Funnel control in the presence of infinite-dimensional internal dynamics

Thomas Berger, Marc Puche and Felix L. Schwenninger

***Abstract*— We consider output trajectory tracking for a class of uncertain nonlinear systems whose internal dynamics may be modelled by infinite-dimensional systems which are bounded-input, bounded-output stable. We describe under which assumptions these systems belong to an abstract class of systems for which funnel control is known to be feasible. As an illustrative example, we show that for a system whose internal dynamics are modelled by a transport equation, which is not exponentially stable, we obtain prescribed performance of the tracking error.**

***Index Terms*— Adaptive control, infinite-dimensional systems, funnel control, BIBO stability.**

I. Introduction

We study output trajectory tracking for uncertain nonlinear systems by funnel control. As a crucial assumption, we require that the internal dynamics of the system, typically arising from a partial differential equation (PDE) in our framework, are *bounded-input, bounded-output (BIBO)* stable.

Funnel control has been developed in [1] for systems with relative degree one, see also the survey [2]. The funnel controller is a low-complexity model-free output-error feedback of high-gain type; it is an adaptive controller since the gain is adapted to the actual needed value by a time-varying (non-dynamic) adaptation scheme. Note that no asymptotic tracking is pursued, but a prescribed tracking performance is guaranteed over the whole time interval. The funnel controller proved to be the appropriate tool for tracking problems in various applications, such as temperature control of chemical reactor models [3], control of industrial servo-systems [4] and underactuated multibody systems [5], speed control of wind turbine systems [6], [7], DC-link power flow control [8], voltage and current control of electrical circuits [9], oxygenation control during artificial ventilation therapy [10] and adaptive cruise control [11].

A funnel controller for a large class of systems described by functional differential equations with arbitrary relative degree has been developed recently in [12]. While this abstract class appears to allow for fairly general infinite-dimensional systems, cf. also Section II, it is in fact not clear which types of PDE systems are encompassed. As a first result, it was shown in [13] that the linearized model of a moving water tank, where sloshing effects appear, belongs to the aforementioned system class. On the other hand, not even every linear, infinite-dimensional system has a well-defined (integer-valued) relative degree: In that case, results as in [1], [12] cannot be applied. Instead, the feasibility of funnel control has to be investigated directly for the (nonlinear) closed-loop system, see [14] for a boundary controlled heat equation and [15] for a general class of boundary control systems.

The present paper is devoted to systems which have a relative degree, but in the presence of internal dynamics that are modelled by a PDE system. We generalize the findings from [13] and develop a general system class containing PDE models for which funnel control is feasible; this result is presented in Section III. As an example, we consider a system internally driven by a transport equation, Section IV, and illustrate the funnel controller by a simulation. Some conclusions are given in Section V.

A. System class

In the remainder of the present paper we consider abstract differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big) \\ &\quad + \Gamma\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big)\, u(t) \\ y|_{[-h,0]} &= y^0 \in W^{r-1,\infty}([-h,0];\mathbb{R}^m), \end{aligned} \tag{1}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the relative degree, and

(N1) the disturbance satisfies $d \in L^\infty(\mathbb{R}_{\ge 0};\mathbb{R}^p)$, $p \in \mathbb{N}$;

(N2) $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q;\mathbb{R}^m)$, $q \in \mathbb{N}$;

(N3) the high-frequency gain matrix function $\Gamma \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q;\mathbb{R}^{m\times m})$ satisfies $\Gamma(d,\eta) + \Gamma(d,\eta)^\top > 0$ for all $(d,\eta) \in \mathbb{R}^p \times \mathbb{R}^q$;

(N4) $T : \mathcal{C}([-h,\infty);\mathbb{R}^{rm}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0};\mathbb{R}^q)$ is an operator with the following properties: a),leftmargin=0.5cm

a) T maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathcal{C}([-h,\infty);\mathbb{R}^{rm})$,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \le c_1 \Rightarrow \sup_{t\ge 0} \|T(\zeta)(t)\| \le c_2,$$

b) T is causal, i.e, for all $t \ge 0$ and all $\zeta, \xi \in \mathcal{C}([-h,\infty);\mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{\text{a.e.}}{=} T(\xi)|_{[0,t)}.$$

