularized $*$-Frobenius algebras.

[5] One just needs to use the naturality of the braiding, the definition of $P_a$ and equation 1.19.

## 1.5 Classification of Strongly Continuous RFAs

In this section we will see how the additional requirement of strong continuity allows an explicit description of the eigenvalues of $P_a$ and the classification of regularized $*$-Frobenius algebras.[6]

**Proposition 20.** *Let $\mathcal{F}$ be a strongly continuous RFA, and use the notation of proposition 15. Then the eigenvalues of $P_a$ are of the form $e^{-a\sigma}$ for $\sigma \in \mathbb{R}$.*

*Proof.* Let $f \in E_n$ be a common eigenvector of $P_a$, $P_b$ and $P_{a+b}$ with eigenvalues $c(a)$, $c(b)$ and $c(a+b)$ respectively, none of which is zero. We have

$$P_a P_b f = c(a)c(b)f = P_{a+b}f = c(a+b)f \implies c(a)c(b) = c(a+b) \tag{1.30}$$

as in proposition 15. Furthermore we have

$$\|P_a f - P_b f\| = \|c(a)f - c(b)f\| = |c(a) - c(b)| \cdot \|f\|, \tag{1.31}$$

which, by strong continuity, goes to 0 as $a \to b$, hence the function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

The unique continuous solution of the functional equation 1.30 is

$$c(a) = e^{-a\sigma} \tag{1.32}$$

for some $\sigma \in \mathbb{R}$, see [18] theorem 3.5. □

**Corollary 21.** *Using the notation of proposition 20 we have*

$$\lim_{a\to 0} P_a = id_{\mathcal{F}} \tag{1.33}$$

*in the strong operator topology.*

*Proof.* Let $f \in \mathcal{F}$ and write $f = \sum_{n=1}^{\infty} f_n$, where $f_n \in E_n$. Notice that

$$\left\|P_a f^2\right\| = \sum_{n=1}^{\infty} e^{-2a\sigma_n} \|f_n\|^2 \xrightarrow{a\to 0} \left\|f^2\right\|$$

using propostion 20. We can calculate

$$\begin{aligned} \|(P_a - id_{\mathcal{F}})f\|^2 &= \langle (P_a - id_{\mathcal{F}})f, (P_a - id_{\mathcal{F}})f\rangle = \langle f, (P_a - id_{\mathcal{F}})^2 f\rangle \\ &= \sum_{n=1}^{\infty} e^{-2a\sigma_n} \|f_n\|^2 - 2\sum_{n=1}^{\infty} e^{-a\sigma_n} \|f_n\|^2 + \sum_{n=1}^{\infty} \|f_n\|^2 . \end{aligned}$$

Each of the three infinite sums individually go to $\|f\|^2$ as $a \to 0$, hence

$$\|(P_a - id_{\mathcal{F}})f\|^2 \xrightarrow{a\to 0} \|f\|^2 - 2\|f\|^2 + \|f\|^2 = 0.$$

□

**Proposition 22.** *A strongly continuous RFA $\mathcal{F}$ decomposes into the direct sum of a family of $*$-Frobenius algebras $\Phi_{\mathcal{F}} = \left\{(E_k^{\Phi}, \sigma_k)\right\}_{k=1}^{\infty}$, where $E_k$ is an eigenspace of $P_a$ with eigenvalue $e^{-a\sigma_k}$.*

[6] The idea of decomposing an area-dependent quantum field theory into the sum of topological field theories was explained to us by André Henriques, for which I am indebted to him. This idea is used to decompose regularized Frobenius algebras into direct sums $*$-Frobenius algebras, yielding their classification.

*Proof.* We know from proposition 15 that $\mathcal{F} = \overline{\bigoplus_{n=1}^{\infty} E_n}$, with $E_n$ the eigenspace of $P_a$ and proposition 20 shows that the corresponding eigenvalue is of the form $e^{-a\sigma_n}$. Let us assume that the $\sigma_n$'s are pairwise distinct. We need to show that $E_n$ are $*$-Frobenius algebras.

Claim: Fix $n$ and write $E = E_n$, $\sigma = \sigma_n$ and $R_E$ for the projection onto $E$. Then $E$ becomes a $*$-Frobenius algebra via the following maps:

- $\eta_E = R_E \circ \eta_a \cdot e^{+a\sigma}$, $\mu_E = \mu_a \restriction_{E\hat{\otimes}E} \cdot e^{+a\sigma}$,
- $\varepsilon_E = \varepsilon_a \restriction_E \cdot e^{+a\sigma}$, $\Delta_E = \Delta_a \restriction_E \cdot e^{+a\sigma}$.

We will show this in the following steps.

- First we need to show that $\mu_E$, $\eta_E$, $\Delta_E$ and $\varepsilon_E$ are independent of the choice of $a$. This can be done by the following calculation by equation 1.23:

  $$\begin{aligned}\eta_E =& R_E \circ \eta_a \cdot e^{+a\sigma} = R_E P_{a-a'} \circ \eta_{a'} \cdot e^{+a\sigma} = P_{a-a'} R_E \circ \eta_{a'} \cdot e^{+a\sigma}\\ =& e^{-(a-a')\sigma} R_E \circ \eta_{a'} \cdot e^{+a\sigma} = R_E \circ \eta_{a'} \cdot e^{+a'\sigma},\end{aligned}$$

  and similarly for the rest.

- Let $v \in E_n$, $w \in E_m$ and let us show the following:

  $$\mu_a(v\hat{\otimes}w) \begin{cases} = 0, & \text{if } n \neq m,\\ \in E_n & \text{if } n = m.\end{cases} \tag{1.34}$$

  We can write by equation 1.21 that

  $$\begin{aligned}&\mu_a(v\hat{\otimes}w) = \mu_{a-a'} \circ (P_{xa'}\hat{\otimes}P_{(1-x)a'})(v\hat{\otimes}w)\\ =&\mu_{a-a'}(v\hat{\otimes}w)e^{-a'(x\sigma_n+(1-x)\sigma_m)},\end{aligned}$$

  for any $0 \leq x \leq 1$, but on the other hand we have by equation 1.19 that

  $$\mu_a(v\hat{\otimes}w) = P_{a'} \circ \mu_{a-a'}(v\hat{\otimes}w) = \sum_{k=1}^{\infty} c_k e^{-a'\sigma_k} f_k,$$

  with $\mu_{a-a'}(v\hat{\otimes}w) = \sum_{k=1}^{\infty} c_k f_k$, $f_k \in E_k$. So for every $k$ and $x$ we have

  $$c_k \left(e^{-a'\sigma_k} - e^{-a'(x\sigma_n+(1-x)\sigma_m)}\right) = 0,$$

  so either $c_k = 0$ or $\sigma_k = (x\sigma_n + (1-x)\sigma_m)$. The latter can only happen if $\sigma_{k_0} = \sigma_n = \sigma_m$ for some $k_0$ and if this is not the case, then all $c_k = 0$, hence $\mu_a(v\hat{\otimes}w) = 0$.

- Equation 1.34 shows that $\mu_E$ lands in $E$ and we can use this equation to calculate

  $$\mu_E \circ (id_E\hat{\otimes}\eta_E) = \mu_a \circ (id_E\hat{\otimes}\eta_{a'}) \restriction_E \cdot e^{(a+a')\sigma} = P_{a+a'} \restriction_E \cdot e^{(a+a')\sigma} = id_E,$$

  by the equations in (1.17), so $\eta_E$ is the unit of $\mu_E$.

- Let us show now that the coproduct $\Delta_E$ lands in $E\hat{\otimes}E$. We have again by equation 1.22 for $v \in E$

$$\Delta_E(P_a v) = e^{a\sigma}\Delta_E(v) = \left(P_{a'}\hat{\otimes}P_{a-a'}\right)(\Delta_E(v)),$$

which shows by an argument similar to the proof of equation 1.34 that $\Delta_E(v) \in E\hat{\otimes}E$. This implies counitality via equation 1.17.

- Finally equation 1.18 provides the Frobenius relation.

□

**Corollary 23.** *A commutative RFA $\mathcal{F}$ decomposes into $\Phi_{\mathcal{F}} = \{(\mathbb{C}_{\epsilon_i}, \sigma_i)\}_{i=1}^{\infty}$.*

*Proof.* Corollary 9 shows that commutative $*$-Frobenius algebras decompose into direct sums of $\mathbb{C}_\epsilon$'s. □

**Corollary 24.** *The center $C$ of a strongly continuous RFA $\mathcal{F}$ decomposes as*

$$C = \overline{\bigoplus_{n=1}^{\infty} Z(E_n)}, \tag{1.35}$$

*where $Z(E_n)$ is the center of $E_n$, using the notation of proposition 22.*

*Proof.* Show $C \subset \overline{\bigoplus_{n=1}^{\infty} Z(E_n)}$: Let $z \in C$, then we can write it as $z = \sum_{n=1}^{\infty} z_n$ with $z_n \in E_n$. Calculate for $y_n \in E_n$

$$\mu_a(z\hat{\otimes}y_n) = \mu_a(z_n\hat{\otimes}y_n),$$

by equation 1.34. On the other hand we have

$$\mu_a(z\hat{\otimes}y_n) = \mu_a(y_n\hat{\otimes}z) = \mu_a(y_n\hat{\otimes}z_n),$$

because $z \in C$, so $z_n \in Z(E_n)$.

To show the converse inclusion let $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in Z(E_n)$. Then

$$\mu_a(x\hat{\otimes}y) = \sum_{m,n=1}^{\infty} \mu_a(x_n\hat{\otimes}y_m) = \sum_{m,n=1}^{\infty} \mu_a(y_m\hat{\otimes}x_n) = \mu_a(y\hat{\otimes}x),$$

because $\mu_a$ is a bounded linear map, so $x \in C$. □

Now we would like to investigate, under which circumstances can we build RFAs from $*$-Frobenius algebras and for this we need a few considerations first. Note that in a $*$-Frobenius algebra $\|\eta\| = \|\varepsilon\|$ and $\|\mu\| = \|\Delta\|$.

**Proposition 25.** *Let $\Psi = \{(F_k, \sigma_k)\}_{k=1}^{\infty}$ be a family of $*$-Frobenius algebras and real numbers such that for all $a > 0$*

$$\sum_{k=1}^{\infty} e^{-2a\sigma_k} \|\eta_k\|^2 < \infty, \tag{1.36}$$

$$\sup_k \left\{e^{-a\sigma_k} \|\mu_k\|\right\} < \infty. \tag{1.37}$$

*This data defines a strongly continuous RFA $\mathcal{G}_\Psi$ via*

$$
\begin{aligned}
\mathcal{G}_\Psi &:= \overline{\bigoplus_{k=1}^\infty F_k}, \\
\eta_a^\Psi := \sum_{k=1}^\infty e^{-a\sigma_k}\eta_k, &\quad \varepsilon_a^\Psi := \sum_{k=1}^\infty e^{-a\sigma_k}\varepsilon_k, \\
\mu_a^\Psi := \sum_{k=1}^\infty e^{-a\sigma_k}\mu_k, &\quad \Delta_a^\Psi := \sum_{i=1}^\infty e^{-a\sigma_k}\Delta_k.
\end{aligned}
$$

*Here $\eta_k$, $\varepsilon_k$, $\mu_k$ and $\Delta_k$ are the unit, counit, product and coproduct of $F_k$ pre- and postcomposed with the projections $R_k : \mathcal{G}_\Psi \to F_k$ and injections $F_k \to \mathcal{G}_\Psi$ mutatis mutandis.*

*Proof.* We need to show that the above defined maps are bounded, check the relations in the definition of RFAs and that the assignment $a \mapsto P_a$ is strongly continuous and that $P_a$ is injective for every $a$. Write $\mathcal{G} = \mathcal{G}_\Psi$, $\eta_a = \eta_a^\Psi$, etc... for simplicity.

- Equation 1.36 shows that $\eta_a$ is bounded:

$$
\begin{aligned}
\|\eta_a\|^2 = \|\eta_a(1)\|^2 = \left\|\sum_{k=1}^\infty e^{-a\sigma_k}\eta_k(1)\right\|^2 &= \sum_{k=1}^\infty e^{-2a\sigma_k}\|\eta_k(1)\|^2 \\
= \sum_{k=1}^\infty e^{-2a\sigma_k}\|\eta_k\|^2 < \infty.
\end{aligned}
$$

- It also implies that $\varepsilon_a$ is bounded: Define $\varepsilon'_a$ to be the adjoint of $\eta_a$, i.e. $\varepsilon'_a(f) = \langle \varepsilon'_a(f), 1\rangle = \langle f, \eta_a(1)\rangle$ for $f = \sum_{k=1}^\infty f_k \in \mathcal{G}$, with $f_k \in F_k$. This is a bounded linear map $\mathcal{G} \to \mathbb{C}$. Then observe that

$$
\begin{aligned}
\varepsilon'_a(f) =& \langle f, \eta_a(1)\rangle = \left\langle \sum_{j=1}^\infty f_j, \sum_{k=1}^\infty e^{-a\sigma_k}\eta_k(1)\right\rangle = \sum_{j,k=1}^\infty e^{-a\sigma_k}\langle f_j, \eta_k(1)\rangle \\
=& \sum_{k=1}^\infty e^{-a\sigma_k}\langle f_k, \eta_k(1)\rangle = \sum_{k=1}^\infty e^{-a\sigma_k}\langle \varepsilon_k(f_k), 1\rangle = \sum_{k=1}^\infty e^{-a\sigma_k}\varepsilon_k(f_k),
\end{aligned}
$$

as $\eta_k^* = \varepsilon_k$, so $\varepsilon'_a = \varepsilon_a$ and in particular $\eta_a^* = \varepsilon_a$.

- Equation 1.37 shows that $\mu_a$ and $\Delta_a$ are bounded: Let $h = \sum_{i,j=1}^\infty h_{ij} \in \mathcal{G}^{\hat{\otimes} 2}$ with $h_{ij} \in F_i \hat{\otimes} F_j$ and calculate

$$
\begin{aligned}
\|\mu_a(h)\|^2 &= \left\|\sum_{k=1}^\infty e^{-a\sigma_k}\mu_k(h)\right\|^2 = \left\|\sum_{i,j,k=1}^\infty e^{-a\sigma_k}\mu_k(h_{kk})\delta_{i,k}\delta_{j,k}\right\|^2 \\
&= \sum_{k=1}^\infty e^{-2a\sigma_k}\|\mu_k(h_{kk})\|^2 \le \sum_{k=1}^\infty e^{-2a\sigma_k}\|\mu_k\|^2\|h_{kk}\|^2 \\
&\le \left(\sup_k\left\{e^{-a\sigma_k}\|\mu_k\|\right\}\right)^2 \sum_{k=1}^\infty \|h_{kk}\|^2, \\
\|\Delta_a(f)\|^2 &= \left\|\sum_{k=1}^\infty e^{-a\sigma_k}\Delta_k(f)\right\|^2 = \left\|\sum_{i,k=1}^\infty e^{-a\sigma_k}\Delta_k(f_k)\delta_{i,k}\right\|^2 \\
&= \sum_{k=1}^\infty e^{-2a\sigma_k}\|\Delta_k(f_k)\|^2 \le \sum_{k=1}^\infty e^{-2a\sigma_k}\|\Delta_k\|^2\|f_k\|^2 \\
&\le \left(\sup_k\left\{e^{-a\sigma_k}\|\Delta_k\|\right\}\right)^2 \sum_{k=1}^\infty \|f_k\|^2,
\end{aligned}
$$

as $\|\mu_k\| = \|\Delta_k\|$.

- It easily follows from the $*$-Frobenius algebra property of the $F_k$'s that the relations in (1.17), (1.18) and (1.20) hold. So let us write for $a = a_1 + a_2$ that

$$P_a := \mu_{a_1} \circ \left(id_{\mathcal{G}} \hat{\otimes} \eta_{a_2}\right) = \sum_{k=1}^{\infty} e^{-a\sigma_k} R_k. \tag{1.38}$$

  The last equality immediately shows that the relations in (1.19) hold as well and that the $P_a$ are injective as $e^{-a\sigma_k} \neq 0$.

- To show strong continuity calculate

$$\begin{aligned}
\|(P_a - P_{a_0})f\|^2 &= \langle (P_a - P_{a_0})f, (P_a - P_{a_0})f\rangle = \langle f, (P_a - P_{a_0})^2 f\rangle \\
&= \|P_{2a}f\|^2 - 2\,\|P_{a+a_0}f\|^2 + \|P_{2a_0}f\|^2 \\
&\xrightarrow{a\to 0} \|P_{2a_0}f\|^2 - 2\,\|P_{2a_0}f\|^2 + \|P_{2a_0}f\|^2 = 0
\end{aligned}$$

  as in the proof of corollary 21.

□

Note that the two conditions of proposition 25 imply that the set $\{\sigma_k\}_{k=0}^{\infty}$ is bounded from below. This can also be seen from equation 1.38 as $P_a$ is the composition of bounded maps.

**Corollary 26.** *Let $\Psi = \{(\mathbb{C}_{\epsilon_i}, \sigma_i)\}_{i=1}^{\infty}$ be a family of commutative $*$-Frobenius algebras and real numbers such that for all $a > 0$*

$$\sum_{i=1}^{\infty} e^{-2a\sigma_i} |\epsilon_i|^2 < \infty, \tag{1.39}$$

$$\sup_i \left\{ e^{-a\sigma_i} |\epsilon_i|^{-1} \right\} < \infty. \tag{1.40}$$

*This data defines a commutative RFA $\mathcal{G}_\Psi$ via proposition 25.*

*Proof.* From corollary 9 have $\|\eta_i\| = |\epsilon_i|$ and $\|\mu_i\| = |\epsilon_i|^{-1}$, hence the conditions of proposition 25 are satisfied. □

**Theorem 27.** *The above two constructions are inverse to each other in the following sense.*

1. *Let $\Phi$ be a family of $*$-Frobenius algebras and numbers satisfying (1.36) and (1.37). Then we obtain a strongly continuous RFA $\mathcal{G}_\Phi$ from proposition 25. We can use proposition 22 to get a family of $*$-Frobenius algebras and numbers $\Psi_{\mathcal{G}_\Phi}$. Then we have*

$$\Psi_{\mathcal{G}_\Phi} = \Phi. \tag{1.41}$$

2. *Let $\mathcal{F}$ be an RFA. Then we obtain a family of $*$-Frobenius algebras and numbers $\Phi_{\mathcal{F}}$ from proposition 22. Now use proposition 25 to get an RFA $\mathcal{G}_{\Phi_{\mathcal{F}}}$, we have*

$$\mathcal{G}_{\Phi_{\mathcal{F}}} = \mathcal{F}. \tag{1.42}$$

*Proof.* Let us start with proving the second part, write $\Phi_{\mathcal{F}} = \{(F_k, \sigma_k)\}_{k=1}^{\infty}$ and $\{R_k\}_{k=1}^{\infty}$ for the resolution of identity that diagonalizes $P_a$ of $\mathcal{F}$. We need to reassemble the RFA that we have disassembled:

$$\mathcal{G}_{\Phi_{\mathcal{F}}} = \overline{\bigoplus_{k=1}^{\infty} F_k} = \mathcal{F},$$

$$\eta_a^{\Phi_{\mathcal{F}}} = \sum_{k=1}^{\infty} e^{-a\sigma_k} \eta_k = \sum_{k=1}^{\infty} e^{-a\sigma_k} R_k \circ \eta_a \cdot e^{+a\sigma_k} = \left(\sum_{k=1}^{\infty} R_k\right) \circ \eta_a = \eta_a,$$

$$\mu_a^{\Phi_{\mathcal{F}}} = \sum_{k=1}^{\infty} e^{-a\sigma_k} \mu_k = \sum_{k=1}^{\infty} e^{-a\sigma_k} \mu_a \circ (R_k \hat{\otimes} R_k) \cdot e^{+a\sigma_k} = \sum_{l,k=1}^{\infty} \mu_a \circ (R_k \hat{\otimes} R_l) = \mu_a,$$

$$\varepsilon_a^{\Phi_{\mathcal{F}}} = \sum_{k=1}^{\infty} e^{-a\sigma_k} \varepsilon_k = \sum_{k=1}^{\infty} e^{-a\sigma_k} \varepsilon_a \circ R_k \cdot e^{+a\sigma_k} = \sum_{k=1}^{\infty} \varepsilon_a \circ R_k = \varepsilon_a,$$

$$\Delta_a^{\Phi_{\mathcal{F}}} = \sum_{k=1}^{\infty} e^{-a\sigma_k} \Delta_k = \sum_{k=1}^{\infty} e^{-a\sigma_k} \Delta_a \circ R_k \cdot e^{+a\sigma_k} = \sum_{k=1}^{\infty} \Delta_a \circ R_k = \Delta_a,$$

where we used equation 1.34. Note that the conditions of proposition 25 hold: Just calculate the norm of $\eta_a$ to see that the first condition is satisfied:

$$\begin{aligned} \|\eta_a\|^2 = \|\eta_a(1)\|^2 &= \left\| \sum_{k=1}^{\infty} R_k \eta_a(1) \right\|^2 = \left\| \sum_{k=1}^{\infty} \eta_k(1) e^{-a\sigma_k} \right\|^2 \\ &= \sum_{k=1}^{\infty} \|\eta_k(1)\|^2 e^{-2a\sigma_k} = \sum_{k=1}^{\infty} \|\eta_k\|^2 e^{-2a\sigma_k} < \infty. \end{aligned}$$

For the second let $F \in \mathcal{F} \hat{\otimes} \mathcal{F}$, then we have

$$\left\| \mu_k(F) e^{-a\sigma_k} \right\| = \|\mu_a(F)\| \leq \|\mu_a\| \, \|F\|,$$

hence $\|\mu_k\| \, e^{-a\sigma_k} \leq \|\mu_a\|$.

