THE EVEN CYCLE THEOREM

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ABSTRACT. We present Verstraëte's proof of the Bondy–Simonovits theorem giving an upper bound on the extremal number of even cycles.

§1 INTRODUCTION

We consider the extremal problem for even cycles. To this end, we denote by $\mathrm{ex}(n, C_{2k})$ the maximum number of edges over all n-vertex graphs not containing a cycle of length $2k$. Bondy and Simonovits [1] proved the so-called *Even cycle theorem*, which was stated earlier by Erdős [3].

Theorem 1.1 (Bondy & Simonovits)**.** *For every integer $k \geqslant 2$, there exists $c = c(k)$ such that for sufficiently large n every bipartite graph with minimum degree at least $cn^{1/k}$ and at most n vertices contains C_{2k}.*

Since every graph can be made bipartite by removing at most half of its edges and since every graph with average degree d contains a subgraph with minimum degree at least $d/2$, Theorem 1.1 implies

$$\mathrm{ex}(n, C_{2k}) \leqslant 2cn^{1+1/k}\,.$$

Let $H = (X \mathbin{\dot\cup} Y, E)$ be a bipartite graph with at most n vertices and $\delta(H) \geqslant cn^{1/k}$ for some $c > 1$. Considering any *breadth-first search tree* in H easily yields an even cycle of length at most $2k$. This might give some evidence for the validity of Theorem 1.1. However, since H is relatively sparse, this simple argument falls short of guaranteeing a cycle of length exactly $2k$. For that a more careful argument is required. In this short note we follow the strategy laid out by Verstraëte [7].

Regarding the growth of $c(k)$ in Theorem 1.1 for $k \longrightarrow \infty$, we remark that the proofs of Bondy and Simonovits [1] and of Verstraëte [7], as well as Pikhurko's refinement of the latter [6], established a linear dependence on k. For example, for the proof presented here, $c(k) = 4k$ will be sufficient for large n. The best current upper bound in this direction is due to the work of Bukh and Jiang [2], in which it was shown that $c(k) \leqslant C\sqrt{k\,\log(k)}$ suffices for some constant C independent of k. However, matching asymptotic lower bounds

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for $\mathrm{ex}(n, C_{2k})$ are only known for $k \in \{2, 3, 5\}$, and we refer to the survey of Füredi and Simonovits [4] for more details.

§2 CYCLES WITH A CHORD

In the proof of Theorem 1.1 we make use of so-called θ-graphs, which simply consist of a cycle together with a chord. The order of such a graph is its number of vertices, which coincides with the length of its defining cycle. It follows from a standard maximal path argument that θ-graphs of a given order arise in graphs whose average degree is comparable to that order. More precisely, we shall employ the following bipartite version of this observation.

Fact 2.1. *For every integer $k \geqslant 3$, every bipartite graph B with average degree at least $2k$ contains a bipartite θ-graph Θ of order at least $2k$.*

Proof. We pass to a subgraph $B' \subseteq B$ with minimum degree at least k and we consider a maximal path P in B'. Let x be an end-vertex of P. The maximality of P implies that the neighbourhood of x is contained in $V(P)$. Since $|N_{B'}(x)| \geqslant k \geqslant 3$, this gives rise to a θ-graph Θ containing $N_{B'}(x)$. Moreover, since $B' \subseteq B$ is bipartite, no two neighbours of x can be consecutive on the path P and $|V(\Theta)| \geqslant 2\,|N_{B'}(x)| \geqslant 2k$ follows. ☐

The next lemma yields many different even-length paths connecting pairs of vertices across a fixed partition under mild assumptions. This will allow us to generate a sequence of consecutive even lengths of cycles in the proof of Theorem 1.1, which will be flexible enough to ensure that the desired length $2k$ is among them.

Lemma 2.2. *Let $k \geqslant 3$ be an integer, and let Θ be a bipartite θ-graph of order at least $2k$ and let $X \cupdot Y = V(\Theta)$ be a partition with $X \neq \varnothing$ and Y not an independent set in Θ. Then for every $\ell \in [k-1]$ there exists an X-Y-path in Θ of length 2ℓ.*

We remark that the conclusion of the lemma can be strengthened to cover all possible path lengths $< 2k$ – independent of the parity – but we shall only use the even lengths.

Proof of Lemma 2.2. Since Θ is a bipartite θ-graph, the number of vertices $n = |V(\Theta)|$ is even and satisfies $n \geqslant 2k$. We identify the vertices of the defining cycle C with the cyclic group $\mathbb{Z}/n\mathbb{Z}$, whose elements we write as $\overline{0}, \dots, \overline{n-1}$. Moreover, since Θ is bipartite, we may assume the chord $e_\star$ is given by the edge $\{\overline{0}, \overline{s}\}$ for some $s \in \{3, 5, \dots, n-3\}$. In fact, by symmetry we may assume $s \leqslant n/2$.