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c) T is locally Lipschitz continuous in the following sense: for all $t \geq 0$ and all $\xi \in \mathcal{C}([-h,t];\mathbb{R}^{rm})$ there exist $\tau, \delta, c > 0$ such that, for all $\zeta_1, \zeta_2 \in \mathcal{C}([-h,\infty);\mathbb{R}^{rm})$ with $\zeta_i|_{[-h,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1, 2$, we have

$$\left\| (T(\zeta_1) - T(\zeta_2))\,|_{[t,t+\tau]} \right\|_\infty \leq c \left\| (\zeta_1 - \zeta_2)|_{[t,t+\tau]} \right\|_\infty .$$

In [1], [12], [16]–[18] it is shown that the class of systems (1) encompasses linear and nonlinear systems with strict relative degree r and BIBO stable internal dynamics. The operator T allows for infinite-dimensional (linear) systems, systems with hysteretic effects or nonlinear delay elements, and combinations thereof. Note that T is typically the solution operator corresponding to a (partial) differential equation which describes the internal dynamics of the system. The linear infinite-dimensional systems that are considered in [1], [18] are in a special Byrnes-Isidori form that is discussed in detail in [19]. While the internal dynamics in these systems is allowed to correspond to a strongly continuous semigroup, all other operators are assumed to be bounded and to satisfy additional restrictive conditions. In contrast to this, in the present paper we consider nonlinear equations which, in particular, involve unbounded operators. This complements and generalizes the findings in [13].

B. Control objective

The objective is to design an output error feedback

$$u(t) = F\big(t, e(t), \dot{e}(t), \ldots, e^{(r-1)}(t)\big),$$

where $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ is a reference signal, which applied to (1) results in a closed-loop system where the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{\ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \ \mid \ \varphi(t)\|e\| < 1\ \}, \qquad (2)$$

which is determined by a function φ belonging to

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0};\mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, all signals $u, e, \dot{e}, \ldots, e^{(r-1)}$ should remain bounded.

The funnel boundary is given by $1/\varphi$, see Fig. 1. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

An important property is that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi_r$ is bounded away from zero, because boundedness of φ implies existence of $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see also [20, Sec. 3.2].

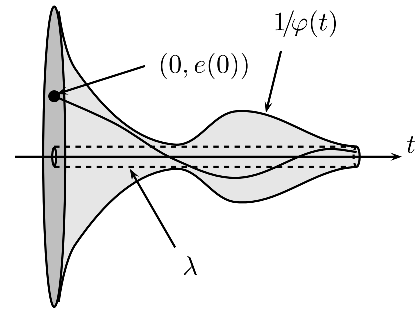

Fig. 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$.

II. Funnel control

It was shown in [12] that the funnel controller

$$\boxed{\begin{aligned} u(t) &= -k_{r-1}(t)\, e_{r-1}(t), \\ e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t), \\ &\ \vdots \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), \\ k_i(t) &= \frac{1}{1 - \varphi_i(t)^2 \|e_i(t)\|^2}, \quad i = 0, \ldots, r-1, \end{aligned}} \qquad (3)$$

where

$$\varphi_0 \in \Phi_r, \varphi_1 \in \Phi_{r-1}, \ldots, \varphi_{r-1} \in \Phi_1, \qquad (4)$$

achieves the control objective described in Section I-B for any system which belongs to the class (1). We stress that while the derivatives $\dot{e}_0, \ldots, \dot{e}_{r-2}$ appear in (3), they only serve as short-hand notations and may be resolved in terms of the tracking error, the funnel functions and the derivatives of these, cf. [12, Rem. 2.1].

The existence of solutions of the initial value problem resulting from the application of the funnel controller (3) to a system (1) must be treated carefully. By a *solution* of (3), (1) on $[-h, \omega)$ we mean a function $y \in \mathcal{C}^{r-1}([-h,\omega);\mathbb{R}^m)$, $\omega \in (0, \infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is weakly differentiable and satisfies the differential equation in (1) with u defined in (3) for almost all $t \in [0, \omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [1] for instance.

The following result is from [12]. Note that in [12] a slightly stronger version of conditions (N3) and (N4) c) is used. However, the proof does not change; in particular, regarding (N4) c), the existence part of the proof in [12] relies on a result from [17] where the version from the present paper is used.