To prove the first part, let us first write $\Phi = \{(F_k, \sigma_k)\}_{k=1}^{\infty}$. Then $\mathcal{G}_\Phi = \overline{\bigoplus_{j=1}^{\infty} F_j}$ and let $R_k$ denote the projection $\mathcal{G}_\Phi \to F_k$. Then we have

$$\begin{aligned} P_a^{\Phi} &= \left(\sum_{k=1}^{\infty} e^{-a\sigma_k} \mu_k\right) \circ \left(\sum_{j=1}^{\infty} e^{-a\sigma_j} (id_{F_j} \hat{\otimes} \eta_j)\right) \\ &= \left(\sum_{j,k=1}^{\infty} e^{-a\sigma_k} \mu_k \circ (id_{F_k} \hat{\otimes} \eta_k) \delta_{j,k}\right) = \sum_{k=1}^{\infty} e^{-a\sigma_k} R_k, \end{aligned}$$

from which we can directly read off the diagonalizing resolution of identity

$\{R_k\}_{k=1}^{\infty}$ of $P_a^{\Phi}$. So we have

$$F_k^{\mathcal{G}_\Phi} = R_k\left(\overline{\bigoplus_{j=1}^{\infty} F_j}\right) = F_k,$$

$$\eta_k^{\mathcal{G}_\Phi} = R_k \circ \sum_{j=1}^{\infty} e^{-a\sigma_j}\eta_j \cdot e^{+a\sigma_k} = \eta_k,$$

$$\mu_k^{\mathcal{G}_\Phi} = \sum_{j=1}^{\infty} e^{-a\sigma_j}\mu_j \circ \left(R_k \hat{\otimes} R_k\right) \cdot e^{+a\sigma_k} = \mu_k,$$

$$\varepsilon_k^{\mathcal{G}_\Phi} = \sum_{j=1}^{\infty} e^{-a\sigma_j}\varepsilon_j \circ R_k \cdot e^{+a\sigma_k} = \varepsilon_k,$$

$$\Delta_k^{\mathcal{G}_\Phi} = \sum_{j=1}^{\infty} e^{-a\sigma_j}\Delta_j \circ R_k \cdot e^{+a\sigma_k} = \Delta_k.$$

□

# Chapter 2

# Area-Dependent Quantum Field Theories

## 2.1 Topological Field Theories in 2 Dimensions

We start with a short summary of 2-dimensional topological field theories as the area-dependent theories will heavily rely on notions form topological field theories. First we need to introduce the notion of smooth surfaces and their boundaries.

Let $\mathbb{S}_1$ denote a unit circle on the complex plane $\mathbb{C}$. In order to be able to speak about smooth manifolds, we need the notion of a collar of $\mathbb{S}_1$, which is a neighborhood of $\mathbb{S}_1$. Two collars are equivalent, if they are the same on a neighborhood of $\mathbb{S}_1$, and we call a germ of collars an equivalence class of collars. Let $U$ be such a disjoint union of circles with collars and let $U_{in} \subset U$ denote the disjoint union of points $|z| \leq 1$ of each collar and $U_{out} \subset U$ the disjoint union of points $|z| \geq 1$ of each collar.

A bordism $M : U \to V$ between circles with collars $U$ and $V$ is a compact oriented 2-dimensional smooth manifold, whose boundary $\partial M$ is identified by a boundary parametrization with the circles of $-U \amalg V$. A boundary parametrization is a choice of representatives $U'$, $V'$ of germs of $U$ and $V$ and injective smooth maps $f_{in} : U'_{in} \to M$, $f_{out} : V'_{out} \to M$, which preserve orientation and boundary and the images of the circles in $U'$ and $V'$ are disjoint and cover $\partial M$. We will refer to $U$ and $V$ for a bordism $M : U \to V$ as the ingoing and outgoing boundary of $M$. We say that two bordisms are equivalent, if the 2-manifolds are diffeomorphic, such that the diffeomorphism is compatible with the boundary parametrization.

We can compose bordisms ($\circ$): Just glue two bordisms together along their boundary parametrizations, i.e. the outgoing boundary of the first bordism to the ingoing boundary of the second bordism. We can permute the circles with collars and compose a permutation with a bordism, the result will be a bordism with permuted boundaries.

**Definition 28.** *Let* $\mathbf{Cob_2}$ *denote the category of 2-dimensional cobordisms, which has*

- *Objects: Disjoint union of a finite number of unit circles* $\mathbb{S}_1$ *in* $\mathbb{C}$*, each of them equipped with a germ of collars.*

- <u>Morphisms:</u> *disjoint unions of equivalence classes of bordisms and permutations, which permute circles.*

By the latter we mean the following. A morphism $X : U \to V$ consists of

- a disjoint decomposition $U = B \sqcup P$ and $V = B' \sqcup P'$, such that $P$ and $P'$ have the same number of circles,
- a bordism $\Sigma : B \to B'$ and
- a bijection $\rho : \pi_0(P) \to \pi_0(P')$.

This becomes a monoidal category with the disjoint union ($\sqcup$) as the tensor product and the permutations endow this category with a symmetric structure. This is the definition used in [9] section 2.3 without defect conditions and metric on the surfaces. Note that the cylinders are not the identity morphisms, but are idempotents.

By abuse of notation we write $\mathbb{S}_1^{\sqcup n}$ for an object in $\mathbf{Cob_2}$, which has $n$ circles and let $S_{n,m}$ denote the bordism in $\mathbf{Cob_2}$, which consists of an $n+m$ holed sphere ($\mathbb{S}_2$) with $m$ ingoing and $n$ outgoing boundaries. We will use this notation to obtain another description of $\mathbf{Cob_2}$. For the details see [2] section 1.4.[7]

**Proposition 29.** *The bordisms $S_{0,1}$, $S_{1,0}$, $S_{2,1}$, $S_{1,2}$ and $S_{1,1}$ generate $\mathbf{Cob_2}$ as a symmetric monoidal category. These generators obey the following relations:*

- $S_{1,1} \circ S_{1,1} = S_{1,1}$,
- $S_{1,2} \circ (S_{1,1} \sqcup S_{1,0}) = S_{1,2} \circ (S_{1,0} \sqcup S_{1,1}) = S_{1,1}$,
- $(S_{1,1} \sqcup S_{0,1}) \circ S_{2,1} = (S_{0,1} \sqcup S_{1,1}) \circ S_{2,1} = S_{1,1}$,
- $S_{2,1} \circ S_{1,2} = (S_{1,1} \sqcup S_{1,2}) \circ (S_{2,1} \sqcup S_{1,1}) = (S_{1,2} \sqcup S_{1,1}) \circ (S_{1,1} \sqcup S_{2,1})$,

*and we also have that $S_{1,2} \circ \tau = S_{1,2}$ for $\tau$ the transposition of two circles.*

Let $Z$ denote a symmetric monoidal functor $\mathbf{Cob_2} \to \mathbf{Vect}_{\Bbbk}$. By proposition 29 the cylinders are not necessarily identities, but are idempotents, therefore their image under $Z$ will be idempotents, but not necessarily identities. So we have $P^2 = P$ with $P = Z(S_{1,1})$. We can write any bordism $M : \mathbb{S}_1^{\sqcup m} \to \mathbb{S}_1^{\sqcup n}$ as $M = S_{1,1}^{\sqcup m} \circ M \circ S_{1,1}^{\sqcup n}$, hence $Z(M)$ will land in the $m$-th tensor product of $\mathrm{Im}P = Z(\mathbb{S}_1)/\mathrm{Ker}P$. Therefore we can define another symmetric monoidal functor $Z'$ by setting $Z'(\mathbb{S}_1) := \mathrm{Im}P = Z(\mathbb{S}_1)/\mathrm{Ker}P$ and $Z'(M) := Z(M) \restriction_{\mathrm{Im}Z(S_{1,1})^{\otimes m}}$. $Z'(M)$ lands in $Z(\mathbb{S}_1)^{\otimes n}$, because $Z(M)$ sends any element in $Z(\mathbb{S}_1)^{\otimes m}$ to zero, which has a factor in $\mathrm{Ker}P$. This new functor sends cylinders to identities.

**Definition 30.** *A 2-dimensional topological field theory (TFT in short) over $\Bbbk$ is a symmetric monoidal functor*

$$Z : \mathbf{Cob_2} \to \mathbf{Vect}_{\Bbbk},$$

*for which $Z(S_{1,1}) = id_{Z(\mathbb{S}_1)}$.*

[7] Note that [2] uses a different definition of $\mathbf{Cob_2}$, but it treats cylinders as real generators and also gives the relations involving cylinders.

By proposition 29 it is enough to define a TFT on the generators of $\mathbf{Cob_2}$, and check that the corresponding relations are satisfied. We will refer to the values of the TFT on the generators as the generators of the TFT and call $Z(\mathbb{S}_1)$ the state-space of the TFT. Denote the category of TFTs over $\Bbbk$ with $\mathbf{TFT}_\Bbbk$. It has objects TFTs over $\Bbbk$ and morphisms monoidal natural transformations. Note that a natural transformation $\theta : Z \to Z'$ being monoidal means that $\theta_n := \theta_{Z(\mathbb{S}_1)^{\otimes n}} = \theta_1^{\otimes n}$ for every $n \geq 1$. We are ready to state the main theorem of this section, which shows in particular that the state space of a TFT is finite dimensional.[8]

**Theorem 31.** *There is an equivalence of categories* $\mathbf{TFT}_\Bbbk \xrightarrow{\sim} \mathbf{cFrob}_\Bbbk$.

*Proof (sketch).* Let $Z$ be a TFT and assign

$$
\begin{aligned}
&Z(\mathbb{S}_1) \mapsto F,\ Z(\tau) = \tau_{F,F},\\
&Z(S_{1,2}) \mapsto \mu_F,\ Z(S_{1,0}) \mapsto \eta_F,\\
&Z(S_{2,1}) \mapsto \Delta_F,\ Z(S_{0,1}) \mapsto \varepsilon_F,
\end{aligned}
$$

where by abuse of notation $\tau$ denotes the transposition of two circles in $\mathbf{Cob_2}$ and $\tau_{F,F}$ denotes the symmetric braiding on $F \otimes F$ in $\mathbf{Vect}_\Bbbk$. Then by the relations in proposition 29 $F$ is a commutative Frobenius algebra. A monoidal natural transformation $\theta : Z \to Z'$ gets mapped to $\theta_{Z(\mathbb{S}_1)}$ and naturality implies that this is a morphism of Frobenius algebras. For the rest of the details see [2] theorem 3.3.2. □

A $*$-structure on $\mathbf{Cob_2}$ is given as follows. Let us use the previous notation for a morphism $X : U \to V$ in $\mathbf{Cob_2}$. Then $X^* : V \to U$ is the morphism, which consists of

- the same disjoint decomposition of objects as $X$,
- the bordism $\Sigma^* : B' \to B$, which is obtained by changing the orientation of $\Sigma$ and exchanging the roles of the in- and outgoing boundaries,
- the bijection $\rho^{-1} : \pi_0(P') \to \pi_0(P)$.

Let us keep this $*$-structure on $\mathbf{Cob_2}$ fixed from now on. We have in particular $S^*_{m,n} = S_{n,m}$. We can consider TFTs $\mathbf{Cob_2} \to \mathbf{Hilb}$, which are additionally $*$-functors. Such TFTs correspond to $*$-Frobenius algebras and vice versa. Instead of giving details, let us turn to the main subject of the thesis. In section 2.4 we will see this statement as a special case of theorem 36.

## 2.2 Area-Dependent Quantum Field Theories

We introduce the notion of the area of a bordism, which we think of as positive numbers assigned to connected components of a bordism. A bordism with area is a bordism $M : U \to V$ in $\mathbf{Cob_2}$ together with a map $\mathcal{A}_M : \pi_0(M) \to \mathbb{R}_+$, which assigns to each connected component of $M$ a positive real number. $\mathcal{A}_M$ is called the area map of $M$ and the number it assigns to a connected component of $M$ is the area of that connected component. Two bordisms with area are equivalent, if the bordisms are equivalent and the area maps are the same.

[8] See page 13.

We can compose bordisms with areas just as in $\mathbf{Cob_2}$, with the area map of the composition given as follows. It assigns to each of the connected components of the composition the sum of the numbers, which were assigned to the factors of the components previously. An example is given in figure 2.1. The areas assigned to connected components are written on them and the in- and outgoing boundaries are marked. We define the composition of a permutation and a bordism with area to have the same area map as before.

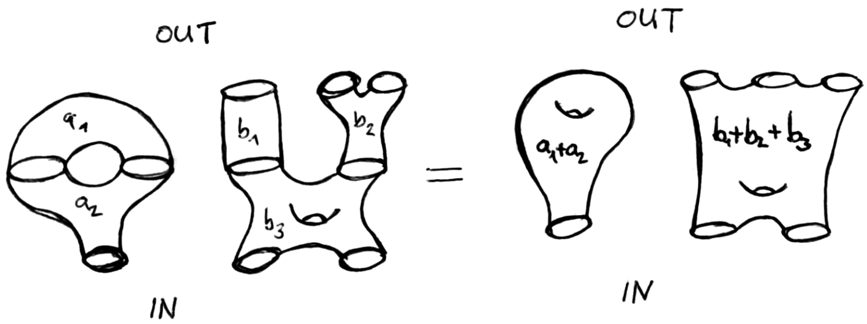


Figure 2.1: Example for composition in $\mathbf{Cob_2^{area}}$

**Definition 32.** *Let* $\mathbf{Cob_2^{area}}$ *denote the category of 2-dimensional bordisms with area, whose*

- *objects are the same as the objects of* $\mathbf{Cob_2}$*,*
- *morphisms are disjoint union of permutations as in* $\mathbf{Cob_2}$ *or equivalence classes of bordisms with areas.*

This category becomes a symmetric monoidal category with the disjoint union as tensor product on objects and for morphisms as follows. For bordisms with areas take the disjoint union on bordisms and let the area map of this bordism assign the same numbers to each connected component, which they were assigned before. Again, by a slight abuse of notation, let $(S_{n,m}, a)$ denote the bordism with area in $\mathbf{Cob_2^{area}}$, which is the $n+m$ holed sphere with $m$ ingoing and $n$ outgoing boundaries and has area $a = \mathcal{A}_{S_{n,m}}(S_{n,m})$.

**Proposition 33.** *The bordisms with area* $(S_{0,1}, a)$*,* $(S_{1,0}, a)$*,* $(S_{2,1}, a)$*,* $(S_{1,2}, a)$ *and* $(S_{1,1}, a)$ *generate* $\mathbf{Cob_2^{area}}$ *as a symmetric monoidal category. These generators obey the following relations:*

$$(S_{1,1}, a_1) \circ (S_{1,1}, a_2) = (S_{1,1}, a_1 + a_2), \tag{2.1}$$

$$\begin{aligned} &(S_{1,2}, a_1) \circ ((S_{1,1}, a_2) \sqcup (S_{1,0}, a_3)) \\ =&(S_{1,2}, b_2) \circ ((S_{1,0}, b_2) \sqcup (S_{1,1}, b_3)) = (S_{1,1}, a), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &((S_{1,1}, a_1) \sqcup (S_{0,1}, a_2)) \circ (S_{2,1}, a_3) \\ =&((S_{0,1}, b_1) \sqcup (S_{1,1}, b_2)) \circ (S_{2,1}, b_3) = (S_{1,1}, a), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &((S_{1,1}, c_1) \sqcup (S_{1,2}, c_2)) \circ ((S_{2,1}, c_3) \sqcup (S_{1,1}, c_4)) \\ =&((S_{1,2}, d_1) \sqcup (S_{1,1}, d_2)) \circ ((S_{1,1}, d_3) \sqcup (S_{2,1}, d_4)) = (S_{2,1}, e_1) \circ (S_{1,2}, e_2), \end{aligned} \tag{2.4}$$

*where* $\sum_{i=1}^{3} a_i = \sum_{i=1}^{3} b_i = a$ *and* $\sum_{i=1}^{4} c_i = \sum_{i=1}^{4} d_i = e_1 + e_2$ *and* $(S_{1,2}, a) \circ \tau = (S_{1,2}, a)$*, where* $\tau$ *denotes the transposition of two circles in* $\mathbf{Cob_2^{area}}$.

Some of these relations are shown on figure 2.2. The proof is essentially the same as of proposition 29, observe how the addition of areas required by the composition in $\mathbf{Cob_2^{area}}$ is respected.

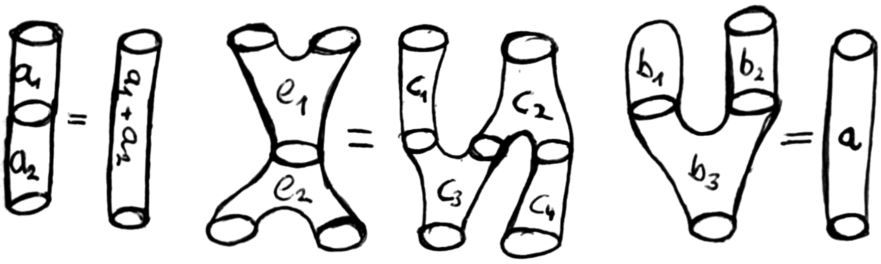


Figure 2.2: Some relations in $\mathbf{Cob_2^{area}}$

To define a $*$-structure on $\mathbf{Cob_2^{area}}$, first note that $\pi_0(M) = \pi_0(M^*)$, for $M$ a bordism in $\mathbf{Cob_2}$ and where $M^*$ is given by the previously fixed $*$-structure on $\mathbf{Cob_2}$. Let $(M, \mathcal{A}_M)$ be a bordism with area, then we set $(M, \mathcal{A}_M)^* := (M^*, \mathcal{A}_M)$. For a permutation $P$ we set $P^* = P^{-1}$ as we did in $\mathbf{Cob_2}$. Let us keep this $*$-structure on $\mathbf{Cob_2^{area}}$ fixed from now on.

**Definition 34.** *An area-dependent quantum field theory or AQFT in short is a symmetric monoidal $*$-functor $Z : \mathbf{Cob_2^{area}} \to \mathbf{Hilb}$ such that $Z(S_{1,1}, a)$ is injective for every $a$.*

The condition that $Z(S_{1,1}, a)$ is injective seems to be a restrictive condition on $Z$ at this point, however in the following we show that it can always be achieved using the $*$-property of the functor. So let $Z$ now denote a symmetric monoidal $*$-functor $\mathbf{Cob_2^{area}} \to \mathbf{Hilb}$ and let us introduce the following notation:

$$\mathcal{H} = Z(\mathbb{S}_1), \qquad P_a = Z(S_{1,1}, a),$$
$$\beta_a = Z(S_{2,0}, a) \in \mathcal{B}(\mathcal{H}^{\hat{\otimes}2}, \mathbb{C}), \quad \gamma_a = Z(S_{0,2}, a) \in \mathcal{B}(\mathbb{C}, \mathcal{H}^{\hat{\otimes}2}),$$

and call $\mathcal{H}$ the state space of $Z$. The following lemma gives us the core of our analysis.

**Lemma 35.** *The operators $P_a$ are of trace class and hence compact.*

*Proof.* Let $(\phi_i)_{i\in I}$ be a complete orthonormal set of $\mathcal{H}$. We need to show that the sum

$$\mathrm{Tr} P_a = \sum_{i\in I} \langle \phi_i, P_a \phi_i \rangle$$

converges absolutely. Write $P_a\phi_i = \sum_{j\in I} P_a^{ji} \phi_j$, $\gamma_a(1) = \sum_{i,j\in I} \gamma_a^{ij} \phi_i \hat{\otimes} \phi_j$ and $\beta_a^{ij} = \beta_a(\phi_i \hat{\otimes} \phi_j)$. Let us cut a cylinder as on figure 2.3. From functoriality we have

$$(P_{a_5''} \hat{\otimes} \beta_{a_2})(P_{a_5'} \hat{\otimes} P_{a_3} \hat{\otimes} P_{a_1'})(\gamma_{a_4}(1) \hat{\otimes} P_{a_1''}) = P_a$$

with $\tilde{a}_1 = a_1' + a_1''$, $a_5 = a_5' + a_5''$, $a_1 = \tilde{a}_1 + a_5$ and $\sum_{i=1}^4 a_i = a$. Use this to calculate

$$
\begin{aligned}
\mathrm{Tr} P_a &= \sum_{i,k,l \in I} \langle \phi_i, (P_{a_5''} \hat{\otimes} \beta_{a_2})(P_{a_5'} \hat{\otimes} P_{a_3} \hat{\otimes} P_{a_1''})(\gamma_{a_4}^{k,l} \phi_k \hat{\otimes} \phi_l \hat{\otimes} (P_{a_1'} \phi_i)) \rangle \\
&= \sum_{i,k,l \in I} \langle \phi_i, (P_{a_5''} \hat{\otimes} \beta_{a_2})(\gamma_{a_4}^{k,l} (P_{a_5'} \phi_k) \hat{\otimes} (P_{a_3} \phi_l) \hat{\otimes} (P_{a_1''} P_{a_1'} \phi_i)) \rangle \\
&= \sum_{i,k,l \in I} \langle \phi_i, (id_{\mathcal{H}} \hat{\otimes} \beta_{a_2})(\gamma_{a_4}^{k,l} (P_{a_5} \phi_k) \hat{\otimes} (P_{a_3} \phi_l) \hat{\otimes} (P_{\tilde{a}_1} \phi_i)) \rangle \\
&= \sum_{i,k,l,m,n,p \in I} \langle \phi_i, P_{a_5}^{pk} \gamma_{a_4}^{kl} P_{a_3}^{ml} \beta_{a_2}^{mn} P_{\tilde{a}_1}^{ni} \phi_p \rangle \\
&= \sum_{i,k,l,m,n,p \in I} P_{a_5}^{pk} \gamma_{a_4}^{kl} P_{a_3}^{ml} \beta_{a_2}^{mn} P_{\tilde{a}_1}^{ni} \delta^{ip} \\
&= \sum_{k,l,m,n \in I} \gamma_{a_4}^{kl} P_{a_3}^{ml} \beta_{a_2}^{mn} P_{a_1}^{nk},
\end{aligned}
$$

in the last step we used that $P$ is a semigroup homomorphism ($P_{\tilde{a}_1} \circ P_{a_5} = P_{a_1}$). Now calculate

$$
\begin{aligned}
&\beta_{a'} \circ \tau_{\mathcal{H},\mathcal{H}} \circ \gamma_{a-a'}(1) = \beta_{a_2} \circ \tau_{\mathcal{H},\mathcal{H}} \circ (P_{a_1} \hat{\otimes} P_{a_3}) \circ \gamma_{a_4}(1) \\
=&\beta_{a_2} \circ \tau_{\mathcal{H},\mathcal{H}} \circ (P_{a_1} \hat{\otimes} P_{a_3}) \left( \sum_{k,l \in I} \gamma_{a_4}^{kl} \phi_k \hat{\otimes} \phi_l \right) \\
=&\beta_{a_2} \circ \tau_{\mathcal{H},\mathcal{H}} \left( \sum_{k,l,m,n \in I} \gamma_{a_4}^{kl} P_{a_1}^{nk} \phi_n \hat{\otimes} P_{a_3}^{ml} \phi_m \right) \\
=& \sum_{k,l,m,n \in I} \gamma_{a_4}^{kl} P_{a_3}^{ml} \beta_{a_2}^{mn} P_{a_1}^{nk},
\end{aligned}
$$

which is a convergent sum since it is a composition of bounded maps and it is equal to $\mathrm{Tr} P_a$ by the former calculation. To see that trace class implies compact look at problem 5.66 c) and d) in [17]. $\square$

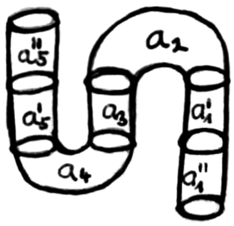


Figure 2.3: Decomposition of a cylinder

The operators $P_a$ are self-adjoint:

$$
P_a^* = Z(S_{1,1}, a)^* = Z(S_{1,1}^*, a) = Z(S_{1,1}, a) = P_a.
$$

They also form a semi-group ($P_a \circ P_b = P_{a+b}$), so by proposition 15 the kernels of the operators $P_a$ coincide. This is the main motivation for including the $*$-property in the definition of AQFTs. Thus by the same argumentation that we used to arrive at definition 30 we can always build a new symmetric monoidal $*$-functor $Z'$ from $Z$ without losing anything in the following way:

- Set $Z'(\mathbb{S}_1) := Z(\mathbb{S}_1)/\mathrm{Ker}P_a = \overline{\mathrm{Im}P_a}$ and
- $Z'(M) := Z(M) \restriction_{Z'(\mathbb{S}_1)^{\hat{\otimes}n}}$

for any morphism $\mathbb{S}_1^{\hat{\otimes}n} \xrightarrow{M} \mathbb{S}_1^{\hat{\otimes}m}$ in $\mathbf{Cob_2^{area}}$. As before, $Z'(M)$ lands in $Z'(\mathbb{S}_1)^{\hat{\otimes}m}$ and $Z'(S_{1,1}, a)$ is injective on $Z'(\mathbb{S}_1)$, hence $Z'$ is an AQFT. It is also worth noting that the state space of any AQFT is a separable Hilbert space.[9]

We would like to have a characterization of AQFTs similar to what is given in theorem 31 for TFTs. Therefore let us first introduce the category of AQFTs, denoted **AQFT**. It has objects AQFTs and morphisms monoidal natural transformations.