We shall analyse the indicator $\mathbb{1}_X \colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \{0,1\}$ of the given set $X = V(\Theta) \smallsetminus Y$. We call a path (and likewise a walk) in Θ *cyclic* if it avoids $e_\star$, and *chordal* otherwise.

Suppose for a contradiction that the lemma fails. In particular, there is a minimal integer $\ell \in [k-1]$, for which there is no cyclic path of length $L = 2\ell$. In other words, both end-vertices of any cyclic path of length L belong to the same set. Equivalently, we have

$$\mathbb{1}_X(a) = \mathbb{1}_X(a + \overline{L}) \tag{2.1}$$

for every $a \in \mathbb{Z}/n\mathbb{Z}$, i.e., the indicator function $\mathbb{1}_X$ is constant on the subgroup $\langle \overline{L} \rangle \subseteq \mathbb{Z}/n\mathbb{Z}$ generated by $\overline{L}$. It follows from Bézout's lemma that $\langle \overline{\gcd(L,n)} \rangle = \langle \overline{L} \rangle$ and, therefore, $\mathbb{1}_X(a) = \mathbb{1}_X(a + \overline{\gcd(L,n)})$ for every $a \in \mathbb{Z}/n\mathbb{Z}$. Consequently, the minimal choice of L implies $\gcd(L,n) = L$. In particular, we have $L \mid n$ and, hence, also $L \leqslant n/2$.

Note that the minimal choice of L combined with the periodicity (2.1) yields X-Y-paths of any even length $M \not\equiv 0 \pmod L$ with $M \leqslant 2k-2$. In fact, for $M = qL + r$ with integers $q \geqslant 0$ and $r \in \{2, \dots, L-2\}$ the minimality of L yields some vertex $a \in \mathbb{Z}/n\mathbb{Z}$ such that

$$\mathbb{1}_X(a) \neq \mathbb{1}_X(a + \overline{r})$$

and the periodicity tells us

$$\mathbb{1}_X(a) \neq \mathbb{1}_X(a + \overline{r}) \overset{(2.1)}{=} \mathbb{1}_X(a + \overline{L + r}) \overset{(2.1)}{=} \dots \overset{(2.1)}{=} \mathbb{1}_X(a + \overline{qL + r}) = \mathbb{1}_X(a + \overline{M})\,.$$

Consequently, it suffices to locate X-Y-paths of every length that is a multiple of L and at most $2k-2$. Depending on the (odd) value of s, we consider two cases.

First case: $s < L \leqslant n/2$.

The minimal choice of L yields a vertex $a \in \mathbb{Z}/n\mathbb{Z}$ such that $\mathbb{1}_X(a) \neq \mathbb{1}_X(a + \overline{s-1})$. Owing to the periodicity (2.1), we may assume

$$a \in \big\{\overline{-L+1}, \overline{-L+2}, \dots, \overline{-1}, \overline{0}\big\}$$

and we infer

$$\mathbb{1}_X(a) \neq \mathbb{1}_X(a + \overline{s-1}) \overset{(2.1)}{=} \mathbb{1}_X(a + \overline{s-1} + \overline{L})\,.$$

Consequently, the chordal a-$(a + \overline{s-1} + \overline{L})$-path has length L and constitutes a desired X-Y-path of length L. For lengths that are multiples of L, we use the assumption of the case, which allows us to extend this chordal path and replace $a + \overline{s-1} + \overline{L}$ by $a + \overline{s-1} + \overline{qL}$ for every $q \in \{2, \dots, (n/L) - 1\}$.

Second case: $L < s \leqslant n/2$.

In this case, we consider for every even $m \in \{2, \dots, L-2\}$ the chordal walk given by the vertex sequence

$$W(m) = (\overline{m}, \overline{m-1}, \dots, \overline{0}, \overline{s}, \overline{s+1}, \dots, \overline{n-1}, \overline{0}, \overline{s}, \overline{s-1}, \dots, \overline{m+2})\,.$$

Note that for every fixed m the walk $W(m)$ passes through the chord $\{\overline{0}, \overline{s}\}$ twice. In particular, it consists of n edges, which is a multiple of L. Moreover, every subwalk of L edges is indeed a path, since by the assumption of the case both $\overline{0}$-$\overline{s}$-paths in C have length