Theorem 1: Consider a system (1) with properties (N1)–(N4) for some $r \in \mathbb{N}$ and $h > 0$. Let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, $\varphi_0, \ldots, \varphi_{r-1}$ as in (4) and $y^0 \in W^{r-1,\infty}([-h, 0]; \mathbb{R}^m)$ be an initial condition such that $e_0, \ldots, e_{r-1}$ defined in (3) satisfy

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \text{for } i = 0, \ldots, r-1.$$

Then the funnel controller (3) applied to (1) yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution $y : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, which has the following properties:

(i) The solution is global, i.e., $\omega = \infty$.

(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $k_0, \ldots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $y, \dot{y}, \ldots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are bounded.

(iii) The functions $e_0, \ldots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the sense

$$\forall\, i = 0, \ldots, r-1 \;\exists\, \varepsilon_i > 0 \;\forall\, t > 0 : \quad \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i.$$

III. A class of operators for funnel control

While the class of functional differential equations (1) appears to be rather general and funnel control is feasible for these systems by Theorem 1, it is not clear exactly which kind of systems that contain PDEs are encompassed by the class (1). In this section we develop a description for a class of operators T which include certain BIBO stable linear PDEs and satisfy condition (N4). The aforementioned PDEs may either be coupled with a nonlinear observation operator which is polynomially bounded, or with a linear observation operator which is possibly unbounded, but with respect to which the system is regular well-posed and the inverse Laplace transform of the corresponding transfer function defines a measure with bounded total variation. This structure is illustrated in Fig. 2.

We give a precise definition of the operator class in the following.

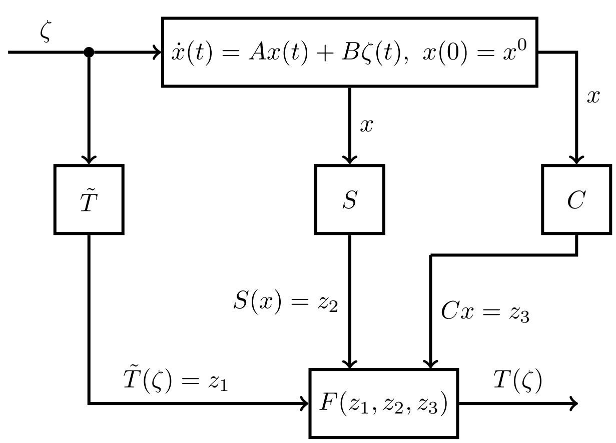

Fig. 2: Structure of an operator $T \in \mathcal{T}_h^{\ell,q}$.

Definition 1: Let $h \geq 0$ and $\ell, q \in \mathbb{N}$. Then $\mathcal{T}_h^{\ell,q}$ is defined as the set of all operators

$$T : \mathcal{C}([-h, \infty); \mathbb{R}^\ell) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$$

which, for any $\zeta \in \mathcal{C}([-h, \infty); \mathbb{R}^\ell)$, are given by

$$T(\zeta)(t) = F\big(\tilde{T}(\zeta)(t), S(x)(t), (Cx)(t)\big), \quad t \geq 0,$$

where x, for some $x^0 \in \mathcal{D}(A)$, is the mild solution[1] of the PDE

$$\dot{x}(t) = Ax(t) + B\zeta(t), \quad x(0) = x^0, \tag{5}$$

where

(P1) $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a bounded $\mathcal{C}_0$-semigroup in X, X a real Hilbert space, and $B \in \mathcal{L}(\mathbb{R}^\ell; X_{-1})$ is an L^2-admissible control operator such that $\dot{x}(t) = Ax(t) + B\zeta(t)$ is BIBO stable, i.e., there exists $\gamma \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ such that for all $\zeta \in \mathcal{C}([-h, \infty); \mathbb{R}^\ell)$ the mild solution of (5) satisfies

$$\forall\, t \geq 0 : \quad \|x(t)\|_X \leq \gamma(\|\zeta|_{[-h,t]}\|_\infty);$$

(P2) $F \in \mathcal{C}^1(\mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \times \mathbb{R}^{q_3} \to \mathbb{R}^q)$;

(P3) $\tilde{T} : \mathcal{C}([-h, \infty); \mathbb{R}^\ell) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{q_1})$ satisfies condition (N4) in Section I-A with $\ell = rm$;