**Theorem 36.** *There is an equivalence of categories* **AQFT** $\to$ **cRFrob**.

*Proof (sketch).* Observe that we are in a very similar situation as at theorem 31. The generators of an AQFT $Z$ together with the relations in proposition 33 define the morphisms for an RFA $\mathcal{F}$ and give the relations in equations 1.17, 1.18 and 1.19, via the following assignments.

$$\begin{aligned} &Z(S_{1,1}, a) \mapsto P_a,\ Z(\tau) \mapsto \tau_{\mathcal{F},\mathcal{F}} \\ &Z(S_{1,2}, a) \mapsto \mu_a,\ Z(S_{1,0}, a) \mapsto \eta_a, \\ &Z(S_{2,1}, a) \mapsto \Delta_a,\ Z(S_{0,1}, a) \mapsto \varepsilon_a, \end{aligned}$$

where $\tau$ denotes the transposition of two circles in $\mathbf{Cob_2^{area}}$ and $\tau_{\mathcal{F},\mathcal{F}}$ denotes the symmetric braiding on $\mathcal{F} \otimes \mathcal{F}$ in **Hilb**. The equations in (1.20) are satisfied because $Z$ is a $*$-functor.

A morphism $\Phi : Z \to Z'$ in **AQFT** is a monoidal natural transformation, and it is determined by $\Phi_{Z(\mathbb{S}_1)}$, this is assigned to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ in **cRFrob**. Naturality implies the relations in equations 1.27 and 1.28, i.e. it is a morphism of RFAs.

It is clear that this gives an equivalence the same way as in the case of theorem 31. $\square$

## 2.3 Strong Continuity

Theorem 36 tells us that if we want to classify AQFTs, then it is enough to classify RFAs. In order to be able to use the classification result in theorem 27, we need to specify the AQFTs we are dealing with. Let $\mathcal{K}, \mathcal{K}' \in$ **Hilb** and $N$ be a positive integer. Then a map $Q_N : (\mathbb{R}_+)^N \to \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is strongly continuous, if for every $(r_1, \dots, r_N) \in (\mathbb{R}_+)^N$ and every $1 \le i \le N$ the map

$$x \mapsto Q_N(r_1, \dots, r_{i-1}, x, r_{i+1}, \dots, r_N)$$

is strongly continuous.

Let now $(M, \mathcal{A}_M)$ denote a bordism with area with $n$ circles on the ingoing and $m$ circles on the outgoing boundary. Recall that $\pi_0(M)$ is the set of connected components of $M$ and that $\mathcal{A}_M$ is a map $\pi_0(M) \to \mathbb{R}_+$.

[9]See the comment after lemma 13 on page 18.

**Definition 37.** *We call a functor $Z : \mathbf{Cob_2^{area}} \to \mathbf{Hilb}$ strongly continuous, if the map*

$$\begin{aligned} (\mathbb{R}_+)^{\times|\pi_0(M)|} &\to \mathcal{B}(\mathcal{H}^{\hat{\otimes} n}, \mathcal{H}^{\hat{\otimes} m}) \\ (\mathcal{A}_M(c))_{c\in\pi_0(M)} &\mapsto Z(M, \mathcal{A}_M) \end{aligned} \tag{2.5}$$

*is strongly continuous for all $M$.*

**Lemma 38.** *A monoidal functor $Z : \mathbf{Cob_2^{area}} \to \mathbf{Hilb}$ is strongly continuous, if and only if the map $a \mapsto P_a = Z(S_{1,1}, a)$ is strongly continuous.*

*Proof.* The direction ($\Rightarrow$) is of course clear. To show the other direction, first note that the strong continuity of the map in equation 2.5 for all bordism with area is equivalent to requiring it for all bordisms with area with one connected component only. This is clear after establishing the following claim.

Claim: Let $A : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H}, \mathcal{H}')$, $B : \mathbb{R}_+ \to \mathcal{B}(\mathcal{K}, \mathcal{K}')$. Then the following are equivalent:

1. $A$ and $B$ are strongly continuous maps,
2. the map $(a, b) \mapsto A(a)\hat{\otimes}B(b) \in \mathcal{B}(\mathcal{H}\hat{\otimes}\mathcal{K}, \mathcal{H}'\hat{\otimes}\mathcal{K}')$ is strongly continuous.

So let us fix $b_0 \in \mathbb{R}_+$. We need to check that for any $T \in \mathcal{H}\hat{\otimes}\mathcal{K}$

$$\left\|A(a)\hat{\otimes}B(b_0)T - A(a_0)\hat{\otimes}B(b_0)T\right\| \xrightarrow{a\to a_0} 0.$$

Choose a sequence $T_n \xrightarrow{n\to\infty} T$ such that for every $n$ $T_n$ is in the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$. We know that if $x \to x_0$ in $\mathcal{H}$ and $y \to y_0$ in $\mathcal{K}$ then $x\hat{\otimes}y \to x_0\hat{\otimes}y_0$ in $\mathcal{H}\hat{\otimes}\mathcal{K}$. Therefore for every $n$

$$\left\|A(a)\hat{\otimes}B(b_0)T_n - A(a_0)\hat{\otimes}B(b_0)T_n\right\| \xrightarrow{a\to a_0} 0.$$

Let us write $A(a_0) = A_0$ and $B(b_0) = B_0$, then

$$\begin{aligned} \left\|A(a)\hat{\otimes}B_0T - A_0\hat{\otimes}B_0T\right\| &= \left\|(A(a) - A_0)\hat{\otimes}B_0T_n + (A(a) - A_0)\hat{\otimes}B_0(T - T_n)\right\| \\ &\leq \left\|(A(a) - A_0)\hat{\otimes}B_0T_n\right\| + \left\|A_0\hat{\otimes}B_0(T - T_n)\right\| \\ &+ \left\|A(a)\hat{\otimes}B_0(T - T_n)\right\|. \end{aligned} \tag{2.6}$$

Now for every $\varepsilon > 0$ we can choose $N_\varepsilon$ such that for every $n > N_\varepsilon$

$$\|T - T_n\| < \frac{\varepsilon}{4\left\|A_0\hat{\otimes}B_0\right\|}.$$

For such an $n$ the first term in (2.6) can be made smaller than $\varepsilon/4$ for $|a - a_0|$ sufficiently small. Note that $\left\|A(a)\hat{\otimes}B_0\right\| = \|A(a)\| \|B_0\|$. By strong continuity of $A$, $\|A(a)\|$ can be made smaller than $2\|A_0\|$ for $|a - a_0|$ sufficiently small[10], so altogether

$$\begin{aligned} \left\|A(a)\hat{\otimes}B_0T - A_0\hat{\otimes}B_0T\right\| &\leq \left\|(A(a) - A_0)\hat{\otimes}B_0T_n\right\| + \left\|A_0\hat{\otimes}B_0\right\| \cdot \|(T - T_n)\| \\ &+ \left\|A(a)\hat{\otimes}B_0\right\| \cdot \|(T - T_n)\| \\ &< \frac{\varepsilon}{4}\left(1 + 1 + \frac{\left\|(A(a)\hat{\otimes}B_0\right\|}{\left\|A_0\hat{\otimes}B_0\right\|}\right) < \varepsilon, \end{aligned}$$

[10] We have $\|A(a)f\| \leq \|(A(a) - A_0)f\| + \|A_0f\|$ and the second term on the right hand side can be made arbitrarily small for any $f \in \mathcal{H}$.

so we just proved our claim.

Let us consider now connected bordisms. We will consider three cases: when the bordism with area has one ingoing boundary component, when it has one outgoing boundary component, and when it has no in or outgoing boundary. When one has more than one in- and outgoing boundaries the argumentation is analogous. So fix $0 < \varepsilon < a_0$ and $0 < x_0$ such that $\varepsilon + x_0 < a_0$.

Let $(M_{n,1}, a)$ denote a bordism with area $a$ and 1 in- and $n$ outgoing boundary. We can write $Z(M_{n,1}, a) = Z(M_{n,1}, a - x) \circ P_x$ for $x > 0$ by functoriality. So calculate for $f \in \mathcal{H}$

$$\begin{aligned} &\|(Z(M_{n,1}, a) - Z(M_{n,1}, a_0))\, f\| = \|Z(M_{n,1}, a_0 - \varepsilon)\,(P_{a-a_0+\varepsilon} - P_\varepsilon)\, f\| \\ \leq\ &\|Z(M_{n,1}, a_0 - \varepsilon)\|\, \|(P_{a-a_0+\varepsilon} - P_\varepsilon)\, f\| \xrightarrow{a \to a_0} 0, \end{aligned}$$

because $P$ is strongly continuous.

Now let $(M_{1,m}, a)$ denote a bordism with area $a$ and $m$ in- and 1 outgoing boundary. We can write $Z(M_{1,m}, a) = P_x \circ Z(M_{1,m}, a - x)$ for $x > 0$ by functoriality. So calculate for $g \in \mathcal{H}^{\hat{\otimes} m}$

$$\|(Z(M_{1,m}, a) - Z(M_{1,m}, a_0))\, g\| = \|(P_{a-a_0+\varepsilon} - P_\varepsilon)\, Z(M_{1,m}, a_0 - \varepsilon) g\| \xrightarrow{a \to a_0} 0,$$

because $P$ is strongly continuous.

Now let $(N, a)$ be a bordism with area $a$ and no boundary. We can cut off a disk, so we have $Z(N, a) = Z(S_{0,1}, x_0) \circ Z(\hat{N}, a - x_0)$. From the Riesz representation theorem (see [17] theorem 5.62) we have a unique vector $e_{x_0} \in \mathcal{H}$, such that $Z(S_{0,1}, x_0)(f) = \langle e_{x_0}, h \rangle$ for all $h \in \mathcal{H}$. So calculate

$$\begin{aligned} &\|Z(N, a) - Z(N, a_0)\| = \left| \left\langle e_{x_0}, \left( Z(\hat{N}, a)(1) - Z(\hat{N}, a_0)(1) \right) \right\rangle \right| \\ \leq\ &\|e_{x_0}\| \left\| Z(\hat{N}, a)(1) - Z(\hat{N}, a_0)(1)) \right\| \xrightarrow{a \to a_0} 0, \end{aligned}$$

by Cauchy–Schwarz and the previous argument. □

**Definition 39.** *A strongly continuous AQFT is an AQFT, which is a strongly continuous functor as well.*

Let $\mathbf{AQFT_{sc}}$ denote the category of strongly continuous AQFTs, which is a full subcategory of $\mathbf{AQFT}$.

**Theorem 40.** *There is an equivalence of categories* $\mathbf{AQFT_{sc}} \xrightarrow{\sim} \mathbf{cRFrob_{sc}}$.

*Proof.* The proof is essentially the same as the proof of theorem 36, we need to check that a strongly continuous AQFT is sent to a strongly continuous RFA. But this is clear, because $Z(S_{1,1}, a)$ for $Z$ an AQFT is sent to $P_a$ of the RFA $Z(\mathbb{S}_1)$, conversely use lemma 38. □

## 2.4 Zero Area Limits

We can alternatively allow bordisms to have zero areas. That is, we can define a category $\mathbf{Cob_2^{area,0}}$, which is given by the same definition as $\mathbf{Cob_2^{area}}$, except that we allow the area maps $\mathcal{A}_M$ ($M$ a bordism) to take values in $\mathbb{R}_{\geq 0}$. Note that $\mathbf{Cob_2^{area}} \subsetneq \mathbf{Cob_2^{area,0}}$. We can consider symmetric monoidal $*$-functors $\mathbf{Cob_2^{area,0}} \to \mathbf{Hilb}$, which send the cylinders $(S_{1,1}, a)$, for $a \geq 0$ to injective maps. One can then ask the question: When does an AQFT extend to such a functor? By this we mean the following: Let $Z : \mathbf{Cob_2^{area}} \to \mathbf{Hilb}$ be an AQFT. Is there a symmetric monoidal $*$-functor $\tilde{Z} : \mathbf{Cob_2^{area,0}} \to \mathbf{Hilb}$, with $\tilde{Z}(S_{1,1}, a)$ injective, such that

$$\tilde{Z} \restriction_{\mathbf{Cob_2^{area}}} = Z? \tag{2.7}$$

If this is the case, we say that $Z$ has a zero area limit. Observe that restricting $\tilde{Z}$ to the subcategory of $\mathbf{Cob_2^{area,0}}$, where all bordisms have zero area gives a TFT, which is a $*$-functor as well, hence $\tilde{Z}(\mathbb{S}_1)$ is a commutative $*$-Frobenius algebra and in particular it is finite dimensional. Hence restricting $\tilde{Z}$ to $\mathbf{Cob_2^{area}}$ gives an AQFT with finite dimensional state space.

**Proposition 41.** *A AQFT has a zero area limit, if and only if its state space is finite dimensional.*

*Proof.* Let $Z$ denote an AQFT. The "only if" direction is clear from the previous argumentation. For the other direction first use theorem 36 to describe the AQFT $Z$ as an RFA $\mathcal{F}$, then decompose $\mathcal{F}$ into the eigenspaces of $P_a$ as in proposition 22. Now note that we have a finite number of distinct nonzero eigenvalues and corresponding eigenspaces of $P_a$, because of finite dimensionality, say $\{E_n\}_{n=1}^N$, which are $*$-Frobenius algebras with maps $\mu_n$, $\eta_n$, $\Delta_n$, $\varepsilon_n$. Note that we only know that the eigenvalues $c_n(a)$ of $P_a$ obey $c_n(a)c_n(b) = c_n(a+b)$, $n = 1, \dots, N$. Now the maps

$$\begin{array}{ll} \mu_0 := \sum_{n=1}^N \mu_n, & \eta_0 := \sum_{n=1}^N \eta_n, \\ \Delta_0 := \sum_{n=1}^N \Delta_n, & \varepsilon_0 := \sum_{n=1}^N \varepsilon_n \end{array}$$

are automatically bounded. Observe that $\mathcal{F}$, together with the maps $\mu_a$, $\eta_a$, $\Delta_a$, $\varepsilon_a$, for $a$ now non-negative give the generators of the desired functor $\tilde{Z}$. $\square$

**Corollary 42** (to corollary 21)**.** *Let $Z$ be a strongly continuous AQFT. Then*

$$\lim_{a \to 0} Z(S_{1,1}, a) = id_{Z(\mathbb{S}_1)} \tag{2.8}$$

*in the strong operator topology.*

*Proof.* Theorem 40 defines a strongly continuous RFA structure on $Z(\mathbb{S}_1)$ with $P_a = Z(S_{1,1}, a)$. Then one can apply corollary 21. $\square$

# Chapter 3

# Lattice Construction

Theorem 27 provides all possible examples of AQFTs. Another way of obtaining examples of AQFTs is through the lattice construction familiar from TFTs.

## 3.1 Bordisms with Cell Decompositions

We will follow [19] and [9] in order to define a cell decomposition of compact 1- and 2-manifolds into $k$-cells ($k = 0, 1, 2$) called PLCW decomposition. A $k$-cell is the homeomorphic image of the interior of the unit ball $B_k$ in $\mathbb{R}^k$. A cell decomposition of a compact $n$-manifold $M$ is the disjoint union of $k$-cells ($k = 0, \ldots, n$), such that their union is $M$ and they have the following property. For any $k$-cell $C$ there is a homeomorphism $\phi : B_k \to M$ such that

- $C$ is the homeomorphic image if the interior of $B_k$,
- $S_k$, the boundary of $B_k$, decomposes in a way that this decomposition is mapped by $\phi$ to the decomposition of $M$,
- $\phi$ restricted to each cell in the above decomposition of $S_k$ is a homeomorphism.

Let $M$ now denote a compact 2-manifold. We write $C(M)$ for the cell decomposition of $M$, $C_2(M)$ for the set of 2-cells of $M$ and call the 0-, 1- and 2-cells vertices, edges and faces respectively.

Let $\mathbf{Cob_2^{cw}}$ denote the category of bordisms with cell decomposition, which is $\mathbf{Cob_2}$ with the following extra structure on objects and morphisms. Each circle of an objects in $\mathbf{Cob_2}$ is endowed with a decomposition into a single 0- and 1-cell; the bordisms are endowed with a decomposition into 0-, 1- and 2-cells, such that the each circle of the boundary of a bordism is decomposed into a single 0- and 1-cell.[11] Composition of bordisms with cell decomposition $M$, $N$ is performed by gluing along the boundary parametrization, which needs to respect the cell decomposition of the boundaries. The cell decomposition of the composition $C(M \circ N)$ is the disjoint union of the cell decompositions $C(M)$ and $C(N)$, with the one of the two identical 0- and 1 cells for each glued boundary component removed. Let $F : \mathbf{Cob_2^{cw}} \to \mathbf{Cob_2}$ denote the forgetful

[11] Note that this is not the most general cell decomposition (one could allow more cells on objects), nonetheless we will use it because of its simplicity.

functor that forgets the cell decomposition. It is surjective and full. Note that $\mathbf{Cob_2^{cw}}$ is a $*$-category the same way as $\mathbf{Cob_2}$ is and $F$ is a $*$-functor.

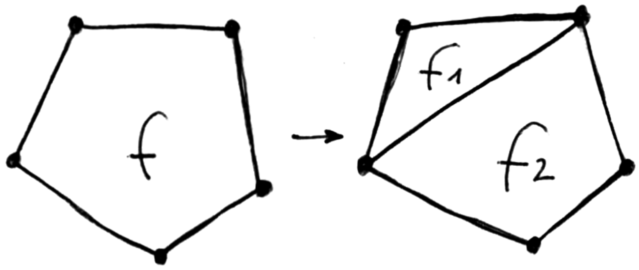


Figure 3.1: Splitting a face by inserting a new edge

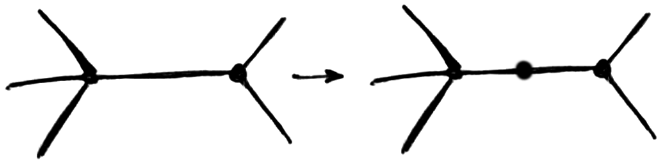

Figure 3.2: Splitting an edge by inserting a new vertex

Let $M$ be a bordism with cell decomposition $C(M)$. Then we can refine the cell decomposition by

- splitting an internal edge by adding a new vertex on it,
- splitting a face by adding a new edge between two of its vertices,

illustrated on figure 3.1 and 3.2. The refinement $C'(M)$ of a cell decomposition is again a cell decomposition of $M$. We have the following result, which is explained in detail in [19].

**Proposition 43.** *Let $C$ and $D$ denote two cell decompositions of $M$. There exists a cell decomposition $E$ of $M$ which is the refinement of both $C$ and $D$.*

Now we would like to extend this notation to include areas, so let $\mathbf{Cob_2^{area,cw}}$ denote the category of bordisms with area and cell decomposition, which has objects same as $\mathbf{Cob_2^{cw}}$, and the morphisms are generated from permutations and bordisms with area and cell decomposition. The latter are pairs $(M, \mathcal{A}_M)$ where $M$ is a bordism with cell decomposition and $\mathcal{A}_M : C_2(M) \to \mathbb{R}_+$ is a map called the area map and the number it assigns to a face is called the area of the face. The area map of disjoint union and composition of such bordisms assigns the same numbers as before. This category comes with a forgetful functor $F' : \mathbf{Cob_2^{area,cw}} \to \mathbf{Cob_2^{area}}$ as well, it forgets the cell decomposition on objects and bordisms and sends the area map of a bordism with area and cell decomposition to a map, which assigns to each connected component of the bordism the sum of the areas of its faces. Similarly, $\mathbf{Cob_2^{area,cw}}$ is a $*$-category the same way as $\mathbf{Cob_2^{area}}$ was and $F'$ is a $*$-functor, moreover $F'$ is surjective and full.

Now we would like to build TFTs the following way. We build a symmetric monoidal functor $Z^{cw} : \mathbf{Cob_2^{cw}} \to \mathbf{Vect}_{\Bbbk}$, and the show that it is independent of the details of the cell decomposition i.e. there is a symmetric monoidal functor $Z : \mathbf{Cob_2} \to \mathbf{Vect}_{\Bbbk}$, such that $Z^{cw} = Z \circ F$. Since $F$ is surjective and full, $Z$ is unique. This will be referred to as the lattice construction of $Z$. Then we would like to adjust this construction to build a symmetric monoidal $*$-functor $\mathbf{Cob_2^{area,cw}} \to \mathbf{Hilb}$ and show that it factorizes through $F'$.

## 3.2 Data for the Lattice Construction of TFTs

The data needed for the lattice construction is

- a finite dimensional vector space $V \in \mathbf{Vect}_{\Bbbk}$,
- a linear map $\beta : V \otimes V \to \Bbbk$, called the pairing,
- a family of linear maps $W^{(n)} : \Bbbk \to V^{\otimes n}$ for $n \geq 1$, called the weights.