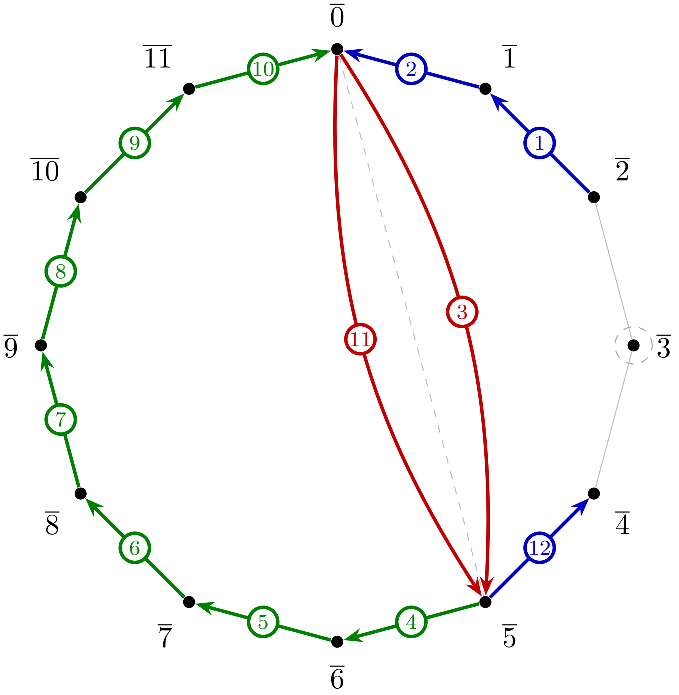

FIGURE 2.1. The walk $W(m)$ for $n = 12$, $s = 5$, $L = 4$, and $m = 2$. Edges are numbered in traversal order, beginning at $\overline{m} = \overline{2}$ and ending at $\overline{m+2} = \overline{4}$. The chord $\{\overline{0}, \overline{s}\}$ is traversed twice (steps 3 and 11), and accordingly the vertices $\overline{0}$ and $\overline{s} = \overline{5}$ are visited twice each. The only vertex of Θ that is not visited is $\overline{m+1} = \overline{3}$ (dashed circle).

at least $\min\{s, n-s\} > L$. If we additionally exclude chordal X-Y-paths of length L as well – i.e., assume that there is no X-Y-path of length L at all – then

$$\mathbb{1}_X(\overline{m}) = \mathbb{1}_X(\overline{m+2})\,.$$

Repeating this argument for every $m \in \{2, \dots, L-2\}$ tells us

$$\mathbb{1}_X(\overline{2}) = \mathbb{1}_X(\overline{4}) = \mathbb{1}_X(\overline{6}) = \cdots = \mathbb{1}_X(\overline{L-2}) = \mathbb{1}_X(\overline{L})$$

and the periodicity (2.1) implies that $\mathbb{1}_X$ is constant on the set of even residues $\langle\overline{2}\rangle \subseteq \mathbb{Z}/n\mathbb{Z}$.

Similarly, repeating the argument above by considering the walks $W(m)$ for every odd $m \in \{1, \dots, L-3\}$ implies

$$\mathbb{1}_X(\overline{1}) = \mathbb{1}_X(\overline{3}) = \mathbb{1}_X(\overline{5}) = \cdots = \mathbb{1}_X(\overline{L-3}) = \mathbb{1}_X(\overline{L-1})$$

and, again by (2.1) it also follows that $\mathbb{1}_X$ is constant on the odd residues $\overline{1} + \langle\overline{2}\rangle$. In particular, either the even or the odd residues must be X and the other class is Y. However, these two classes define a valid 2-colouring of Θ, contradicting our assumption on Y.

Hence the additional assumption above must be false: a chordal X-Y-path P of length L does exist. We now show that P can be extended to yield X-Y-paths of every length that is a multiple of L and at most $2k-2$. Specifically, we may successively extend P by appending L consecutive cycle edges at each of its endpoints until fewer than L unused cycle vertices remain on each side. By the periodicity (2.1), each such extension preserves the value of $\mathbb{1}_X$ at the moving endpoint, so all the extended paths remain X-Y-paths. Consequently, we obtain an X-Y-path of every length that is a multiple of L and at most $2k-2$. □

§3 Proof of the Bondy–Simonovits theorem

For $k = 2$, the result for $C_4 = K_{2,2}$ is a special case of the Kövari–Sós–Turán theorem [5] and, hence, we may assume $k \geqslant 3$. We set $c = 4k$, and for $n \geqslant 2^k$ let $H = (V, E)$ be a bipartite graph on at most n vertices with minimum degree $\delta(H) \geqslant cn^{1/k}$. Fix some vertex $x \in V$ and consider the breadth-first search tree T of H with root x. This defines the family of pairwise disjoint *level sets* $V_0, V_1, \dots$ given by

$$V_i = \left\{y \in V\colon \operatorname{dist}(x,y) = i\right\},$$

where $\operatorname{dist}(x,y)$ denotes the number of edges of a shortest x-y-path in H. In particular,

$$|V_0| = |\{x\}| = 1 \qquad \text{and} \qquad |V_1| = |N(x)| = d_H(x) \geqslant cn^{1/k} \tag{3.1}$$

and $\varnothing = V_m = V_{m+1} = \dots$ for some integer $m > 1$.

