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Summer term 2018

Fourier Analysis – Exercise sheet 7
(to be discussed on July 9)

Important information:
This last exercise class will be held in a slightly different format than what has been the case so far: During the exercise class the participants should work (alone or in groups) on some given exercises and I will assist with problems and difficulties. Some exercises on the topics that are currently discussed in the lecture are already included below. Other exercises — dealing with topics that were already covered by previous exercise sheets — will be provided on Monday.

Ex 7.1: *Discuss whether the following objects are tempered distributions:*

(a) *the functionals*

$$L_f : \phi \mapsto \int_{\mathbb{R}} \phi(s) f(s)\, ds$$

for the choices of functions $f(x) = c$ *for all* $x \in \mathbb{R}$ *or* $f(x) = e^{x^2}$.

(b) L_f *for* $f \in L^p(\mathbb{R})$, $p \in [1, \infty]$.

(c) L_f *for* $f \in L^1_{loc}(\mathbb{R})$

(d) *the functional* δ_t *given by* $\phi \mapsto \phi(t)$ *for fixed* $t \in \mathbb{R}$

(e) L_μ *for any finite Borel measure* μ, *where* L_μ *is defined by*

$$L_\mu \phi = \int_{\mathbb{R}} \phi(s)\, d\mu(s)$$

(f) $L_{\log|\cdot|}$

(g) *the functional given by*

$$\phi \mapsto \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\phi(s)}{s}\, ds$$

Ex 7.2:*(Derivative and Fourier transform of tempered distribution)*
Let $n \in \mathbb{N}$. *We define the* n*-th derivative* $\partial^n u \in S'(\mathbb{R})$ *of a tempered distribution* $u \in S'(\mathbb{R})$ *by*

$$\langle \partial^n u, \phi \rangle = \langle u, (-1)^n \phi \rangle$$

(recall that by definition $\langle u, \phi \rangle = u(\phi)$*). Also, define the Fourier transform* $\mathcal{F}u$ *by*

$$\langle \mathcal{F}u, \phi \rangle = \langle u, \mathcal{F}\phi \rangle$$

and similarly the inverse Fourier transform by

$$\langle \mathcal{F}^\star u, \phi \rangle = \langle u, \mathcal{F}^\star \phi \rangle$$

Show the following

(a) *This definition of the Fourier transform is consistent with the definition we have seen for functions* u *in* L^p *for* $p = 1$ *and* $p = 2$ *(note what we have shown in Ex. 7.1).*

(b) *Compute the Fourier transform of* $\partial \delta_0$ *where* δ_0 *is defined as in Ex 7.2*

(c) *Compute the derivative of the function step function* $f(s) = \begin{cases} 1 & |s| \leq 1 \\ 0 & |s| > 1 \end{cases}$

(d) *Compute the Fourier transform of the distributions defined by the functions* sin *and* cos.