This set of data needs to satisfy the following conditions

1. $\beta = \beta \circ \tau_{V,V}$ ($\beta$ is symmetric),
2. $\pi_n \circ W^{(n)} = W^{(n)}$ for any $n \geq 1$, where $\pi_n(v_1 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$ ($W^{(n)}$ is cyclic invariant),
3. $(id_{V^{\otimes(n-1)}} \otimes \beta \otimes id_{V^{\otimes(m-1)}}) \circ \big(W^{(n)} \otimes W^{(m)}\big) = W^{(n+m-2)}$ (gluing property),
4. $(id_V \otimes \beta) \circ \big(W^{(2)} \otimes id_V\big) = id_V$ (unitality requirement),
5. $(\beta \otimes id_{V^{\otimes(n-2)}}) \circ W^{(n)} = W^{(n-2)}$ for $n \geq 3$ (self gluing property).

We refer to a set of data satisfying this set of conditions as the data for the lattice construction. A similar set of data has been given in [20, 9]. We will see in the following how this data can be encoded in Frobenius algebras. We introduce a graphical notation for the above maps on figure 3.3 and the show the conditions 1-5 on figure 3.4. Note that


Figure 3.3: Graphical notation of the pairing and the weights

- $\beta$ determines $W^{(2)}$ via (4) uniquely,
- $\beta$ and $W^{(3)}$ determine $W^{(1)}$ via (5),
- $\beta$ and $W^{(3)}$ determine $W^{(n)}$ for $n \geq 3$ via (3).

So the independent data is $\beta$ and $W^{(3)}$, but we still keep this formulation as it provides a better picture. Using the self gluing property (5) and cyclicity (2) we also have $(id_{V^{\otimes k}} \otimes \beta \otimes id_{V^{\otimes(n-k-2)}}) \circ W^{(n)} = W^{(n-2)}$ for any $0 \leq k \leq n-2$.

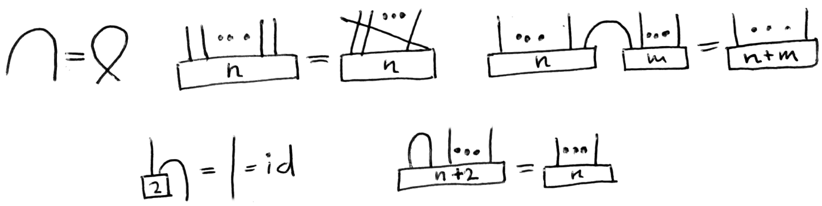


Figure 3.4: Graphical notation of the conditions 1-5

**Proposition 44.** *The data for the lattice construction determines a $\Delta$-separable symmetric Frobenius algebra by setting*

- $\eta := W^{(1)}$,
- $\varepsilon := \beta \circ \big(id_V \otimes W^{(1)}\big)$,
- $\mu := (id_V \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_V) \circ \big(W^{(3)} \otimes id_{V^{\otimes 2}}\big)$,
- $\Delta := (id_{V^{\otimes 2}} \otimes \beta) \circ \big(W^{(3)} \otimes id_V\big)$.

*Proof.* We show that the tuple $(V, \eta, \mu, \varepsilon, \Delta)$ is a Frobenius algebra. Calculate:

$$\begin{aligned} \mu \circ (id_V \otimes \eta) &= (id_V \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_V) \circ \Big(W^{(3)} \otimes W^{(1)} \otimes id_V\Big) \\ &= (id_V \otimes \beta) \circ \Big(W^{(2)} \otimes id_V\Big) = id_V, \end{aligned}$$

by conditions 3 and 4; to show that $\mu \circ (id_V \otimes \eta) = id_V$, condition 2 needs to be used. So $\eta$ is the unit of $\mu$. For the counit calculate

$$\begin{aligned} (\varepsilon \otimes id_V) \circ \Delta &= (\beta \otimes id_V \otimes \beta) \circ \Big(W^{(1)} \otimes W^{(3)} \otimes id_V\Big) \\ &= (id_V \otimes \beta) \circ \Big(W^{(2)} \otimes id_V\Big) = id_V, \end{aligned}$$

by conditions 3 and 4; to show that $(id_V \otimes \varepsilon) \circ \Delta = id_V$, condition 2 needs to be used. So $\varepsilon$ is the counit of $\Delta$. Now we need to check the Frobenius relation $\Delta \circ \mu = (\mu \otimes id_V) \circ (id_V \otimes \Delta)$. The lhs is

$$\begin{aligned} lhs &= (id_{V^{\otimes 2}} \otimes \beta \otimes \beta) \circ (id_{V^{\otimes 5}} \otimes \beta \otimes id_V) \circ \Big(W^{(3)} \otimes W^{(3)} \otimes id_{V^{\otimes 2}}\Big) \\ &= (id_{V^{\otimes 2}} \otimes \beta) \circ (id_{V^{\otimes 3}} \otimes \beta \otimes id_V) \circ \Big(W^{(4)} \otimes id_{V^{\otimes 2}}\Big), \end{aligned}$$

by using condition 3. The rhs is

$$\begin{aligned} rhs &= (id_V \otimes \beta \otimes id_V \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_{V^{\otimes 4}}) \circ \Big(W^{(3)} \otimes id_V \otimes W^{(3)} \otimes id_V\Big) \\ &= (id_V \otimes \beta \otimes id_V \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_{V^{\otimes 4}}) \circ (\pi_3^{-1} \otimes id_{V^{\otimes 4}}) \\ &\circ \Big(W^{(3)} \otimes id_V \otimes W^{(3)} \otimes id_V\Big) \\ &= (id_{V^{\otimes 2}} \otimes \beta) \circ (id_V \otimes \tau_{V,V} \otimes id_V \otimes \beta) \circ (\tau_{V,V} \otimes \beta \otimes id_V \otimes \tau_{V,V} \otimes id_V) \\ &\circ \Big(W^{(3)} \otimes W^{(3)} \otimes id_{V^{\otimes 2}}\Big) \\ &= (id_{V^{\otimes 2}} \otimes \beta) \circ (id_{V^{\otimes 3}} \otimes \beta \otimes id_V) \circ (\pi_4 \otimes id_{V^{\otimes 2}}) \\ &\circ \Big(W^{(4)} \otimes id_{V^{\otimes 2}}\Big) = lhs \end{aligned}$$

by using conditions 2, 3 and the naturality of the braiding. This calculation is also shown on figure 3.5. One similarly shows $\Delta \circ \mu = (id_V \otimes \mu) \circ (\Delta \otimes id_V)$.

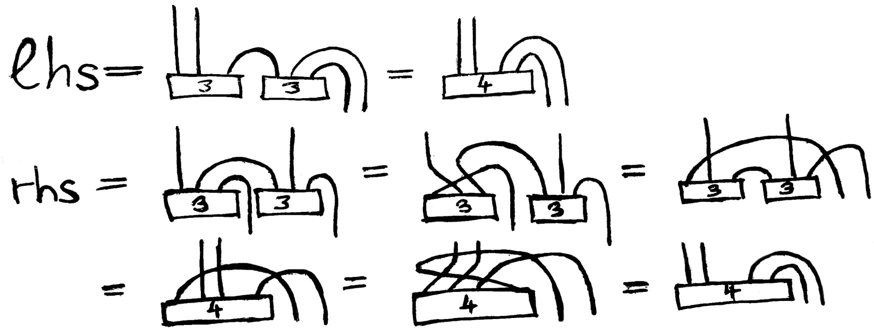


Figure 3.5: Calculation showing that the Frobenius relation holds

To show that $V$ is a symmetric Frobenius algebra calculate

$$\begin{aligned}
\varepsilon \circ \mu &= (\beta \otimes \beta) \circ (id_{V^{\otimes 3}} \otimes \beta \otimes id_V) \circ \left(W^{(1)} \otimes W^{(3)} \otimes id_{V^{\otimes 2}}\right) \\
&= \beta \circ (id_V \otimes \beta \otimes id_V) \circ \left(W^{(2)} \otimes id_{V^{\otimes 2}}\right) \\
&= \beta \circ (id_V \otimes \beta \otimes id_V) \circ (\tau_{V,V} \otimes id_V) \circ \left(W^{(2)} \otimes id_{V^{\otimes 2}}\right) \\
&= \beta \circ (id_V \otimes \beta \otimes id_V) \circ \left(W^{(2)} \otimes \tau_{V,V}\right) \\
&= \varepsilon \circ \mu \circ \tau_{V,V},
\end{aligned}$$

by conditions 3, 2 and the naturality of the braiding.

To show that $V$ is $\Delta$-separable calculate

$$\begin{aligned}
\mu \circ \Delta &= (id_V \otimes \beta \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_{V^{\otimes 3}}) \circ \left(W^{(3)} \otimes W^{(3)} \otimes id_V\right) \\
&= (id_V \otimes \beta \otimes \beta) \circ \left(W^{(4)} \otimes id_V\right) \\
&= (id_V \otimes \beta) \circ \left(W^{(2)} \otimes id_V\right) = id_V,
\end{aligned}$$

by using conditions 3, 5 and 4. □

Let now $\Delta^{(n)} := (\Delta \otimes id_{F^{\otimes(n-2)}})$ with $\Delta^{(1)} = id_F$ and $\Delta^{(2)} = \Delta$ denote the $n$-fold coproduct on a Frobenius algebra $F$.

**Proposition 45.** *A $\Delta$-separable symmetric Frobenius algebra $F$ determines the data for the lattice construction by setting*

- *the vector space to be $F$,*
- $\beta := \varepsilon \circ \mu$,
- $W^{(n)} := \Delta^{(n)} \circ \eta$.

*Proof.* We need to check if the above defined set of data satisfies the conditions 1-5.

- Condition 1 follows from $F$ being symmetric.
- To show condition 2, first note that the cyclic invariance of $W^{(2)} = \Delta \circ \eta$ also follows from $F$ being symmetric. We can use this calculate

$$\begin{aligned} W^{(n)} = \Delta^{(n)} \circ \eta &= \left(\Delta^{(n-1)} \otimes id_F\right) \circ \Delta \circ \eta \\ &= \left(\Delta^{(n-1)} \otimes id_F\right) \circ \tau_{F,F} \circ \Delta \circ \eta = \pi_n^{-1} \circ W^{(n)}, \end{aligned}$$

  by coassociativity and again by the naturality of the braiding.
- To show the gluing property (3) calculate using the Frobenius relation, unitality and counitality that

$$\begin{aligned} &(id_{V^{\otimes(n-1)}} \otimes \beta \otimes id_{V^{\otimes(m-1)}}) \circ \left(W^{(n)} \otimes W^{(m)}\right) \\ &= \left(\Delta^{(n-1)} \otimes \varepsilon \otimes \Delta^{(m-1)}\right) \circ (id_F \otimes \mu \otimes id_F) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta) \\ &= \left(\Delta^{(n-1)} \otimes \Delta^{(m-1)}\right) \circ (\mu \otimes \varepsilon \otimes id_V) \circ (\eta \otimes \Delta \otimes id_F) \circ \Delta \circ \eta \\ &= \left(\Delta^{(n-1)} \otimes \Delta^{(m-1)}\right) \circ \Delta \circ \eta = W^{(n+m-2)}. \end{aligned}$$

- To show that the unitality requirement (4) is satisfied use the Frobenius relation, unitality and counitality:

$$\begin{aligned} &(id_F \otimes \beta) \circ \left(W^{(2)} \otimes id_F\right) \\ &= (id_F \otimes \varepsilon) \circ (id_F \otimes \mu) \circ (\Delta \otimes id_F) \circ (\eta \otimes id_F) \\ &= (id_F \otimes \varepsilon) \circ \Delta \circ \mu \circ (\eta \otimes id_F) = id_F. \end{aligned}$$

- For the last condition (5) calculate for $k \geq 1$ that

$$\begin{aligned} (\beta \otimes id_{F^{\otimes k}}) \circ W^{(k+2)} &= \left(\varepsilon \otimes \Delta^{(k)}\right) \circ ((\mu \circ \Delta) \otimes id_F) \circ \Delta \circ \eta \\ &= \left(\varepsilon \otimes \Delta^{(k)}\right) \circ \Delta \circ \eta = \Delta^{(k)} \circ \eta = W^{(k)}, \end{aligned}$$

  where we used $\Delta$-separability and counitality of $F$.

□

Let us fix a finite dimensional vector space $V$ and denote the sets

- $\mathbf{L} := \{\text{data for lattice construction on } V\}$,
- $\mathbf{F} := \{\Delta\text{-separable symmetric Frobenius algebra structures on } V\}$.

Proposition 44 and 45 define maps of sets $\alpha$ and $\omega$ respectively:

$$\mathbf{L} \underset{\omega}{\overset{\alpha}{\rightleftarrows}} \mathbf{F}. \tag{3.1}$$

**Theorem 46.** *The maps in (3.1) are inverse to each other.*

*Proof.* First show that $\omega \circ \alpha = id_{\mathbf{L}}$, note that it is enough to show that $W^{(1)}$, $W^{(3)}$ and $\beta$ get mapped to themselves. Let $L \in \mathbf{L}$, i.e. it is a tuple $L = \left(V, \beta, W^{(n)}\right)$, then $\alpha(L) = (V, \eta, \mu, \varepsilon, \Delta)$ is an FA.

- It is clear that $W^{(1)}$ gets mapped to itself.
- $W^{(3)}$ gets mapped to $(\Delta \otimes id_V) \circ \Delta \circ \eta$, this equals

$$\begin{aligned}&(id_{V^{\otimes 2}} \otimes \beta \otimes id_V \otimes \beta) \circ \left(W^{(3)} \otimes W^{(3)} \otimes W^{(1)}\right)\\ =&(id_{V^{\otimes 2}} \otimes \beta \otimes id_V) \circ \left(W^{(3)} \otimes W^{(2)}\right) = W^{(3)},\end{aligned}$$

  by the gluing property.

- $\beta$ gets mapped to $\varepsilon \circ \mu$, which equals

$$\begin{aligned}&(\beta \otimes \beta) \circ (id_{V^{\otimes 3}} \otimes \beta \otimes id_V) \circ \left(W^{(1)} \otimes W^{(3)} \otimes id_{V^{\otimes 2}}\right)\\ =&\beta \circ (id_V \otimes \beta \otimes id_V) \circ \left(\otimes W^{(2)} \otimes id_{V^{\otimes 2}}\right) = \beta,\end{aligned}$$

  by the gluing property and the unitality requirement.

An analogous result can be found in [9].

Now show that $\alpha \circ \omega = id_{\mathbf{F}}$. Let $F \in \mathbf{F}$, i.e. $F = (V, \eta, \mu, \varepsilon, \Delta)$ and $\omega(F) = \left(V, \beta, W^{(n)}\right)$.

- It is clear that $\eta$ is mapped to itself.
- $\mu$ is mapped to $(id_V \otimes \beta) \circ (id_{V^{\otimes 2}} \otimes \beta \otimes id_V) \circ \left(W^{(3)} \otimes id_{V^{\otimes 2}}\right)$, which equals

$$\begin{aligned}&(id_V \otimes (\varepsilon \circ \mu)) \circ (\Delta \otimes (\varepsilon \circ \mu) \otimes id_V) \circ ((\Delta \circ \eta) \otimes id_{V^{\otimes 2}})\\ =&(id_V \otimes (\varepsilon \circ \mu)) \circ (\Delta \otimes \varepsilon \otimes id_V) \circ ((\Delta \circ \mu) \otimes id_V) \circ (\eta \otimes id_{V^{\otimes 2}})\\ =&(id_V \otimes (\varepsilon \circ \mu)) \circ (\Delta \otimes id_V) = (id_V \otimes \varepsilon) \circ \Delta \circ \mu = \mu,\end{aligned}$$

  by using the Frobenius relation, unitality and counitality.

- $\varepsilon$ is mapped to $\beta \circ \left(id_V \otimes W^{(1)}\right)$, which equals $\varepsilon \circ \mu \circ (id_V \otimes \eta) = \varepsilon$ by unitality.
- $\Delta$ is sent to $(id_{V^{\otimes 2}} \otimes \beta) \circ \left(W^{(3)} \otimes id_V\right)$, which equals

$$(\Delta \otimes (\varepsilon \circ \mu)) \circ ((\Delta \circ \eta) \otimes id_V) = (\Delta \otimes \varepsilon) \circ \Delta \circ \mu \circ (\eta \otimes id_V) = \Delta,$$

  by the Frobenius relation, unitality and counitality.

□

Note that there is a similar statement in [21] section 4.3, but there triangulations are used instead of an arbitrary cell decomposition.

## 3.3 Lattice Construction of TFTs

Using the data for the lattice construction we are ready to define a functor $Z^{cw} : \mathbf{Cob_2^{cw}} \to \mathbf{Vect}_{\Bbbk}$. Similar constructions has been given in [20, 9], the main difference here is that the we first assign elements to faces and then contract along edges. Consider now a bordism with a cell decomposition $M$ that has $N_i$ in- and $N_o$ outgoing boundaries, $N_c$ internal edges, $N_f$ faces, let $N_b := N_i + N_o$ and follow the next steps.

1. Take the tensor product of $W^{(n_p)}$ for each face $p$ with $n_p$ edges

$$W := \bigotimes_{p=1}^{N_f} W^{(n_p)} : \mathbb{k} \to \bigotimes_{p=1}^{N_f} V^{\otimes n_p} \tag{3.2}$$

   and note that for each internal edge we have two factors of $V$ in the above tensor product.

2. Consider the permutation that brings

   - each pair of $V$ assigned to an internal edge next to each other and to the left side of the tensor product,
   - each copy of $V$ assigned to outgoing boundaries right of the previous bunch,
   - each copy of $V$ assigned to ingoing boundaries to the right side of the tensor product:

$$\rho : \bigotimes_{p=1}^{N_f} V^{\otimes n_p} \to \bigotimes_{c=1}^{N_c} V^{\otimes 2} \otimes \bigotimes_{b_o=1}^{N_o} V \otimes \bigotimes_{b_i=1}^{N_i} V, \tag{3.3}$$

   where $b_i$ and $b_o$ run over in- and outgoing boundary edges respectively and $c$ runs over internal edges.

3. Apply $\beta$'s on internal edges

$$C := \beta^{\otimes N_c} \otimes id_{V^{\otimes N_b}} : V^{\otimes 2N_c} \otimes V^{\otimes N_o} \otimes V^{\otimes N_i} \to V^{\otimes N_o} \otimes V^{\otimes N_i} \tag{3.4}$$

4. Let $B_k := id_{V^{\otimes(N_i-k)}} \otimes \beta \otimes id_{V^{\otimes(N_i-k)}}$ and $\tilde{B}^{N_i} := B_{N_i} \circ B_{N_i-1} \circ \cdots \circ B_2 \circ B_1$. The corresponding graphical notation is shown on figure 3.6.

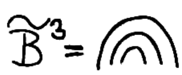

Figure 3.6: Graphical notation of $\tilde{B}^3$

5. Take the composition of the maps 3.2, 3.3, 3.4 and form the following

$$\left(id_{V^{\otimes N_o}} \otimes \tilde{B}^{N_i}\right) \circ ((C \circ \rho \circ W) \otimes id_{V^{\otimes N_i}}) : V^{\otimes N_i} \to V^{\otimes N_o}, \tag{3.5}$$

   which we denote $Y(M) : V^{\otimes N_i} \to V^{\otimes N_o}$.

These steps will be referred to as the lattice construction. Observe that as a consequence of our implementation of gluing via $\beta$ we have

$$Y(M \circ M') = Y(M) \circ Y(M'), \tag{3.6}$$

for $M'$ another bordism with cell decomposition, i.e. this assignment respects composition. We now show that $Y(M)$ is independent of the cell decomposition of $M$.

**Proposition 47.** *$Y(M)$ does not change if we refine the cell decomposition of $M$.*

*Proof.* The key idea is to use the gluing property and the self-gluing property.

Take a face $f$ in the cell decomposition and split it into the faces $f_1$ and $f_2$ as seen on figure 3.1. According to the defining steps of $Y(M)$ we will assign $W^{n_{f_1}} \otimes W^{n_{f_2}}$ to these faces and then apply $\beta$:

$$\left(id_{V^{n_{f_1}-1}} \otimes \beta \otimes id_{V^{n_{f_2}-1}}\right) \circ \left(W^{n_{f_1}} \otimes W^{n_{f_2}}\right),$$

which equals $W^{n_f}$ by the gluing property. This is was originally assigned to the face $f$, so $Y(M)$ has not changed.

Instead of showing that $Y(M)$ is not changed by splitting an internal edge, we show that $Y(M)$ is not changed if we add an edge and a vertex as shown on figure 3.7. On figure 3.8 it is shown how adding a vertex and an edge together with the freedom of adding and removing edges can be used to split an edge by adding a vertex. Conversely we can first add an edge and then split it, so these two pair of moves are equivalent. For details refer to [9] section 3.6.

Figure 3.7: Adding an edge and a vertex

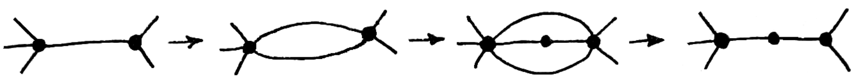

Figure 3.8: Another way of adding a vertex

$W^{(2)}$ is assigned to the left hand side of figure 3.7, on the right hand side we follow the defining steps of $Y(M)$: The assigned morphism is

$$\begin{aligned}
&(id_V \otimes \beta \otimes id_V) \circ (\tau_{V,V} \otimes \beta \otimes \tau_{V,V}) \circ \left(W^{(3)} \otimes W^{(3)}\right) \\
&= (id_V \otimes \beta \otimes id_V) \circ (\tau_{V,V} \otimes \tau_{V,V}) \circ W^{(4)} \\
&= (id_V \otimes \beta \otimes id_V) \circ (\tau_{V,V} \otimes \tau_{V,V}) \circ \pi_4^{-1} \circ W^{(4)} \\
&= (\beta \circ \tau_{V,V} \otimes id_{V^{\otimes 2}}) \circ W^{(4)} = (\beta \otimes id_{V^{\otimes 2}}) \circ W^{(4)} = W^{(2)},
\end{aligned}$$

by the gluing property, cyclic invariance and the self-gluing property of the weights and the symmetry of the pairing. □

Before completing the definition of $Z^{cw}$, we need a few more considerations. We use the result of theorem 46 and write everything in terms of the Frobenius algebra $V$. Let us calculate what is assigned to the cylinder $\mathbb{S}_1 \to \mathbb{S}_1$. Consider the cylinder decomposed into a single face, one vertex on each boundary and one edge connecting these vertices as on figure 3.9. The map $P$ assigned to this cylinder is shown on figure 3.10, which can also be written as on figure 3.11, the calculation is shown on figure 3.12.