For $r \geqslant 1$ we denote by $H_r = H[V_0 \cup V_1 \cup \dots \cup V_r]$ the subgraph of H induced on the union of the first $r+1$ level sets. Obviously H_r is connected and has radius at most r. Moreover, since H contains no odd cycle, the definition of the level sets yields

$$E(H_r) = \dot{\bigcup_{i=1}^{r}} E\big(V_{i-1}, V_i\big)\,, \tag{3.2}$$

i.e., all sets V_i are independent in H (since any such edge would yield an odd closed walk in H) and there are no edges between V_i and V_j for $j > i+1$ (by the property of shortest paths defining the level sets). We show that for some $r \leqslant k$ the graph H_r has moderately large average degree.

Claim 3.1. For some $r \leqslant k$ the graph H_r has average degree at least $c = 4k$.

Proof. In view of (3.1), there is some minimal integer $r > 1$ such that

$$|V_r| \leqslant n^{1/k} \cdot |V_{r-1}|\,.$$

We note that V_r might be the empty set. However, by definition,

$$|V_{r-1}| > n^{\frac{1}{k}} \cdot |V_{r-2}| > \dots > n^{\frac{r-1}{k}} \cdot |V_0|\,,$$

and, therefore, $r \leqslant k$ follows. Moreover, by (3.2),

$$e(H_r) \geqslant \sum_{y \in V_{r-1}} d_H(y) \geqslant cn^{1/k} \cdot |V_{r-1}| \,.$$

On the other hand, since $|V_i| < |V_{i+1}| \cdot n^{-1/k}$ for every $i < r-1$ we have

$$\begin{aligned} |V(H_r)| &= |V_r| + \sum_{i=0}^{r-1} |V_i| \\ &< n^{1/k} \cdot |V_{r-1}| + |V_{r-1}| \cdot \sum_{i=0}^{r-1} n^{-i/k} < (n^{1/k} + 2) \cdot |V_{r-1}| \leqslant 2n^{1/k} \cdot |V_{r-1}| \,, \end{aligned}$$

where we used the assumption $n \geqslant 2^k$ for the last two estimates. Finally, comparing the lower bound on $e(H_r)$ and the upper bound on $|V(H_r)|$ concludes the proof of the claim. □

A standard averaging argument invoking the claim and the edge decomposition property (3.2) tells us that for some $i \in [r]$ the induced bipartite graph $B = H[V_{i-1}, V_i]$ has average degree at least $c/2 = 2k$, as the contrapositive leads to the absurd situation

$$c \cdot |V(H_r)| \leqslant 2\, e(H_r) = \sum_{i=1}^{r} 2e(V_{i-1}, V_i) < \sum_{i=1}^{r} \frac{c}{2} \cdot \big(|V_{i-1}| + |V_i|\big) < c \cdot |V(H_r)| \,.$$

Consequently, Fact 2.1 provides a θ-graph $\Theta \subseteq B$ of order at least $2k$. Let U_{i-1} and U_i be the intersections of $V(\Theta)$ with V_{i-1} and V_i, respectively. The set U_{i-1} defines a minimal subtree T_Θ of T having U_{i-1} as its set of leaves. Let x_Θ be the root of T_Θ. Owing to

$$|U_{i-1}| = \frac{1}{2}|V(\Theta)| \geqslant k \,,$$

the degree of x_Θ in T_Θ is at least 2 and the components of $T_\Theta - x_\Theta$ induce a nontrivial partition of U_{i-1} (by considering the membership in the subtrees of $T_\Theta - x_\Theta$).

Let $X \subseteq U_{i-1}$ be one of those partition classes and set $Y = V(\Theta) \smallsetminus X$. In particular, the induced subgraph $\Theta[Y]$ contains some edge and we can invoke Lemma 2.2. The lemma provides us for every $\ell \in [k-1]$ an X-$(U_{i-1} \smallsetminus X)$-path $P_{2\ell}$ of length 2ℓ. Since the end-vertices of $P_{2\ell}$ are in different partition classes of U_{i-1}, the unique path in T_Θ connecting those end-vertices must pass through x_Θ. Therefore, all such paths in T_Θ have a fixed length $2h_\Theta$, where $h_\Theta \geqslant 1$ is the height of the tree T_Θ. This way we obtain even cycles in H of any length of the form $2h_\Theta + 2\ell$ for every $\ell \in [k-1]$. Since $1 \leqslant h_\Theta \leqslant r-1 < k$ we are guaranteed to have a C_{2k} among them. □

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