(P4) $S : X \to \mathbb{R}^{q_2}$ is a Fréchet differentiable operator with continuous Fréchet derivative and satisfies

$$\forall\, x \in X : \ \|S(x)\| \leq p(\|x\|_X)$$

for some polynomial $p(s)$;

(P5) $C \in \mathcal{L}(\mathcal{D}(A); \mathbb{R}^{q_3})$ is an L^2-admissible observation operator such that the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\zeta(t),\\ \nu(t) &= Cx(t)\end{aligned}$$

is well-posed, i.e., for some $\omega \in \mathbb{R}$ the transfer function $H : \mathbb{C}_\omega \to \mathbb{C}^{q_3 \times \ell}$, which is uniquely determined (up to a constant) by

$$\frac{1}{t-s}(H(s) - H(t)) = C\big((sI - A)^{-1}(tI - A)^{-1}\big)B$$

for all $s, t \in \mathbb{C}_\omega,\ s \neq t$, exists and is proper, that is $\sup_{s \in \mathbb{C}_\omega} \|H(s)\| < \infty$. Furthermore, we require that the system is regular, i.e., $\lim_{\mathrm{Re}\, s \to \infty} H(s)v$ exists for all $v \in \mathbb{C}^\ell$, and we require that H satisfies that the inverse Laplace transform of its components $h_{ij} = \mathcal{L}^{-1}(H_{ij})$ is a real-valued measure with bounded total variation for all $i = 1, \ldots, q_3$ and $j = 1, \ldots, \ell$.

Remark 1:

(i) We note that the notion of *admissible* operators is well-known in infinite-dimensional linear systems theory with unbounded control and observation operators, see e.g. [21], and is motivated by interpreting a PDE on a larger space in order to define solutions. Further, note that any operator T as given in Definition 1 with the properties (P1)–(P5) is indeed well-defined from $\mathcal{C}([-h, \infty); \mathbb{R}^\ell)$ to $L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$.

[1]See e.g. [21] for a definition of the mild solution.

(ii) We emphasize that the assumption of BIBO stability of (5) as in (P1) is quite weak. A sufficient condition for this is *input-to-state stability*, which has been introduced by Sontag [22]. This concept was studied extensively for nonlinear systems, see [23], and for systems containing PDEs it is investigated in [24], [25]. However, the state of an input-to-state stable system converges to zero whenever the input is zero, which is not required for BIBO stable systems considered here.

In the following main result we show that any operator which belongs to the class $\mathcal{T}_h^{\ell,q}$ satisfies the condition (N4) in Section I-A.

Theorem 2: Any $T \in \mathcal{T}_h^{\ell,q}$ satisfies condition (N4) in Section I-A.

Proof: Step 1: We show property (N4) a). To this end, observe that by continuity of F it suffices to show this for the maps $\zeta \mapsto \tilde{T}(\zeta)$, $\zeta \mapsto S(x)$ and $\zeta \mapsto Cx$. By (P3), $\tilde{T}$ satisfies (N4) a) and by (P2) together with (P1) we have

$$\|S(x)(t)\| \le p\big(\|x(t)\|_X\big) \le p\left(\gamma(\|\zeta\|_\infty)\right)$$

for all bounded $\zeta \in \mathcal{C}([-h,\infty);\mathbb{R}^\ell)$. It remains to show that Cx is bounded. From (P5) the system (A,B,C) is regular well-posed, from which it follows by the variation of constants formula, see e.g. [26], that

$$Cx(\cdot) = CT_A(\cdot)x_0 + (h * \zeta)(\cdot),$$

where $(T_A(t))_{t\ge 0}$ is the $\mathcal{C}_0$-semigroup generated by A and $h = (h_{ij})_{i=1,\dots,q_3;j=1,\dots,\ell}$ is the inverse Laplace transform of the transfer function H. By assumption we have that $h \in \mathrm{M}(\mathbb{R}_{\ge 0};\mathbb{R}^{q_3\times\ell})$. Thus, for all $t \ge 0$,