Figure 3.9: Decomposition of a cylinder into a single 2-cell

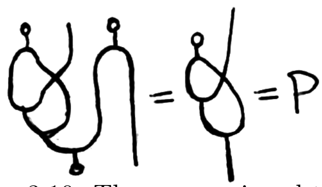

Figure 3.10: The map assigned to a cylinder

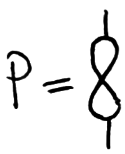

Figure 3.11

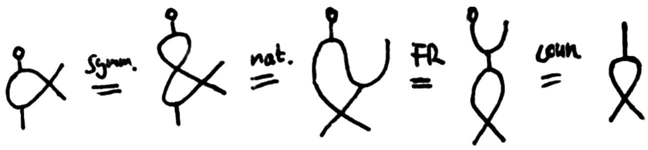


Figure 3.12: Calculation of the alternative form of $P$

**Lemma 48.** *We have $P^2 = P$ and $P$ projects onto the center of $V$.*

*Proof.* On one hand, since $Y$ respects composition we have $P^2 = P$, see figure 3.13. On the other hand we would like to illustrate this with a direct calculation,

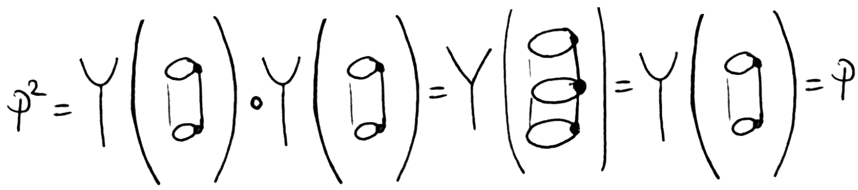

Figure 3.13

see figure 3.14.

To see that $P$ indeed projects onto the center of $V$ we will show that

- the center of $V$ is contained in the image of $P$, more precisely we show that $z = P(z)$ for all $z$ in the center, see figure 3.15 for the calculation,
- the image of $P$ is contained in the center of $V$, i.e. $P(x)y = yP(x)$ for all $x, y \in V$, see figure 3.16 for this calculation.

□

Now since $Y(M)$ is independent of the cell decomposition of $M$, let us consider another cell decomposition of $M$, which is obtained by the old cell decomposition glued together with cylinders on all boundaries. Since $Y$ respects composition we have $Y(M) = P^{\otimes N_o} \circ Y(M) \circ P^{\otimes N_i}$, hence there is no harm done if we restrict $Y(M)$ to the appropriate tensor power of the image of $P$ and $Y(M)$ lands in the appropriate tensor power of the image of $P$.

We define $Z^{cw}$ to send an object with $n$ circles to the $n$-th tensor power of the image of $P$, bordisms to $Y(M)$ restricted to the appropriate tensor power of the image of $P$ and permutations to permutations. It is clear from the lattice

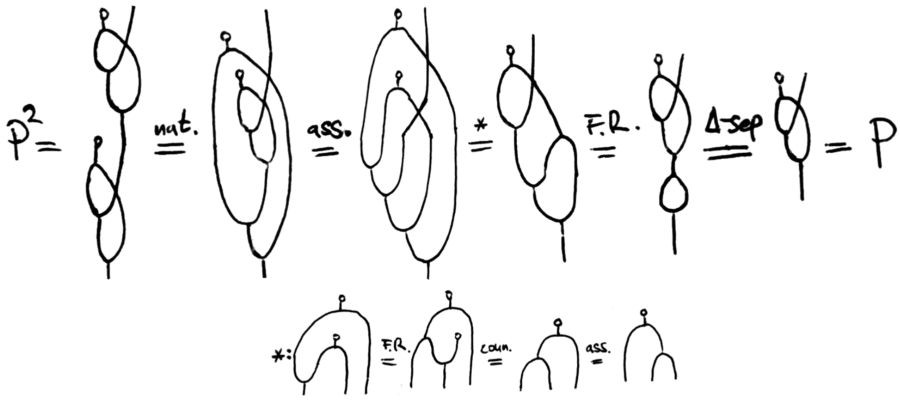


Figure 3.14

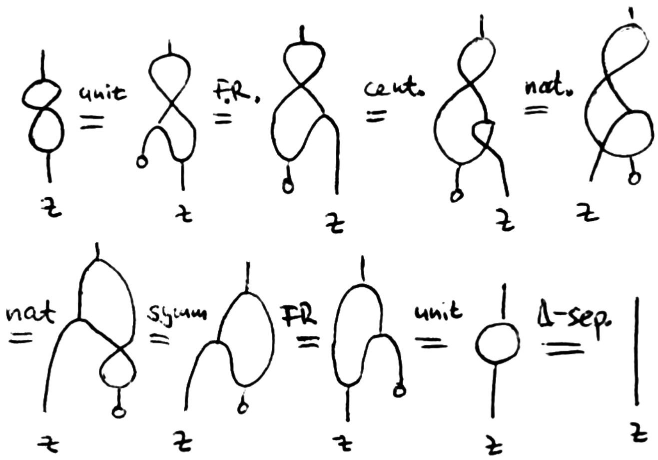


Figure 3.15: Calculation of $z = P(z)$ for $z$ in the center

construction that we obtain a symmetric monoidal functor and that it factorizes through $F$, i.e. it is independent of the cell decomposition of bordisms. We have just shown the following theorem, also to be found in [9].

**Theorem 49.** *The lattice construction using $V$ defines a TFT, whose state space is the center of $V$.*

## 3.4 Data for the Lattice Construction of AQFTs

In order to generalize the above construction to build an AQFT, we need a slightly different set of data as a starting point. We fix the following set of data:

- a Hilbert space $\mathcal{L} \in \mathbf{Hilb}$ and
- the pairing $\beta_a \in \mathcal{B}(\mathcal{L}^{\hat{\otimes} 2}, \mathbb{C})$,
- the weights $W_a^{(n)} \in \mathcal{B}(\mathbb{C}, \mathcal{L}^{\hat{\otimes} n})$ for every $n \geq 1$ integer,

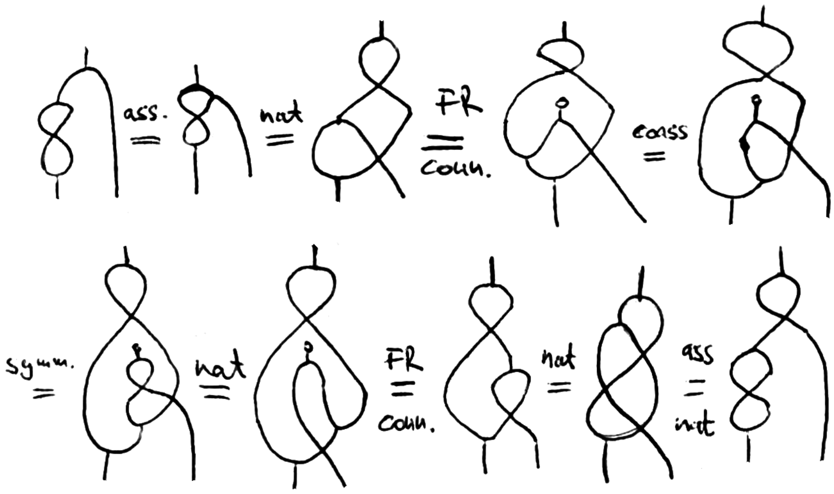


Figure 3.16: Calculation of $P(x)y = yP(x)$

which are two families of maps, where the parameter $a$ takes positive real values. This data has to satisfy the following adjusted conditions:

1. $\beta_a = \beta_a \circ \tau_{\mathcal{L},\mathcal{L}}$,
2. $\pi_n \circ W_a^{(n)} = W_a^{(n)}$, where $\pi_n$ is defined as in condition 2 in the TFT case,
3. $\left(id_{\mathcal{L}^{\hat{\otimes}(n-1)}} \hat{\otimes} \beta_{a_0} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}(m-1)}}\right) \left(W_{a_1}^{(n)} \hat{\otimes} W_{a_2}^{(m)}\right) = W_{a_0+a_1+a_2}^{(n+m-2)}$,
4. $\left(\beta_{a_1} \hat{\otimes} id_{\mathcal{L}}\right) \circ \left(id_{\mathcal{L}} \hat{\otimes} W_{a_2}^{(2)}\right) =: P_{a_1+a_2}$ is injective,
5. $\left(\beta_{a_1} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}n}}\right) \circ W_{a_2}^{(n+2)} = W_{a_1+a_2}^{(n)}$,

for any $a, a_0, a_1, a_2$ positive and $n, m \geq 1$. Note again that $\beta_a$, $W_a^{(1)}$ and $W_a^{(3)}$ determine all $W_a^{(n)}$ for $n \geq 1$. Furthermore $P_a$ is a semigroup: $P_{a_1} \circ P_{a_2} = P_{a_1+a_2}$;

$$P_a = \left(id_{\mathcal{L}} \hat{\otimes} \beta_{a_1}\right) \circ \left(W_{a_2}^{(2)} \hat{\otimes} id_{\mathcal{L}}\right), \tag{3.7}$$

$$\left(P_{a_1} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}n-1}}\right) \circ W_{a_2}^{(n)} = W_{a_1+a_2}^{(n)}. \tag{3.8}$$

Therefore it is natural to require that

6. $\beta_{a_1} \circ \left(P_{a_2} \hat{\otimes} id_{\mathcal{L}}\right) = \beta_{a_1+a_2}$.

These last two equations show that we can freely distribute the parameter among the pairing and the weights, whenever they are contracted. So far these are straightforward generalizations of the data and the conditions for the lattice construction of TFTs. Let us introduce the following notation: $B_a^k := \left(id_{\mathcal{L}^{\hat{\otimes}(k-1)}} \hat{\otimes} \beta_a \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}(k-1)}}\right)$, $\tilde{B}_a^n := B_{a/n}^1 \circ B_{a/n}^2 \circ \cdots \circ B_{a/n}^n$. In order to later fit the definition of an AQFT we need to require additionally that

7. $\left(W_a^{(1)}\right)^* = \beta_{a_1} \circ \left(W_{a_2}^{(1)} \hat{\otimes} id_{\mathcal{L}}\right)$, $\left(W_a^{(3)}\right)^* = \tilde{B}_{a_1}^3 \circ \left(W_{a_2}^{(3)} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}3}}\right)$ and $\beta_a^* = W_a^{(2)}$.

A straightforward consequence of the last condition and equation 3.7 is that $P_a$ is self-adjoint. If we want to build an AQFT, which is strongly continuous we need to additionally require that

8. the assignments $a \mapsto \beta_a$ and $a \mapsto W_a^{(n)}$ are strongly continuous.

**Proposition 50.** *The data for the lattice construction of AQFTs on $\mathcal{L}$ subject to conditions 1-7 determines a $\Delta$-separable symmetric regularized $*$-Frobenius algebra by setting*

- $\eta_a := W_a^{(1)}$,
- $\mu_a := \left(id_{\mathcal{L}} \hat{\otimes} \tilde{B}_{a_1}^2\right) \circ \left(W_{a_2}^{(3)} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 2}}\right)$, *with* $a_1 + a_2 = a$,
- $\varepsilon_a := \beta_{b_1} \circ \left(W_{b_2}^{(1)} \hat{\otimes} id_{\mathcal{L}}\right)$, *with* $b_1 + b_2 = a$,
- $\Delta_a := \left(id_{\mathcal{L}^{\hat{\otimes} 2}} \hat{\otimes} \beta_{c_1}\right) \circ \left(W_{c_2}^{(3)} \hat{\otimes} id_{\mathcal{L}}\right)$, *with* $c_1 + c_2 = a$.

*If we also require condition 8 then this RFA is strongly continuous.*

*Proof.* Then the proof is essentially the same as of proposition 44, we just need to keep in mind that the parameters need to add up on each side of the equations.

We need to check the injectivity of $\tilde{P}_a := \mu_{a_1} \circ \left(\eta_{a_2} \hat{\otimes} id_{\mathcal{L}}\right)$, so calculate

$$\begin{aligned}
\tilde{P}_a &= \mu_{a_1} \circ \left(\eta_{a_2} \hat{\otimes} id_{\mathcal{L}}\right) = \left(id_{\mathcal{L}} \hat{\otimes} \tilde{B}_{b_1}^k\right) \circ \left(W_{b_2}^{(3)} \hat{\otimes} W_{b_3}^{(1)} \hat{\otimes} id_{\mathcal{L}}\right) \\
&= \left(id_{\mathcal{L}} \hat{\otimes} \beta_{c_1}\right) \circ \left(W_{c_2}^{(2)} \hat{\otimes} id_{\mathcal{L}}\right) = P_a,
\end{aligned}$$

by using the gluing property ($a = a_1 + a_2 = b_1 + b_2 + b_3 = c_1 + c_2$), so $\tilde{P}_a$ is clearly injective. This also shows that $a \mapsto \tilde{P}_a$ is strongly continuous, if condition 8 holds.

Additionally we need to check if $\eta_a^* = \varepsilon_a$ and $\mu_a^* = \Delta_a$. The first equation is automatically satisfied by the definition of $\varepsilon_a$ and condition 7. For the second equation calculate using condition 7 and 4 that

$$\begin{aligned}
\mu_a^* &= \left(\tilde{B}_{a_1}^3 \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 2}}\right) \circ \left(W_{a_2}^{(3)} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 5}}\right) \circ \left(id_{\mathcal{L}^{\hat{\otimes} 2}} \hat{\otimes} W_{a_3}^{(2)} \hat{\otimes} id_{\mathcal{L}}\right) \circ \left(id_{\mathcal{L}} \hat{\otimes} W_{a_4}^{(2)}\right) \\
&= \left(id_{\mathcal{L}^{\hat{\otimes} 2}} \hat{\otimes} \beta_{b_1}\right) \circ \left(W_{b_2}^{(3)} \hat{\otimes} id_{\mathcal{L}}\right) = \Delta_a,
\end{aligned}$$

so $\mathcal{L}$ is a regularized $*$-Frobenius algebra.

□

**Proposition 51.** *A $\Delta$-separable symmetric regularized $*$-Frobenius algebra $\mathcal{F}$ determines the data for the lattice construction of AQFTs by setting*

- *the Hilbert space to be* $\mathcal{F}$,
- $\beta_a := \varepsilon_{a_1} \circ \mu_{a_2}$,
- $W_a^{(n)} := \Delta_{a_1}^{(n)} \circ \eta_{a_2}$,

*with* $a_1 + a_2 = a$. *If $\mathcal{F}$ is furthermore strongly continuous, then condition 8 holds.*

*Proof.* Proving conditions 1-5 can be done as in the proof of proposition 45, one just has to keep in mind that the parameters should add up on each side of the equations.

To show condition 6 calculate

$$\beta_{a_1} \circ \left(P_{a_2} \hat{\otimes} id_{\mathcal{L}}\right) = \varepsilon_{a_1'} \circ \mu_{a_1''} \left(P_{a_2} \hat{\otimes} id_{\mathcal{L}}\right) = \varepsilon_{a_1'} \circ \mu_{a_1''+a_2} = \beta_{a_1+a_2}.$$

To show that condition 7 is satisfied calculate

$$\begin{aligned}
\beta_a^* &= (\eta_{a_1} \circ \mu_{a_2})^* = \Delta_{a_2} \circ \varepsilon_{a_1} = W_a^{(2)}, \\
\left(W_a^{(3)}\right)^* &= \left((\Delta_{a_1} \hat{\otimes} id_{\mathcal{L}}) \circ \Delta_{a_2} \circ \eta_{a_3}\right)^* = \varepsilon_{a_3} \circ \mu_{a_2} \circ (\mu_{a_1} \hat{\otimes} id_{\mathcal{L}}) \\
&= \varepsilon_{a_3} \circ \mu_{a_2} \circ (\mu_{a_1'} \hat{\otimes} \varepsilon_{a_1''} \hat{\otimes} id_{\mathcal{L}}) \circ (id_{\mathcal{L}} \hat{\otimes} \Delta_{a_1'''} \hat{\otimes} id_{\mathcal{L}}) \\
&= \varepsilon_{a_3} \circ \mu_{a_2} \circ (id_{\mathcal{L}} \hat{\otimes} \varepsilon_{a_1''} \circ \mu_{a_1'} \hat{\otimes} id_{\mathcal{L}}) \circ (\Delta_{a_1'''} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 2}}) \\
&= \varepsilon_{a_3} \circ \mu_{a_2} \circ (id_{\mathcal{L}} \hat{\otimes} \varepsilon_{a_1''} \circ \mu_{a_1'} \hat{\otimes} id_{\mathcal{L}}) \\
&\circ (\Delta_{b_1} \hat{\otimes} \varepsilon_{b_2} \circ \mu_{b_3} \hat{\otimes} id_{\mathcal{L}}) \circ (\Delta_{b_4} \circ \eta_{b_5} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 3}}) \\
&= \tilde{B}^3_{c_1} \circ \left(W_{c_3}^{(3)} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 3}}\right).
\end{aligned}$$

To show strong continuity first calculate

$$\tilde{P}_a = \left(id_{\mathcal{L}} \hat{\otimes} \beta_{a_1}\right) \circ \left(W_{a_2}^{(2)} \hat{\otimes} id_{\mathcal{L}}\right) = \left(id_{\mathcal{L}} \hat{\otimes} \varepsilon_{b_1} \circ \mu_{b_2}\right) \circ \left(\Delta_{b_3} \circ \eta_{b_4} \hat{\otimes} id_{\mathcal{L}}\right) = P_a,$$

with $a = a_1 + a_2 = b_1 + b_2 + b_3 + b_4$. Then fix $\epsilon > 0$ such that $a - \epsilon = a' > 0$ and $b - \epsilon = b' > 0$ and calculate:

$$\begin{aligned}
\left|(\beta_a - \beta_b)(f)\right| &= \left|\beta_\epsilon \circ \left((P_{a'} - P_{b'}) \hat{\otimes} id_{\mathcal{L}}\right)(f)\right| \\
&\leq \|\beta_\epsilon\| \left\|\left((P_{a'} - P_{b'}) \hat{\otimes} id_{\mathcal{L}}\right)(f)\right\| \xrightarrow{a \to b} 0,
\end{aligned}$$

as $a - b = a' - b'$ and $a \mapsto P_a$ is strongly continuous. Furthermore

$$\left\|W_a^{(n)} - W_b^{(n)}\right\| = \left\|\left((P_{a'} - P_{b'}) \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} n-1}}\right) \left(W_\epsilon^{(n)}\right)\right\| \xrightarrow{a \to b} 0.$$

□

Let us fix a Hilbert space $\mathcal{L}$ and denote the sets

- $\hat{\mathbf{L}} :=$ {data for lattice construction for AQFTs on $\mathcal{L}$ satisfying conditions 1-7},
- $\hat{\mathbf{F}} :=$ {$\Delta$-separable symmetric regularized $*$-Frobenius algebra structures on $\mathcal{L}$}.

Proposition 50 and 51 define maps of sets $A$ and $\Omega$ respectively:

$$\hat{\mathbf{L}} \underset{\Omega}{\overset{A}{\rightleftarrows}} \hat{\mathbf{F}}. \tag{3.9}$$

**Theorem 52.** *The maps in (3.9) are inverse to each other.*

Note that the proof is essentially the same as in the TFT case, the parameters need to add up on each side of the equations. We see from the previous arguments that condition 8 being satisfied is equivalent to the RFA being strongly continuous. Equipped with this data we show how to adjust the lattice construction of TFTs to produce AQFTs.

## 3.5 Lattice Construction of AQFTs

Consider a bordism with area and cell decomposition $M$ with the notation of section 3.3. Let $n_p$ denote the number of edges of a face $p$, of which $n_p^c$ are internal[12] and $n_p^i$ sit on ingoing boundaries and $a_p$ the area assigned to it. Fix a small[13] $a > 0$, let $\tilde{a}_p := a_p - (n_p^c/2 + n_p^i)a$ and follow the next steps.

1. Take the tensor product of $W_{\tilde{a}_p}^{(n_p)}$ for each face $p$:

$$W = \widehat{\bigotimes}_{p=1}^{N_f} W_{\tilde{a}_p}^{(n_p)} : \mathbb{C} \to \widehat{\bigotimes}_{p=1}^{N_f} \mathcal{L}^{\hat{\otimes} n_p}. \tag{3.10}$$

2. Consider the permutation of step 2 of the lattice construction of TFTs, which now permutes tensor products of $\mathcal{L}$.

$$\rho : \bigotimes_{p=1}^{N_f} \mathcal{L}^{\otimes n_p} \to \bigotimes_{c=1}^{N_c} \mathcal{L}^{\otimes 2} \otimes \bigotimes_{b_o=1}^{N_o} \mathcal{L} \otimes \bigotimes_{b_i=1}^{N_i} \mathcal{L}, \tag{3.11}$$

3. Apply $\beta_a$'s on internal edges

$$C = \beta_a^{\hat{\otimes} N_c} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} N_b}} : \mathcal{L}^{\hat{\otimes} 2N_c} \hat{\otimes} \mathcal{L}^{\hat{\otimes} N_b} \to \mathcal{L}^{\hat{\otimes} N_b}. \tag{3.12}$$

4. Take the composition of the maps 3.10, 3.11, 3.12 and form the following

$$Y(M) := \left( id_{\mathcal{L}^{\hat{\otimes}(N_o)}} \hat{\otimes} \tilde{B}_a^{N_i} \right) \circ \left( (C \circ \rho \circ W) \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} N_i}} \right) : \mathcal{L}^{\hat{\otimes} N_i} \to \mathcal{L}^{\hat{\otimes} N_o} \tag{3.13}$$

It is not hard to see that $Y(M)$ is independent of the choice of $a$ and the details of the cell decomposition of $M$, including the distribution of the total area among faces. To see how this works, let us denote $Q_a := Y(S_{1,1}, a)$. We can calculate this with the cell decomposition on figure 3.9, with $a$ assigned to the single face. Let now $\hat{S}_{1,1}^{[a_1,a_2]}$ be a cylinder decomposed into two faces, with assigned areas $a_1$ and $a_2$, as shown on figure 3.17. We can calculate $Y(\hat{S}_{1,1}^{[a_1,a_2]})$ as shown of figure 3.18, and see that we can freely distribute the area by condition 6 and equation 3.8. To show that $Y(\hat{S}_{1,1}^{[a_1,a_2]}) = Q_{a_1+a_2}$, write everything in terms of the RFA $\mathcal{L}$ and calculate similarly as on figure 3.14. It is also clear from the construction that $Y$ is functorial in the sense that

- $Y((M, \mathcal{A}_M) \circ (M', \mathcal{A}_{M'})) = Y(M, \mathcal{A}_M) \circ Y(M', \mathcal{A}_{M'})$,
- $Y((M, \mathcal{A}_M) \sqcup (M', \mathcal{A}_{M'})) = Y(M, \mathcal{A}_M) \hat{\otimes} Y(M', \mathcal{A}_{M'})$.

Similarly to the TFT case we can write

$$Q_a := Y(S_{1,1}, a) = \mu_{a_1} \circ \tau_{\mathcal{L},\mathcal{L}} \circ \Delta_{a_2}, \tag{3.14}$$

for $a_1 + a_2 = a$, which shows that $Q_a$ is self-adjoint.[14] Functoriality of $Y$ shows that we have a semigroup: $Q_a \circ Q_b = Q_{a+b}$. Proposition 10 shows that $Q_a$

[12] If an edge belongs to only this one face then we count it twice.

[13] It should be sufficiently small for $\tilde{a}_p$ to remain positive for each face $p$.

[14] Recall that $\tau_{\mathcal{L},\mathcal{L}}^* = \tau_{\mathcal{L},\mathcal{L}}^{-1} = \tau_{\mathcal{L},\mathcal{L}}$ from the definition of a braided monoidal $*$-category on page 14 and because the braiding is symmetric.

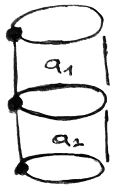


Figure 3.17: Decomposition of a cylinder into two faces

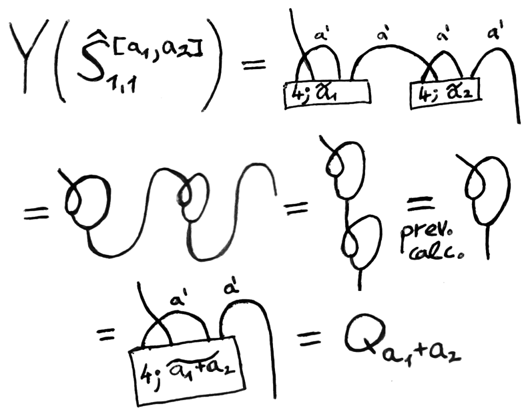


Figure 3.18: An example calculation, which shows independence of dell decomposition and area distribution. After the second equation we write everything in terms of the RFA $\mathcal{L}$. We omit the parameters, since this map only depends on the sum $a_1 + a_2$.

is compact: We have $P_{a_1} \circ Q_{a_2} = Q_a$ and $P_{a_1}$ is compact by proposition 17. This last equation also shows that $a \mapsto Q_a$ is strongly continuous, if $a \mapsto P_a$ is strongly continuous.

We just showed that we can apply proposition 15 on the operators $Q_a$, thus they have the same kernel $K$ for any $a$. By the same argument as in the TFT case,[15] we do not lose anything if we restrict $Y(M)$ to the appropriate tensor power of $\mathcal{H} = \mathcal{L}/K = \overline{\mathrm{Im} Q_a}$ and then $Y(M) \restriction_{\mathcal{H}^{\hat{\otimes} N_i}}$ will land in $\mathcal{H}^{\hat{\otimes} N_o}$.

Now we are ready to define the functor $Z^{cw} : \mathbf{Cob}_{\mathbf{2}}^{\mathbf{area,cw}} \to \mathbf{Hilb}$. It assigns the Hilbert space $\mathcal{H}^{\hat{\otimes} n}$ to an object with $n$ circles, the bounded linear map $Y(M) \restriction_{\mathcal{H}^{\hat{\otimes} N_i}}$ to a bordism $M$ with area and cell decomposition and permutations to permutations. We still need to check if we defined a $*$-functor, it is enough to check this on the generators of $\mathbf{Cob}_{\mathbf{2}}^{\mathbf{area}}$, which we will do in the following lemma. Altogether we have shown the following theorem.

**Theorem 53.** *The data for the lattice construction on $\mathcal{L}$ satisfying conditions 1-7 defines an AQFT. If furthermore condition 8 is satisfied then we obtain a strongly continuous AQFT.*

**Lemma 54.** *We have $Y(S_{1,0}, a)^* = Y(S_{0,1}, a)$ and $Y(S_{1,2}, a)^* = Y(S_{2,1}, a)$.*

[15]See page 48.

*Proof.* Recall the notation on figure 3.3 and 3.6. We will not mark the parameters, but assume that they add up where necessary. Then notice that it is easy to calculate the adjoint of the weights: $\left(W_a^{(n)}\right)^* = \tilde{B}_{a_1}^n \circ \left(W_{a_2}^{(n)} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} n}}\right)$. The proof is shown on figure 3.19, and we use induction over $n \geq 3$. The case $n = 2$ is clear by condition 7, the case $n = 1$ is shown on figure 3.20. This

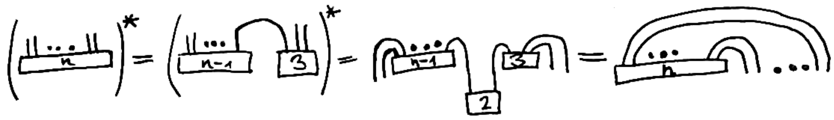


Figure 3.19: The adjoint of a weight

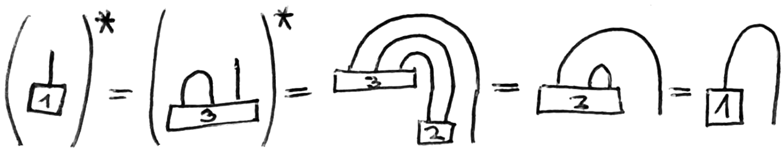


Figure 3.20: Adjoint of $W_a^{(1)}$

immediately shows that

$$Y(S_{1,0}, a)^* = \left(W_a^{(1)}\right)^* = \beta_{a_1} \circ \left(W_{a_1}^{(1)} \hat{\otimes} id_{\mathcal{L}}\right) = Y(S_{0,1}, a).$$

Then also note that we can "pull the legs" of $\left(W_a^{(n)}\right)^*$ to the right as illustrated on figure 3.21. We define $M_a := Y(S_{3,0}, a)$, see figure 3.22. Use this to define

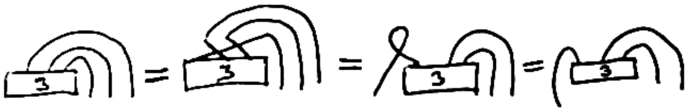


Figure 3.21

$\tilde{\mu}_a := Y(S_{1,2}, a)$ and $\tilde{\Delta}_a := Y(S_{2,1}, a)$ on figure 3.23. Calculate $M_a^* = \tilde{B}_{a_1}^3 \circ \left(M_{a_2} \hat{\otimes} id_{\mathcal{L}^{\hat{\otimes} 3}}\right)$ on figure 3.24. Then we have $\tilde{\mu}_a^* = \tilde{\Delta}_a$ shown on figure 3.25. $\square$

**Proposition 55.** *The center of the regularized $*$-Frobenius algebra $\mathcal{L}$ is $\mathcal{H}$.*

*Proof.* The argument is essentially the same as the TFT case, one just has to keep in mind that the parameters of the maps need to add up. $\square$

By proposition 22 we have that $\mathcal{L} = \overline{\bigoplus_{n=1}^{\infty} E_n}$ and by corollary 24 we have that the center of $\mathcal{L}$ is the completed direct sum of the centers of $E_n$.

**Theorem 56.** *The state space of the AQFT defined in theorem 53 using $\mathcal{L}$ is the center of the RFA $\mathcal{L}$.*

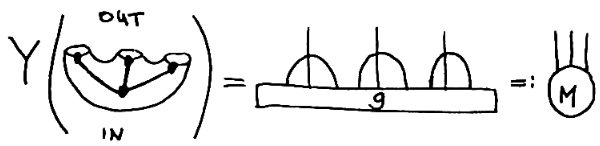


Figure 3.22: Definition of $M_a$

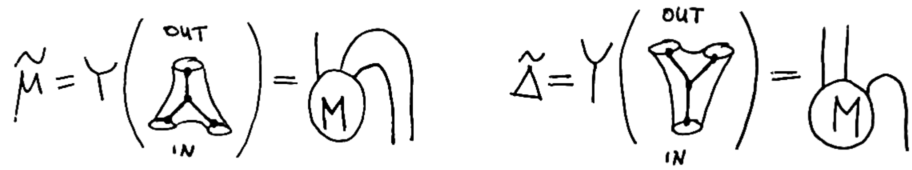


Figure 3.23: Definition of $\tilde{\mu}$ and $\tilde{\Delta}$

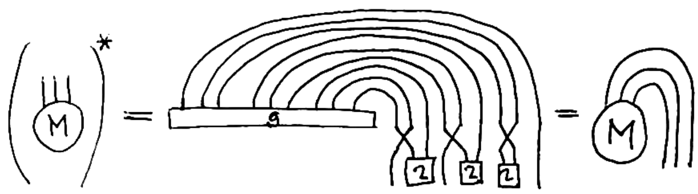


Figure 3.24: The adjoint of $M$

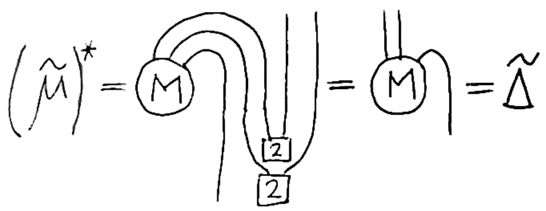


Figure 3.25: The adjoint of $\tilde{\mu}_a$

# Chapter 4

# 2-dimensional Yang–Mills Theory

It is time to see an actual example of an AQFT familiar from physics, called the two dimensional Yang–Mills theory, described for example in [8]. However we will follow a different approach and use the techniques developed in chapter 3. Therefore let us fix a compact semi-simple Lie group $G$.

## 4.1 Representation Theory of Compact Lie Groups

In this section we summarize some results from the representation theory of compact Lie groups, which are needed to define the data for the lattice construction.

A finite dimensional complex vector space $V$ is called a G-module, if the action

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g.v \end{aligned} \tag{4.1}$$

satisfies $g.(h.v) = (gh).v$, $e.v = v$ for $e$ the unit in $G$, linear in $V$ and if it is a smooth map. Alternatively we have a smooth group homomorphism $\rho_V : G \to Gl(V)$ with $g.v = \rho_V(g)(v)$. We call $V$ simple, if it does not contain a proper subspace, which is a $G$-module. Two $G$-modules are isomorphic, if there is a vector space isomorphism between them that commutes with the $G$-action. Note that being isomorphic is an equivalence relation, so let us denote the set of equivalence classes of simple $G$-modules with $I_G$. One could define infinite dimensional $G$-modules as well, but we will not as we will not use them here (as $G$-modules).

There is a measure on $G$, called the Haar measure (or the integral), which assigns to every continuous map $f : G \to \mathbb{R}$ a real number $\int_G f(x)dx$, such that the assignment is linear, left- and right-translation invariant, $\int_G f(x)dx = \int_G f(x^{-1})dx$[16] and $\int_G 1dx = 1$. We can therefore define the integral of any map $G \to U$ where $U$ is a finite dimensional real vector space: pick a basis and integrate the components independently. Linearity guarantees that this

[16] This is a consequence of the Haar measure being both left and right invariant, see [22] for more details.

integral does not depend on the choice of the basis. For details refer to [22] chapter IV.2. The Haar-measure allows to define a scalar product of continuous functions $f, g : G \to \mathbb{C}$:

$$\langle f, g\rangle = \int_G \overline{f(x)} g(x) dx. \tag{4.2}$$

**Proposition 57.** *On each $G$-module $V$ we have a $G$ invariant Hermitian form $(-,-)$, i.e.*

$$(g.v, g.v') = (v, v') = \overline{(v', v)}. \tag{4.3}$$

For the proof refer to [22] chapter IV. proposition 4.6. From now on always assume that each $G$-module is endowed with such a Hermitian form. Let $V$ be a $G$-module, $u, v \in V$. We call the functions of the form

$$\begin{aligned} G &\to \mathbb{C} \\ g &\mapsto (u, g.v) \end{aligned} \tag{4.4}$$

<u>matrix element functions</u> of the $G$-module $V$. Pick an orthonormal basis $\{v_i^V\}_{i=1}^{d_V}$ of $V$, where $d_V = \dim V$. Denote the particular matrix element functions in this basis with

$$\begin{aligned} r_{i,j}^V : G &\to \mathbb{C} \\ g &\mapsto (v_i^V, g.v_j^V). \end{aligned} \tag{4.5}$$

**Definition 58.** *The character of the $G$-module $V$ is defined as the function*

$$\chi_V := \sum_{i=1}^{d_V} r_{i,i}^V = \mathrm{Tr} \circ \rho_V, \tag{4.6}$$

*where* Tr *is the trace.*

In particular the characters are independent of the basis. They also satisfy $\overline{\chi_V(x)} = \chi_V(x^{-1})$. For details refer to [22] chapter IV. equation 4.14.

**Proposition 59.** *Let $V$, $V'$ be two simple $G$-modules, $u, v \in V$, $u', v' \in V'$. Then we have*

$$\int_G \overline{(u, g.v)}(u', g.v') dg = \frac{\delta_{V,V'}}{d_V}(u, u')\overline{(v, v')}, \tag{4.7}$$

$$\int_G \overline{\chi_V(g)} \chi_{V'}(g) dg = \delta_{V,V'}, \tag{4.8}$$

*where $\delta_{V,V'} = 1$ when $V$ is isomorphic to $V'$ and 0 otherwise.*

For the proof refer to [22] chapter IV. corollary 4.10 and corollary 4.16.

**Lemma 60.** *Let $V$ be a simple $G$-module, $x, y \in G$. Then*

$$\int_G \chi_V(xz^{-1})\chi_V(zy) dz = \frac{1}{d_v}\chi_V(xy).$$

*Proof.* Use the invariance of the integral, rewrite the characters in terms of scalar products and use the translation invariance and the Hermitian property of the scalar product on $V$ as follows:

$$\int_G \chi_V(xz^{-1})\chi_V(zy)dz = \int_G \chi_V(z^{-1})\chi_V(zxy)dz$$

$$= \int_G \sum_{i=1}^{\dim V}(e_i, z^{-1}.e_i)\sum_{j=1}^{\dim V}(e_j, zxy.e_j) = \sum_{i=1}^{\dim V}\sum_{j=1}^{\dim V}\int_G \overline{(e_i, z.e_i)}(e_j, zxy.e_j)$$

$$= \sum_{i,j=1}^{\dim V}\frac{1}{\dim V}\overline{(e_i, e_j)}(xy.e_j, e_i) = \sum_{i=1}^{\dim V}\frac{1}{\dim V}(xy.e_i, e_i) = \frac{1}{\dim V}\chi_V(xy).$$

by equation 4.7, where $\{e_i\}$ is an orthonormal base of $V$. □

Let $L^2(G)$ denote the square integrable functions on $G$ (with respect to equation 4.2) and $Cl^2(G)$ the square integrable class functions, which are the functions invariant under conjugation ($f(gxg^{-1}) = f(x)$). Recall the orthogonality relations in equations 4.7 and 4.8.

**Theorem 61** (Peter–Weyl)**.**

- *The functions* $e^V_{i,j} = d_V^{1/2} r^V_{i,j}$ *form a complete orthonormal set of* $L^2(G)$*,*
- *the characters* $\chi_V$ *form a complete orthonormal set of* $Cl^2(G)$*,*

*where* $V$ *denotes simple* $G$*-modules and* $i, j = 1, \dots, d_V$*.*

For the proof refer to [23] theorem 6.4.1 and proposition 6.5.3. We can reformulate this theorem in the following way. Let $M_V$ denote the vector space spanned by matrix element functions of the simple $G$-module $V$ and $\mathbb{C}\chi_V$ the vector space spanned by the character $\chi_V$. Then by theorem 61 we have

$$L^2(G) = \overline{\bigoplus_{V\in I_G} M_V}, \tag{4.9}$$

$$Cl^2(G) = \overline{\bigoplus_{V\in I_G} \mathbb{C}\chi_V}. \tag{4.10}$$

We need one more additional notion before we can give the data for the lattice construction. Let $\mathbf{D} : C^\infty(G) \to C^\infty(G)$ denote the differential operator, called the Laplacian on $G$, where $C^\infty(G)$ denotes the smooth functions on $G$. Instead of giving the definition we look at its crucial property, which is the only thing we need to use. Recall that $M_V \subset C^\infty(G)$. We have for $f \in M_V$ that $Df = -\sigma_V f$, where $\sigma_V$ is the value of the Casimir element on the simple $G$-module $V$. Note that $\sigma_V$ is a non-negative real number. Then we have that

$$e^{aD}f = e^{-a\sigma_V}f. \tag{4.11}$$

For more details see [23] chapter 8.2, especially proposition 8.2.1. The operator $e^{aD}$ extends to a bounded linear operator on $L^2(G)$ for every $a > 0$: Let $\varphi \in L^2(G)$ and write $\varphi = \sum_{V\in I_G}\varphi_V$ with $\varphi_V \in M_V$. Since $e^{-2\sigma_V} \le 1$, we have

$$\left\|e^{aD}\varphi\right\|^2 = \left\|\sum_{V\in I_G} e^{-a\sigma_V}\varphi_V\right\|^2 = \sum_{V\in I_G} e^{-2\sigma_V}\left\|\varphi_V\right\|^2 \le \left\|\varphi\right\|^2. \tag{4.12}$$

## 4.2 The Boltzmann Weights

We set $\mathcal{L} := L^2(G)$ to be the Hilbert space of the data for the lattice construction and we identify $\mathcal{L}^{\hat{\otimes}k} \cong L^2(G^{\times n})$, see equation A.10. The weights need a little bit preparation. Consider the functions

$$w_a^{(k)}(x_1, \dots, x_k) = \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(x_1 \cdots x_k), \tag{4.13}$$

for $a > 0$ and $k = 1, 2, \dots$. We show that these functions are in $\mathcal{L}^{\hat{\otimes}k}$.

**Lemma 62.** *For $k$ a positive integer we have*

$$\langle \chi_V(x_1 \cdots x_k), \chi_W(x_1 \cdots x_k) \rangle = \delta_{V,W}, \tag{4.14}$$

$$\tag{4.15}$$

*Proof.* Calculate using lemma 60:

$$\begin{aligned}
&\langle \chi_V(x_1 \cdots x_k), \chi_W(x_1 \cdots x_k) \rangle \\
=&\int_G \cdots \int_G \overline{\chi_V(x_1 \cdots x_k)} \chi_W(x_1 \cdots x_k) dx_1 \dots dx_k \\
=&\int_G \cdots \int_G \chi_V(x_k^{-1} \cdots x_1^{-1}) \chi_W(x_1 \cdots x_k) dx_1 \dots dx_k \\
=&\delta_{V,W} d_V^{-1} \int_G \cdots \int_G \chi_V(x_k^{-1} \cdots x_2^{-1} x_2 \cdots x_k) dx_2 \dots dx_k \\
=&\delta_{V,W} d_V^{-1} \int_G \cdots \int_G \chi_V(e) dx_2 \dots dx_k = \delta_{V,W} d_V^{-1} \chi_V(e) = \delta_{V,W}
\end{aligned}$$

□

We can use lemma 62 to calculate the norm of $w_a^{(k)}$:

$$\begin{aligned}
\left\| w_a^{(k)} \right\|^2 &= \left\langle \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(x_1 \cdots x_k), \sum_{W \in I_G} e^{-a\sigma_W} d_W \chi_W(x_1 \cdots x_k) \right\rangle \\
&= \sum_{V \in I_G} e^{-a\sigma_V} d_V \sum_{W \in I_G} e^{-a\sigma_W} d_W \langle \chi_V(x_1 \cdots x_k), \chi_W(x_1 \cdots x_k) \rangle \\
&= \sum_{V \in I_G} e^{-2a\sigma_V} d_V^2,
\end{aligned} \tag{4.16}$$

so we need to show that this sum is convergent.

**Proposition 63.** *Using the above notation we have*

$$\sum_{V \in I_G} d_V^2 e^{-2a\sigma_V} < \infty. \tag{4.17}$$

*Proof.* The idea behind the proof of (4.17) is to give some estimates of the quantities in the summand so that at the end we arrive at a familiar convergent sum. We will follow the calculation in [10], which uses some results from [24].

- First label the simple modules with dominant weights: $V \in I_G \leftrightarrow \lambda \in \Pi^+$.

- From [10] equation 3.2 we have that

$$d_\lambda \leq N|\lambda|^m, \tag{4.18}$$

where $N$ is a constant independent of $\lambda$, $2m = \dim G - \mathrm{rank}G$ and $|\cdot|$ is the norm on the weight space, which is a crude estimate of the Weyl dimension formula [25] corollary 24.3.

- From [10] equation 3.5 we have that

$$\sigma_\lambda \geq |\lambda|^2 \Rightarrow e^{-2a\sigma_\lambda} \leq e^{-2a|\lambda|^2}, \tag{4.19}$$

which comes from calculating the value of the Casimir element explicitly.

- There is an $r$-tuple $n \in \mathbb{Z}^r$ for each $\lambda \in \Pi^+$ and constants $k_1$, $k_2$ independent of $n$ and $\lambda$ such that

$$k_1\|n\| \leq |\lambda| \leq k_2\|n\| \tag{4.20}$$

with $\|n\|^2 = \sum_{i=1}^r n_i^2$, which is implicit in the proof of lemma 1.3 in [24].

- Finally let $b(j)$ be the number of $n \in \mathbb{Z}^r$ such that $\|n\|^2 = j$. This can be approximated with the volume of the $r$-dimensional cube with edge length $2\sqrt{j}+1$, i.e.

$$b(j) \leq (2\sqrt{j}+1)^r \tag{4.21}$$

Then approximate the sum 4.17 according to the above steps:

$$\begin{aligned}\sum_{V\in I_G} d_\alpha^2 e^{-2a\sigma_\alpha} &= \sum_{\lambda\in\Pi^+} d_\lambda^2 e^{-2a\sigma_\lambda} \leq N^2 \sum_{\lambda\in\Pi^+} |\lambda|^{2m} e^{-2a|\lambda|^2} \\ &\leq N' \sum_{n\in\mathbb{Z}^r} \|n\|^{2m} e^{-A\|n\|^2} = N' \sum_{j=0}^{\infty} b(j) j^m e^{-Aj} \\ &\leq N' \sum_{j=0}^{\infty} j^m (2\sqrt{j}+1)^r e^{-Aj},\end{aligned} \tag{4.22}$$

with $A, N' > 0$ constants, which is finite by e.g. the integral test. $\square$

The weights for the lattice construction are given by

$$\begin{aligned}W_a^{(k)} : \mathbb{C} &\to \mathcal{L}^{\hat{\otimes}k} \\ 1 &\mapsto w_a^{(k)}.\end{aligned} \tag{4.23}$$

The functions $w_a^{(k)}$ are called Boltzmann weights in the literature [8] and $w_a^{(1)}$ is also the so-called heat kernel of the Lie group $G$, see e.g. [23] section 12.6, for the case $G = U(n)$ the unitary group.

## 4.3 The Pairing

We define the pairing $\beta_a : \mathcal{L}^{\hat{\otimes}2} \to \mathbb{C}$ on the dense subspace $\mathcal{L}^{\otimes 2}$ of $\mathcal{L}^{\hat{\otimes}2}$ as follows. Let $f, g \in \mathcal{L}$ and

$$\beta_a(f\hat{\otimes}g) := \int_G f(x^{-1})(e^{aD}g)(x)dx. \tag{4.24}$$

We can write $f = \sum_{V\in I_G} f_V$ and $g = \sum_{W\in I_G} g_W$, with $f_V, g_V \in M_V$. Then we can write by the linearity of the integral that

$$\begin{aligned}\beta_a(f\hat{\otimes}g) &= \sum_{V,W\in I_G} \int_G f_V(x^{-1})(e^{aD}g_W)(x)dx \\ &= \sum_{V,W\in I_G} \int_G f_V(x^{-1})e^{-a\sigma_W}g_W(x)dx \\ &= \sum_{V\in I_G} e^{-a\sigma_V} \int_G f_V(x^{-1})g_V(x)dx, \end{aligned} \tag{4.25}$$

so by writing $\beta_V(f\hat{\otimes}g) := \int_G f_V(x^{-1})g_V(x)dx$ we have

$$\beta_a = \sum_{V\in I_G} e^{-a\sigma_V}\beta_V. \tag{4.26}$$

To check that this defines a bounded operator we will show that

$$\|\beta_a\| \leq \sum_{V\in I_G} e^{-a\sigma_V} \|\beta_V\| \leq \infty. \tag{4.27}$$

**Lemma 64.** *The dual of $\beta_V$ is $d_V\chi_V(xy)$ and $\|\beta_V\| = d_V$.*

*Proof.* From equation 4.7 we have

$$\beta_V(e^V_{i,j}\hat{\otimes}e^V_{k,l}) = \delta_{j,k}\delta_{i,l}. \tag{4.28}$$

Let us denote $\beta_V^* = \gamma_V$ and write $\gamma_V(1) = \sum_{i,j,k,l=1}^{d_V} \gamma_V^{ijkl} e_V^{i,j}\hat{\otimes}e_V^{k,l}$. Then calculate

$$\begin{aligned}\delta_{j,k}\delta_{i,l} &= \langle 1, \beta_V(e_V^{i',j'}\hat{\otimes}e_V^{k',l'})\rangle = \langle \gamma_V(1), e_V^{i',j'}\hat{\otimes}e_V^{k',l'}\rangle \\ &= \langle \sum_{i,j,k,l=1}^{d_V} \gamma_V^{ijkl} e_V^{i,j}\hat{\otimes}e_V^{k,l}, e_V^{i',j'}\hat{\otimes}e_V^{k',l'}\rangle = \left(\gamma_V^{ijkl}\right)^*,\end{aligned}$$

so $\gamma_V(1)(x,y) = \sum_{i,j=1}^{d_V} e_V^{i,j}(x)\hat{\otimes}e_V^{j,i}(y) = d_V\chi_V(xy)$.

By the Riesz representation theorem (see [17] theorem 5.62) $\|\beta_V\| = \|\gamma_V(1)\|$. Using on the other hand by lemma 62 we have $\|\gamma_V(1)\|^2 = d_V^2\,\|\chi_V(xy)\| = d_V^2$. □

**Proposition 65.** *The pairing $\beta_a$ is bounded for every $a > 0$.*

*Proof.* We can approximate the norm of $\beta_a$ in equation 4.27:

$$\|\beta_a\| \leq \sum_{V\in I_G} e^{-a\sigma_V} d_V \leq \sum_{V\in I_G} e^{-2(a/2)\sigma_V} d_V^2 \leq \infty, \tag{4.29}$$

by using lemma 64, proposition 63 and that $d_V \geq 1$. □

**Remark 66.** *The limit $\lim_{a\to 0}\beta_a$ does not define a bounded functional.*

*Proof.* Using the notations of lemma 64 we have $\|\gamma_V(1)\| = d_V$. By equation 4.26 we have

$$\beta_a(\gamma_V(1)/d_V) = e^{-a\sigma_V} d_V \int_G \chi_V(x^{-1}x)dx = d_V e^{-a\sigma_V}.$$

Assuming that $\lim_{a\to 0}\beta_a = \beta_0$ exists, i.e. it is a bounded linear functional, we must have $\beta_0(\gamma_V(1)/d_V) = d_V$, which immediately shows that $\beta_0$ is unbounded, as the set $\{d_V |\ V \in I_G\}$ is not bounded. □

## 4.4 Conditions on the Pairing and the Weights

Now that we have defined the set of data for the lattice construction, we need to check that the conditions on this set of data are satisfied.

1. To show that the pairing $\beta_a$ is symmetric, it is enough to show that

$$\beta_a(e^V_{i,j}\hat{\otimes}e^W_{k,l}) = \beta_a(e^W_{k,l}\hat{\otimes}e^V_{i,j}).$$

   By equation 4.28, both sides are equal $e^{-a\sigma_V}\delta_{V,W}\delta_{il}\delta_{jk}$.

2. The Boltzmann weights are cyclic invariant, by the property of the trace.

3. To see the gluing property use lemma 60:

$$\begin{aligned}
&\left(id_{\mathcal{L}^{\hat{\otimes}(n-1)}}\hat{\otimes}\beta_{a_0}\hat{\otimes}id_{\mathcal{L}^{\hat{\otimes}(m-1)}}\right)\left(W^{(n)}_{a_1}\hat{\otimes}W^{(m)}_{a_2}\right)\\
=&\int_G\sum_{V,W\in I_G} e^{-a_1\sigma_V}e^{-a_2\sigma_W}d_Vd_W\chi_V(x_1\cdots x_{n-1}z^{-1})e^{a_0D}\chi_W(zy_2\cdots y_m)dz\\
=&\sum_{V,W\in I_G}\int_G e^{-a_1\sigma_V}e^{-a_2\sigma_W}d_Vd_W\chi_V(x_1\cdots x_{n-1}z^{-1})e^{-a_0\sigma_W}\chi_W(zy_2\cdots y_m)dz\\
=&\sum_{V,W\in I_G} e^{-(a_0+a_1+a_2)\sigma_V}d_V\chi_V(x_1\cdots x_{n-1}y_2\cdots y_m)\\
=&W^{(n+m-2)}_{a_0+a_1+a_2}(x_1\cdots x_{n-1}y_2\cdots y_m).
\end{aligned}$$

4. Let $a_1, a_2 > 0$ and $a_1 + a_2 = a$. Calculate $P_a(f) = \left(\beta_{a_1}\hat{\otimes}id_{\mathcal{L}}\right)\circ\left(id_{\mathcal{L}}\hat{\otimes}W^{(2)}_{a_2}\right)(f)$ for $f\in\mathcal{L}$:

$$\begin{aligned}
P_a(f)(x) &= \int_G f(y^{-1})e^{a_1D}\sum_{V\in I_G} d_Ve^{-a_2\sigma_V}\chi_V(yx)dy\\
&= \sum_{V\in I_G} e^{-(a_1+a_2)\sigma_V}d_V\int_G f(y^{-1})\chi_V(yx)dy\\
&= \sum_{V,W\in I_G} e^{-(a_1+a_2)\sigma_V}\sum_{i,j=1}^{d_W}\sum_{k,l=1}^{d_V} f^{ij}_W\int_G e^{ij}_W(y^{-1})e^{kl}_V(y)e^{lk}_V(x)dy\\
&= \sum_{V,W\in I_G} e^{-(a_1+a_2)\sigma_V}\sum_{i,j=1}^{d_W}\sum_{k,l=1}^{d_V} f^{ij}_We^{lk}_V(x)\delta_{V,W}\delta_{jk}\delta_{il}\\
&= \sum_{V\in I_G} e^{-a\sigma_V}\sum_{i,j=1}^{d_V} f^{ij}_Ve^{ij}_V(x),
\end{aligned}$$

   where we used $f=\sum_{W\in I_G} f_W$ and $f_W=\sum_{i,j=1}^{d_V} f^{ij}_We^{ij}_W$, thus

$$P_a(f) = \sum_{V\in I_G} e^{-a\sigma_V}f_V, \tag{4.30}$$

   hence $P_a$ is injective.

5. For the self gluing property calculate:

$$\begin{aligned}
&\left(\beta_{a_1}\hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}(n-2)}}\right)\circ W_{a_2}^{(n)} \\
&= \int_G \sum_{V\in I_G} e^{-a_2\sigma_V} \sum_{i,j=1}^{d_V} e_V^{ij}(y^{-1}) e^{a_1 D} e_V^{ji}(yx_3\cdots x_n)dy \\
&= \int_G \sum_{V\in I_G} e^{-(a_1+a_2)\sigma_V} \sum_{i,j=1}^{d_V} e_V^{ij}(y^{-1}) e_V^{ji}(yx_3\cdots x_n)dy \\
&= \int_G \sum_{V\in I_G} e^{-(a_1+a_2)\sigma_V} d_V \chi_V(y^{-1}yx_3\cdots x_n)dy \\
&= \int_G \sum_{V\in I_G} e^{-(a_1+a_2)\sigma_V} d_V \chi_V(x_3\cdots x_n)dy \\
&= \sum_{V\in I_G} e^{-(a_1+a_2)\sigma_V} d_V \chi_V(x_3\cdots x_n) = W_{a_1+a_2}^{(n-2)}.
\end{aligned}$$

6. It is clear that $\beta_{a_1}\circ\left(P_{a_2}\hat{\otimes} id_{\mathcal{L}}\right) = \beta_{a_1+a_2}$, as $P_a$ scales the components with the right factor, see equation 4.30.

7. To show strong continuity calculate using equation 4.30 and the notation of part 4 that

$$\|(P_a - P_b)f\|^2 = \sum_{V\in I_G}\left(e^{-a\sigma_V} - e^{-b\sigma_V}\right)^2 \|f_V\|^2,$$

which goes to 0 as $a\to b$, as it is a convergent sum and the coefficients vanish.

8. The last thing to show is that $\beta_a^* = W_a^{(2)}$, which was done in lemma 64 and that $(W_a^{(3)})^* = \tilde{B}_{a_1}\circ\left(W_{a_2}^{(3)}\hat{\otimes} id_{\mathcal{L}^{\hat{\otimes}3}}\right)$, with $a = a_1 + a_2$. For the latter calculate:

$$\begin{aligned}
&\left(W_a^{(3)}\right)^*\left(e_{V_1}^{ij}\hat{\otimes} e_{V_2}^{kl}\hat{\otimes} e_{V_3}^{mn}\right) = \overline{\left\langle e_{V_1}^{ij}\hat{\otimes} e_{V_2}^{kl}\hat{\otimes} e_{V_3}^{mn}, W_a^{(3)}(1)\right\rangle} \\
&= \left\langle \sum_{V\in I_G} e^{-a\sigma_V} d_V^{-1/2} \sum_{i',j',k'=1}^{d_V} e_V^{i'j'}\hat{\otimes} e_V^{j'k'}\hat{\otimes} e_V^{k'i'},\ e_{V_1}^{ij}\hat{\otimes} e_{V_2}^{kl}\hat{\otimes} e_{V_3}^{mn}\right\rangle \\
&= \sum_{V\in I_G} e^{-a\sigma_V} d_V^{-1/2}\delta_{V,V_1}\delta_{V,V_2}\delta_{V,V_3} \sum_{i',j',k'=1}^{d_V} \delta_{ii'}\delta_{jj'}\delta_{kj'}\delta_{lk'}\delta_{mk'}\delta_{ni'} \\
&= e^{-a\sigma_{V_1}} d_{V_1}^{-1/2}\delta_{V_1,V_2}\delta_{V_1,V_3}\delta_{in}\delta_{jk}\delta_{lm}
\end{aligned}$$

using 4.7. On the other hand using $\beta_a\left(e_{V_1}^{ij}\hat{\otimes} e_{V_2}^{kl}\right) = \delta_{V_1,V_2}\delta_{il}\delta_{jk}$ we have

$$\tilde{B}_{a_1}^3\circ\left(W_{a_2}^{(3)}(1)\hat{\otimes} e_{V_1}^{ij}\hat{\otimes} e_{V_2}^{kl}\hat{\otimes} e_{V_3}^{mn}\right) = e^{-a\sigma_{V_1}} d_{V_1}^{-1/2}\delta_{V_1,V_2}\delta_{V_1,V_3}\delta_{in}\delta_{jk}\delta_{lm}.$$

## 4.5 The State Space

The last thing we need to do is to calculate what is assigned to the cylinder. Take the usual cell decomposition of a cylinder as on figure 3.9 and calculate the assigned map $Q_a$.

$$
\begin{aligned}
Q_a(f) &= \int_G \int_G \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(xy^{-1}z^{-1}y) f(z) dy dz \\
&= \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(x) \int_G \overline{\chi_V(z)} f(z) dz \\
&= \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(x) \langle \chi_V, f \rangle. \qquad (4.31)
\end{aligned}
$$

**Proposition 67.** *The state space of the 2d Yang–Mills theory is $Cl^2(G)$.*

*Proof.* Equation 4.31 shows that the image of $Q_a$ is spanned by characters, so its closure is $Cl^2(G)$ by theorem 61. □

We can also calculate what is assigned to $(S_{1,2}, a)$. First calculate $M_a$ defined in the proof of lemma 54:

$$
\begin{aligned}
M_a(1)(x, y, z) &= \int_G \int_G \int_G \sum_{V \in I_G} e^{-a\sigma_V} d_V \chi_V(xu^{-1}v^{-1}yvw^{-1}zwu) du dv dw \\
&= \int_G \int_G \sum_{V \in I_G} e^{-a\sigma_V} \chi_V(uxu^{-1}v^{-1}yv) \chi_V(z) du dv \\
&= \int_G \sum_{V \in I_G} e^{-a\sigma_V} d_V^{-1} \chi_V(uxu^{-1}) \chi_V(y) \chi_V(z) du \\
&= \sum_{V \in I_G} e^{-a\sigma_V} d_V^{-1} \chi_V(x) \chi_V(y) \chi_V(z). \qquad (4.32)
\end{aligned}
$$

From this we immediately have for $F \in \mathcal{H}^{\hat{\otimes} 2}$ that

$$
\begin{aligned}
\tilde{\mu}_a(F)(x) &= \int_G \int_G \sum_{V \in I_G} e^{-a\sigma_V} d_V^{-1} \chi_V(x) \chi_V(y^{-1}) \chi_V(z^{-1}) F(y, z) dy dz \\
&= \sum_{V \in I_G} e^{-a\sigma_V} d_V^{-1} \chi_V(x) \langle \chi_V \hat{\otimes} \chi_V, F \rangle. \qquad (4.33)
\end{aligned}
$$

## 4.6 The RFA Structure on $L^2(G)$

In this section we compute the RFA structure on $\mathcal{L} = L^2(G)$ and then the center of $\mathcal{L}$ in order to illustrate how we can calculate the state space alternatively as in theorem 56. Let $a_1, a_2 > 0$, $a = a_1 + a_2$ and $f \in \mathcal{L}$ with $f = \sum_{V \in I_G} f_V$ with $f_V \in M_V$ and $f_V = \sum_{i,j=1}^{d_V} f_V^{ij} e_V^{ij}$, furthermore let $F \in \mathcal{L}^{\hat{\otimes} 2}$ with $F = \sum_{V,W \in I_G} F_{VW}$, $F_{VW} \in M_V \hat{\otimes} M_W$.

**Lemma 68.** *Using the above notation we have*

$$
d_V \int_G \chi_V(xy^{-1}) f_W(y) dy = \delta_{V,W} f_V(x). \qquad (4.34)
$$

*Proof.* Calculate using equation 4.7:

$$d_V \int_G \chi_V(xy^{-1}) f_W(y) dy = \int_G \sum_{i,j=1}^{d_V} \sum_{k,l=1}^{d_W} e_V^{ij}(x) e_V^{ji}(y^{-1}) f_W^{kl} e_W^{kl}(y) dy$$

$$= \sum_{i,j,k,l=1}^{d_V} e_V^{ij}(x) \delta_{V,W} f_V^{kl} \delta_{ik} \delta_{jl} = \delta_{V,W} \sum_{i,j=1}^{d_V} e_V^{ij}(x) f_V^{ij} = \delta_{V,W} f_V(x).$$

□

We need to follow proposition 50 in order to calculate the families of morphisms of the RFA $\mathcal{L}$:

- First set $\eta_a(1) = w_a^{(1)}$.
- Calculate $\varepsilon_a$:

$$\begin{aligned}\varepsilon_a(f) &= \int_G \sum_{V \in I_G} e^{-a_1 \sigma_V} d_V \chi_V(x^{-1}) \left(e^{a_2 D} f\right)(x) dx \\ &= \int_G \sum_{V,W \in I_G} e^{-a_1 \sigma_V} d_V \overline{\chi_V(x)} e^{-a_2 \sigma_W} f_W(x) dx \\ &= \int_G \sum_{V \in I_G} e^{-a \sigma_V} d_V \langle \chi_V, f_V \rangle = \langle w_a^{(1)}, f \rangle.\end{aligned}$$

  This also shows that $\varepsilon_a = \eta_a^*$. Furthermore by lemma 68 we have $\varepsilon_a(f) = \sum_{V \in I_G} e^{-a\sigma_V} f_V(e)$.

- Calculate $\Delta_a$:

$$\begin{aligned}\Delta_a(f)(x) &= \int_G W_{a_1}^{(3)}(x, y, z^{-1}) \left(e^{a_2 D} f\right) dx \\ &= \int_G \sum_{V,W \in I_G} e^{-a_1 \sigma_V} d_V \chi_V(xyz^{-1}) e^{-a_2 \sigma_W} f_W(x) dx \\ &= \sum_{V \in I_G} e^{-a\sigma_V} f_V(xy),\end{aligned}$$

  where we used lemma 68.

- We use this to calculate $\mu_a$:

$$\begin{aligned}\mu_a(F)(x) &= \left(id_{\mathcal{L}} \hat{\otimes} \beta_{a_1}\right) \circ \left(\Delta_{a_2} \hat{\otimes} id_{\mathcal{L}}\right)(F)(x) \\ &= \left(id_{\mathcal{L}} \hat{\otimes} \beta_{a_1}\right) \left( \sum_{V,W \in I_G} e^{-a_2 \sigma_V} F_{VW}(xy, z) \right) \\ &= \sum_{V,W \in I_G} e^{-a_2 \sigma_V} e^{-a_1 \sigma_W} \int_G F_{VW}(xz^{-1}, z) dz \\ &= \sum_{V \in I_G} e^{-a\sigma_V} \int_G F_{VV}(xz^{-1}, z) dz,\end{aligned}$$

  where we used that $\beta_a(f_V \hat{\otimes} f_W) = \delta_{VW} \beta_a(f_V \hat{\otimes} f_V)$ (coming from equation 4.7). We could also show that $\Delta_a^* = \mu_a$, but we will not do it now.

- Finally let us calculate $P_a$:

$$P_a(f)(x) = \mu_{a_1}\left(\eta_{a_2}(1)\hat{\otimes} f\right)(x) = \sum_{V\in I_G} e^{-a\sigma_V} d_V \int_G \chi_V(xz^{-1}) f_V(z)dz$$
$$= \sum_{V\in I_G} e^{-a\sigma_V} f_V(x),$$

by lemma 68, which is what we expected from equation 4.30.

**Proposition 69.** *The RFA structure on $\mathcal{L}$ is given by the following families of maps.*

- $\eta_a(1) = w_a^{(1)} = \sum_{V\in I_G} e^{-a\sigma_V} d_V \chi_V$,
- $\varepsilon_a(f) = \langle w_a^{(1)}, f\rangle = \sum_{V\in I_G} e^{-a\sigma_V} f_V(e)$,
- $\Delta_a(f)(x,y) = \sum_{V\in I_G} e^{-a\sigma_V} f_V(xy)$,
- $\mu_a(F)(x) = \sum_{V\in I_G} e^{-a\sigma_V} \int_G F_{VV}(xz^{-1}, z)dz$,
- $P_a(f) = \sum_{V\in I_G} e^{-a\sigma_V} f_V$.

Now we are ready to calculate the center of $\mathcal{L}$. Let $f$ in the center of $\mathcal{L}$, i.e. $\mu_a(f\hat{\otimes}g) = \mu_a(g\hat{\otimes}f)$ for every $g \in \mathcal{L}$. We can write this equation as

$$\sum_{V\in I_G} e^{-a\sigma_V} \int_G f_V(xz^{-1}) g_V(z)dz = \sum_{V\in I_G} e^{-a\sigma_V} \int_G g_V(zy^{-1}) f_V(y)dy$$
$$= \sum_{V\in I_G} e^{-a\sigma_V} \int_G f_V(z^{-1}x) g_V(z)dz,$$

by using the invariance of the integral (substituting $z = xy^{-1}$ at the second equation). So we have for every $V \in I_G$ that

$$\int_G \left(f_V(xz^{-1}) - f_V(z^{-1}x)\right) g_V(z)dz = 0,$$

which implies by equation 4.7 that

$$f_V(xz^{-1}) = f_V(z^{-1}x)$$

for almost all $x, z \in G$. Since $f_V$ is continuous this means that this holds for all $x, z \in G$, hence $f_V$ is a class function and hence $f$ is a class function. We have just shown that the center is contained in $Cl^2(G)$. The converse inclusion can be easily calculated: Let $f \in Cl^2(G)$ and $g \in \mathcal{L}$, then

$$\mu_a(f\hat{\otimes}g) = \sum_{V\in I_G} e^{-a\sigma_V} \int_G f_V(xz^{-1}) g_V(z)dz = \sum_{V\in I_G} e^{-a\sigma_V} \int_G f_V(z^{-1}x) g_V(z)dz$$
$$= \sum_{V\in I_G} e^{-a\sigma_V} \int_G g_V(zy^{-1}) f_V(y)dy = \mu_a(g\hat{\otimes}f),$$

again by an appropriate substitution of integration variables.

**Proposition 70.** *The center of the RFA $\mathcal{L}$ given in proposition 69 is $Cl^2(G)$.*

**Remark 71.** *Let $\mathcal{H} := Cl^2(G)$ as before. Then $\mu_a \restriction_{\mathcal{H}^{\hat{\otimes}2}} = \tilde{\mu}_a$ from equation 4.33.*

*Proof.* Let $H \in \mathcal{H}^{\hat{\otimes}2}$ and write $H = \sum_{V,W \in I_G} H_{VW} d_V \chi_V \hat{\otimes} \chi_W$. Calculate using lemma 68 that

$$\mu_a(H) = \sum_{V,W \in I_G} H_{VW} d_V \mu_a(\chi_V \hat{\otimes} \chi_W) = \sum_{V \in I_G} H_{VV} e^{-a\sigma_V} \chi_V = \tilde{\mu}_a(H).$$

□

## 4.7 Zero Area Limits

We are investigating, if in some cases the $a \to 0$ limit gives bounded maps. Note that

- $\lim_{a\to 0} P_a = id_{\mathcal{L}}$ by corollary 21,
- proposition 63 shows that $\lim_{a\to 0} W_a^{(n)}$ do not exist for any $n \geq 1$ and
- remark 66 shows that $\lim_{a\to 0} \beta_a$ does not give a bounded functional.

**Proposition 72.** *The limits $\lim_{a\to 0} \mu_a$ and $\lim_{a\to 0} \Delta_a$ are bounded operators.*

*Proof.* Assume that the second limit gives a bounded operator $\Delta_0$. Then $\Delta_0(f)(x, y) = f(xy)$. Let us calculate its norm:

$$\|f(xy)\|^2 = \int_G \overline{f(xy)} f(xy) dx dy = \int_G \overline{f(z)} f(z) dz dy = \|f(x)\|^2,$$

by changing variables $z = xy$, so $\|\Delta_0\| = 1$.

Now calculate the adjoint of $\Delta_0$ and observe that it coincides with $\mu_0$, hence $\mu_0$ is bounded as well.

□

# Appendix A

# Hilb as a Symmetric Monoidal Category

The tensor product of Hilbert spaces is well known and usually in the literature a simple construction is given (see e.g. [16] section II.4), which is defined as the completion of the algebraic tensor product. This is not the most elegant way of giving the definition and it also does not provide a straightforward method for showing functoriality. However it is possible to define the tensor product of Hilbert spaces ($\hat{\otimes}$) with a universal property - similarly to the algebraic tensor product - with just one additional property. Using this universal property one can give the category of Hilbert spaces equipped with this tensor product (**Hilb**,$\hat{\otimes}$) a symmetric monoidal structure analogously to the category of vector spaces. One can also show that the simple construction fits into this definition. We follow [26] chapter 2.6, for proofs not present here the reader can refer to this book.

**Definition 73.** *Let $\mathcal{H} \in$ **Hilb**. Call the set $Y = \{y_i\} \subset \mathcal{H}$ a Hilbert base of $\mathcal{H}$ if $Y$ is a complete orthonormal set of $\mathcal{H}$.*

**Definition 74.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n \in$ **Hilb**. We say that the multilinear functional $\varphi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$ is*

- *bounded, if $\exists c > 0$ such that*

$$|\varphi(x_1, \ldots, x_n)| \leq c \cdot ||x_1|| \ldots ||x_n|| \tag{A.1}$$

  *$\forall x_i \in \mathcal{H}_i$ and write $||\varphi|| := inf\{such\ c\}$ for the norm of $\varphi$,*

- *Hilbert-Schmidt, if it is bounded and*

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \ldots, y_n)|^2 \leq \infty \tag{A.2}$$

  *for some $Y_i$ a Hilbert base of $\mathcal{H}_i$.*

**Lemma 75.** *Let $\mathcal{K}_1, \ldots \mathcal{K}_n, \mathcal{H}_1, \ldots, \mathcal{H}_n \in$ **Hilb**, $A_i \in \mathcal{B}(\mathcal{K}_i, \mathcal{H}_i)$, $\psi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$ a bounded multilinear functional and $\varphi = \psi \circ (A_1 \times \cdots \times A_n)$. Then for $Z_i$ a Hilbert base of $\mathcal{K}_i$ and for $Y_i$ a Hilbert base of $\mathcal{H}_i$ we have*

$$\sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} |\varphi(y_1, \ldots, y_n)|^2 \leq ||A_1||^2 \ldots ||A_n||^2 \sum_{z_1 \in Z_1} \cdots \sum_{z_n \in Z_n} |\psi(z_1, \ldots, z_n)|^2. \tag{A.3}$$

Note that as a special case, this shows that the value of the expression (A.2) is independent of the choice of the Hilbert bases.

**Proposition 76.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n \in$ **Hilb** with Hilbert bases $Y_1 \ldots Y_n$ and consider the set of all Hilbert-Schmidt functionals $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$. It becomes a Hilbert space via*

- *pointwise addition and multiplication by a complex number,*
- *inner product given by the formula*

$$< \varphi, \psi >:= \sum_{y_1 \in Y_1} \cdots \sum_{y_n \in Y_n} \overline{\varphi(y_1, \ldots, y_n)} \psi(y_1, \ldots, y_n),$$

- *and norm $||\varphi||_2 =< \varphi, \varphi >^{1/2}$*

*for $\varphi, \psi$ Hilbert-Schmidt functionals.*

Note that the latter two are independent of choices of Hilbert bases. One can show that $||\varphi|| \leq ||\varphi||_2$ for $\varphi$ a Hilbert-Schmidt functional.

**Definition 77.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n, \mathcal{H} \in$ **Hilb**. A multilinear map $L : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathcal{H}$ is called*

- *bounded, if $\exists c > 0$ such that*

  $$||L(x_1, \ldots, x_n)|| \leq c \cdot ||x_1|| \ldots ||x_n||$$

  *$\forall x_i \in \mathcal{H}_i$ and write $||L|| := inf\{such\ c\}$ for the norm of $L$,*

- *weak Hilbert-Schmidt, if it is bounded and $\forall u \in \mathcal{H}$ the multilinear functional $L_u : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathbb{C}$*

  $$L_u(x_1, \ldots, x_n) =< u, L(x_1, \ldots, x_n) >$$

  *is a Hilbert-Schmidt functional. Denote with $||L||_2 := sup_{u \in \mathcal{H}}\{||L_u||/||u||\}$ the Hilbert-Schmidt norm of $L$.*

One can show that $||L|| \leq ||L||_2$ for a weak Hilbert-Schmidt map. Note that a weak Hilbert-Schmidt map $L : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathbb{C}$ is a Hilbert-Schmidt functional. To familiarize ourselves with this definition let us look at some examples.

**Example 78.** *There exists a notion of a Hilbert-Schmidt map: we say that $A : \mathcal{H} \to \mathcal{K}$ is a Hilbert-Schmidt map if $\sum_{y \in Y} ||Ay||^2 < \infty$ for some Hilbert base $Y \subset \mathcal{H}$. It is a stronger condition than $A$ being a weak Hilbert-Schmidt map:*

$$\sum_{y \in Y} | < Ay, u > |^2 \leq \sum_{y \in Y} ||Ay||^2 ||u||^2 \leq \infty$$

*$\forall u \in \mathcal{H}$, so Hilbert-Schmidt implies weak Hilbert-Schmidt. The converse is however not true: consider the identity $\mathcal{H} \to \mathcal{H}$. It is not a Hilbert-Schmidt map:*

$$\sum_{y \in Y} ||y||^2 = \sum_{y \in Y} 1 = \infty,$$

*but*

$$\sum_{y \in Y} | < y, u > |^2 = ||u||^2 < \infty,$$

*so it is weak Hilbert-Schmidt. Now consider the map that sends every element of $Y$ to the same vector. It is clearly bounded, and clearly not weak Hilbert-Schmidt.*

With this definition we are ready to define the tensor product of Hilbert spaces.

**Theorem 79.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n \in$ **Hilb**. Then*

1. *there is $\mathcal{H} \in$ **Hilb** and a weak Hilbert-Schmidt map $p : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathcal{H}$ such that for every $\mathcal{K} \in$ **Hilb** and weak Hilbert-Schmidt map $L : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathcal{K}$ there is a unique $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $L = T \circ p$ and $||T|| = ||L||_2$,*

2. *if $v_m, w_m \in \mathcal{H}_m$ we have*

$$< p(v_1, \ldots, v_n), p(w_1, \ldots, w_n) > = < v_1, w_1 > \cdots < v_n, w_n >, \tag{A.4}$$

*if $Y_m \subset \mathcal{H}_m$ is a Hilbert base then the set*

$$\{p(y_1, \ldots, y_n) | y_i \in Y_i, \ i = 1, \ldots n\} \tag{A.5}$$

*is a Hilbert base of $\mathcal{H}$ and $||p||_2 = 1$,*

3. *if both $(\mathcal{H}, p)$ and $(\mathcal{H}', p')$ satisfy the conditions set in part 1 then there is a unique unitary map $U : \mathcal{H} \to \mathcal{H}'$ such that $p' = U \circ p$.*

**Definition 80.** *We call the pair $(\mathcal{H}, p)$ defined by the universal property in theorem 79.1 the tensor product of the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and write $\mathcal{H}_1 \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_n = \mathcal{H}$ and $v_1 \hat{\otimes} \ldots \hat{\otimes} v_n = p(v_1, \ldots, v_n)$, $v_i \in \mathcal{H}_i$. We summarize this in the commutative diagram in figure A.1.*

$$\begin{tikzcd}
\mathcal{H}_1 \times \cdots \times \mathcal{H}_n \arrow[rr, "p = \hat{\otimes}"] \arrow[dr, "L"'] & & \mathcal{H} = \mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_n \arrow[dl, dashed, "\exists! T"] \\
& \mathcal{K} &
\end{tikzcd}$$

Figure A.1: Universal property of the tensor product

The tensor product constructed in the proof of theorem 79 is the Hilbert space of conjugate linear Hilbert-Schmidt functionals on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$. The "elementary tensors" of the form $v_1 \hat{\otimes} \ldots \hat{\otimes} v_n$ generate a dense subspace $\mathcal{H}_0 = p(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ (the algebraic tensor product) of $\mathcal{H}$ and the closure of $\mathcal{H}_0$ with respect to the inner product in (A.4) is $\mathcal{H}$. Note that if one of the tensorands $\mathcal{H}_1, \mathcal{H}_2 \in$ **Hilb** is finite dimensional (consider $n = 2$ for simplicity) then the algebraic tensor product is already closed with respect to (A.4) The universal property defines the tensor product on objects in **Hilb** and it would be nice if we could use it to define the tensor product of morphisms in **Hilb**. For this we need to show one additional property of weak Hilbert-Schmidt maps.

**Proposition 81.** *Let $A_i \in \mathcal{B}(\mathcal{K}_i, \mathcal{H}_i)$ and $L : \mathcal{H}_1 \times \ldots \mathcal{H}_n \to \mathcal{H}$ a weak Hilbert-Schmidt map. Then the map $\tilde{L} = L \circ (A_1 \times \cdots \times A_n) : \mathcal{K}_1 \times \cdots \times \mathcal{K}_n \to \mathcal{H}$ is weak Hilbert-Schmidt as well with $||\tilde{L}||_2 \leq ||L||_2||A_1|| \ldots ||A_n||$.*

*Proof.* $L$ being a weak Hilbert-Schmidt map means that $\forall u \in \mathcal{H}$ the multilinear functional $L_u$ is a Hilbert-Schmidt functional. But by lemma 75 then $L_u \circ (A_1 \times \cdots \times A_n)$ is a Hilbert-Schmidt functional with Hilbert-Schmidt norm $||L_u \circ (A_1 \times \cdots \times A_n)||_2 \leq ||L_u||_2||A_1|| \ldots ||A_n||$. Therefore $\tilde{L}$ is a weak Hilbert-Schmidt map with norm $||\tilde{L}||_2 \leq ||L||_2||A_1|| \ldots ||A_n||$. □

**Proposition 82.** *The tensor product is a functor $\hat{\otimes}$ : $\mathbf{Hilb} \times \mathbf{Hilb} \to \mathbf{Hilb}$ .*

*Proof.* First one has to define what it does on morphisms. Let $(A, B) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \times \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, we have tensor products $(\mathcal{H}_1 \hat{\otimes} \mathcal{K}_1, \hat{\otimes}_1)$ and $(\mathcal{H}_2 \hat{\otimes} \mathcal{K}_2, \hat{\otimes}_2)$. Then by proposition 81 the map $\hat{\otimes}_2 \circ (A \times B)$ is weak Hilbert-Schmidt, so we can apply the universal property to the pair $(\mathcal{H}_2 \hat{\otimes} \mathcal{K}_2, \hat{\otimes}_2 \circ (A \times B))$ to get a unique morphism denoted by $A \hat{\otimes} B \in \mathcal{B}(\mathcal{H}_1 \hat{\otimes} \mathcal{K}_1, \mathcal{H}_2 \hat{\otimes} \mathcal{K}_2)$ such that

$$\hat{\otimes}_2 \circ (A \times B) = (A \hat{\otimes} B) \circ \hat{\otimes}_1. \tag{A.6}$$

Note that $||A \hat{\otimes} B|| \leq ||A|| \cdot ||B||$ by proposition 81 and one can show that actually $||A \hat{\otimes} B|| = ||A|| \cdot ||B||$.

To check that it is compatible with compositions let $\mathcal{H}_1 \xrightarrow{A_1} \mathcal{K} \xrightarrow{B_1} \mathcal{L}_1$ and $\mathcal{H}_2 \xrightarrow{A_2} \mathcal{K} \xrightarrow{B_2} \mathcal{L}_2$ be composable morphisms in **Hilb**. By the universal property, the diagram in figure A.2 commutes.

$$\begin{tikzcd}
\mathcal{H}_1 \times \mathcal{H}_2 \arrow[r, "\hat{\otimes}"] \arrow[d, "A_1 \times A_2"] \arrow[dd, bend right=60, "(B_1 \circ A_1) \times (B_2 \circ A_2)"'] & \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2 \arrow[d, "A_1 \hat{\otimes} A_2"] \arrow[dd, bend left=60, "(B_1 \circ A_1) \hat{\otimes} (B_2 \circ A_2)"] \\
\mathcal{K}_1 \times \mathcal{K}_2 \arrow[r, "\hat{\otimes}"] \arrow[d, "B_1 \times B_2"] & \mathcal{K}_1 \hat{\otimes} \mathcal{K}_2 \arrow[d, "B_1 \hat{\otimes} B_2"] \\
\mathcal{L}_1 \times \mathcal{L}_2 \arrow[r, "\hat{\otimes}"] & \mathcal{L}_1 \hat{\otimes} \mathcal{L}_2
\end{tikzcd}$$

Figure A.2: Compatibility with composition

To show that $id_{\mathcal{H}} \hat{\otimes} id_{\mathcal{K}} = id_{\mathcal{H} \hat{\otimes} \mathcal{K}}$ use the diagram in figure A.3 that commutes by the universal property again.

□

Now it is time to give **Hilb** a monoidal structure, therefore we need the following lemma.

**Lemma 83.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_{n+m} \in$ **Hilb**. Then there is a unique invertible map $U : \mathcal{H}_1 \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{n+m} \to (\mathcal{H}_1 \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_n) \hat{\otimes} (\mathcal{H}_{n+1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{n+m})$ such that $U(x_1 \hat{\otimes} \ldots \hat{\otimes} x_{n+m}) = (x_1 \hat{\otimes} \ldots \hat{\otimes} x_n) \hat{\otimes} (x_{n+1} \hat{\otimes} \ldots \hat{\otimes} x_{n_m})$. Note that this map is unitary as well.*

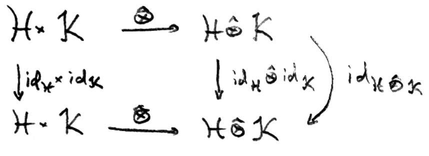

Figure A.3: Compatibility with identities

**Proposition 84.** *The category* $(\mathbf{Hilb}, \hat{\otimes}, \mathbb{C})$ *with tensor unit* $\mathbb{C}$ *is a monoidal category.*

*Proof.* First we need to define the associator $(a)$ and the left and right unit constraints $(l, r)$.

Let $\mathcal{H}, \mathcal{K}, \mathcal{L} \in \mathbf{Hilb}$ and define the invertible map $a_{\mathcal{H},\mathcal{K},\mathcal{L}} : (\mathcal{H}\hat{\otimes}\mathcal{K})\hat{\otimes}\mathcal{L} \to \mathcal{H}\hat{\otimes}(\mathcal{K}\hat{\otimes}\mathcal{L})$ as the composition of the two unique maps in figure A.4 given by lemma 83. Note that this is the unique map that sends the elementary tensors

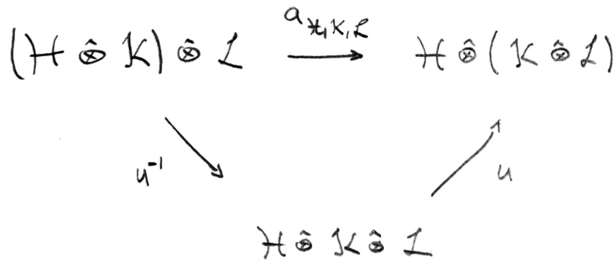

Figure A.4: Definition of the associator

$(h\hat{\otimes}k)\hat{\otimes}l \mapsto h\hat{\otimes}(k\hat{\otimes}l)$. Now the pentagon axiom follows from the uniqueness of the map that sends elementary tensors $((h\hat{\otimes}k)\hat{\otimes}l)\hat{\otimes}i \mapsto h\hat{\otimes}(k\hat{\otimes}(l\hat{\otimes}i))$, naturality follows from the universal property.

Consider the map $\mathbb{C} \times \mathcal{H} \to \mathcal{H}$ on the diagram in figure A.5. It is a weak

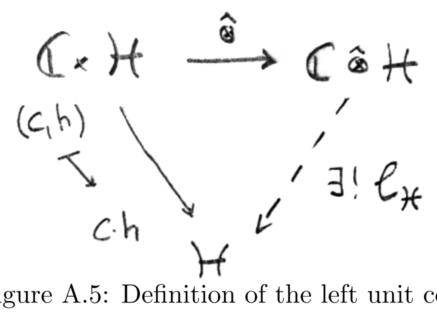

Figure A.5: Definition of the left unit constraint

Hilbert-Schmidt map because

$$\sum_{y \in Y} | < 1 \cdot y, u > |^2 = ||u||^2 \leq \infty$$

$\forall u \in \mathcal{H}$ with $Y \subset \mathcal{H}$ a Hilbert base, so by the universal property we get the left unit constraint $l_{\mathcal{H}}$. Its naturality comes from the definition. One defines the right unit constraint analogously. Then the triangle axiom follows easily from the definition of the associator and the left and right unit constraint.

□

As usual, it will be convenient to work with an equivalent strict monoidal category instead of **Hilb**, see [13] theorem XI.5.3; We use the same notation for this strict monoidal category for simplicity.

**Proposition 85.** $(\mathbf{Hilb}, \hat{\otimes}, \mathbb{C})$ *is a symmetric monoidal category with braiding*

$$\begin{aligned} \tau_{\mathcal{H},\mathcal{K}} : \mathcal{H}\hat{\otimes}\mathcal{K} &\to \mathcal{K}\hat{\otimes}\mathcal{H} \\ h\hat{\otimes}k &\mapsto k\hat{\otimes}h \end{aligned} \tag{A.7}$$

*for* $\mathcal{H}, \mathcal{K} \in \mathbf{Hilb}$.

*Proof.* Clear by using the universal property to $\hat{\otimes}$ composed with the flip map $\mathcal{H} \times \mathcal{K} \to \mathcal{K} \times \mathcal{H}$. □

Note that $||\tau_{\mathcal{H},\mathcal{K}}|| = 1$.

**Corollary 86.** *For each* $\mathcal{H} \in \mathbf{Hilb}$ *and* $n \geq 1$ *integer we have an* $\mathbb{S}_n$ *action on* $\mathcal{H}^{\hat{\otimes}n}$ *permuting the factors.*

Finally we give an example that will be used extensively throughout the following.

**Example 87.** *Let* $(\Omega_1, \mu_1)$ *and* $(\Omega_2, \mu_2)$ *be measure spaces such that* $L^2(\Omega_1, \mu_1)$ *and* $L^2(\Omega_2, \mu_2)$ *are separable. Then there is a unique isometry*

$$L^2(\Omega_1, \mu_1)\hat{\otimes}L^2(\Omega_2, \mu_2) \to L^2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2), \tag{A.8}$$

*which sends* $e_i \otimes f_j \mapsto e_i f_j$*, where* $\{e_i\}$ *and* $\{f_j\}$ *are complete orthonormal sets of* $L^2(\Omega_1, \mu_1)$ *and* $L^2(\Omega_2, \mu_2)$ *respectively and write* $L^2(\Omega_1, \mu_1)\hat{\otimes}L^2(\Omega_2, \mu_2) \cong L^2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$*. Note that one could similarly consider more than two such measure spaces and their tensor product.*

*Proof.* The idea is to show that $\{e_i f_j\}$ form a complete orthonormal set in $L^2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$. For the details of the proof see [16] theorem II.10. □

Consider now $L^2(\Omega^{\times n}, \mu^{\times n})$. $\mathbb{S}_n$ acts on it by permuting the variables i.e. for $\sigma \in \mathbb{S}_n$ and $f(x_1, \ldots, x_n) \in L^2(\Omega^{\times n}, \mu^{\times n})$

$$(\sigma.f)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

**Proposition 88.** *Let* $L^2(\Omega, \mu)$ *be separable* $n \geq 1$*. The isometry*

$$\phi : L^2(\Omega, \mu)^{\hat{\otimes}n} \to L^2(\Omega^{\times n}, \mu^{\times n}) \tag{A.9}$$

*coming from example 87 commutes with the* $\mathbb{S}_n$ *action.*

*Proof.* It is clear that it commutes on the dense subspace generated by elementary tensors so by continuity it commutes on the whole space. □

We can directly apply these results to tensor products of $L^2(G, \mu)$, where $\mu$ is the Haar measure on $G$ to get

$$L^2(G)^{\hat{\otimes} n} \cong L^2(G^{\times n}). \tag{A.10}$$

In most cases it will be convenient to work with $L^2(G^{\times n})$ instead of $L^2(G)^{\hat{\otimes} n}$ and this isometry allows us to jump back and forth as we wish. Therefore we will identify $L^2(G)^{\hat{\otimes} n}$ with $L^2(G^{\times n})$ for simplicity.

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