$$\begin{aligned}\|Cx(t)\| &\le \|CT_A(t)x_0\| + \|(h*\zeta)(t)\|\\ &\le \|C\|_{\mathcal{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|AT_A(t)x_0\|\\ &\quad + \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\|\zeta\|_\infty\\ &= \|C\|_{\mathcal{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|T_A(t)Ax_0\|\\ &\quad + \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\|\zeta\|_\infty\\ &\le \|C\|_{\mathcal{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|T_A(t)\|_{\mathcal{L}(X)}\|Ax_0\|_X\\ &\quad + \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\|\zeta\|_\infty\\ &\le M\|C\|_{\mathcal{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|Ax_0\|_X + \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\|\zeta\|_\infty,\end{aligned}$$

where we have used that $x_0 \in \mathcal{D}(A)$ and $(T_A(t))_{t\ge 0}$ is bounded, that is, $\|T_A(t)\|_{\mathcal{L}(X)} \le M$ for some $M \ge 1$ and all $t \ge 0$. Thus,

$$\|Cx(\cdot)\|_\infty \le M\|C\|_{\mathcal{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|Ax_0\|_X + \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\|\zeta\|_\infty.$$

Step 2: We show property (N4) b). This is a straightforward consequence of the definition of $\tilde{T}$.

Step 3: We show property (N4) c). Fix $t \ge 0$ and $\xi \in \mathcal{C}([-h,t];\mathbb{R}^\ell)$. Let $\tilde{\tau},\tilde{\delta},\tilde{c}$ be the constants given by property (N4) c) of $\tilde{T}$. Set $\tau := \tilde{\tau}$ and $\delta := \tilde{\delta}$. Further let $\zeta_i \in \mathcal{C}([-h,\infty);\mathbb{R}^\ell)$ with $\zeta_i|_{[-h,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t,t+\tau]$ and $i = 1,2$. Let x_i denote the mild solution of (5) corresponding to ζ_i for $i = 1,2$. Then, by linearity, $x_1 - x_2$ is the mild solution corresponding to $\zeta_1 - \zeta_2$. Since S is Fréchet differentiable with continuous Fréchet derivative $DS : X \to \mathcal{L}(X;\mathbb{R}^{q_2})$ by (P4), the mean value theorem implies that it is locally Lipschitz continuous. Therefore, we find that for all $s \in [t,t+\tau]$

$$\begin{aligned}\|S(x_1(s)) - S(x_2(s))\| &\le L_1\big\|\big(x_1 - x_2\big)(s)\big\|\\ &\le L_1\gamma(\big\|\big(\zeta_1-\zeta_2\big)|_{[-h,s]}\big\|_\infty)\\ &\le L_1L_2\big\|\big(\zeta_1-\zeta_2\big)|_{[t,t+\tau]}\big\|_\infty,\end{aligned}$$

where, with $\tilde{x}$ denoting the mild solution of (5) corresponding to $\tilde{\xi}$ for $\tilde{\xi}|_{[-h,t]} = \xi$ and $\tilde{\xi}|_{[t,\infty)} \equiv \xi(t)$, we have $\|x_i(s) - \tilde{x}(t)\|_X \le \gamma(\|\zeta_i|_{[t,s]} - \xi(t)\|_\infty) < \gamma(\delta)$, which justifies to set

$$\begin{aligned}L_1 &:= \sup_{\|x-\tilde{x}(t)\|_X \le \gamma(\delta)} \|DS(x)\|_{\mathcal{L}(X;\mathbb{R}^{q_2})},\\ L_2 &:= \sup_{s\in[0,2\delta]} |\gamma'(s)|.\end{aligned}$$

Furthermore, by linearity and (P5) we have

$$\begin{aligned}\|Cx_1(s) - Cx_2(s)\| &= \|(h*(\zeta_1-\zeta_2))(s)\|\\ &\le L_3\big\|\big(\zeta_1-\zeta_2\big)|_{[t,t+\tau]}\big\|_\infty\end{aligned}$$

for all $s \in [t,t+\tau]$ and $L_3 := \|h\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}$. Now define $\hat{c} := \tilde{c} + L_1L_2 + L_3$ and

$$L_4 := \sup\left\{\ \|F'(z)\|\ \middle|\ \left\| z - \begin{pmatrix}\tilde{T}(\tilde{\xi})(t)\\ S(\tilde{x})(t)\\ C\tilde{x}(t)\end{pmatrix}\right\| \le \hat{c}\delta\ \right\}$$

and set

$$c := \hat{c}L_4.$$

Then we have

$$\|\tilde{T}(\zeta_1)(s) - \tilde{T}(\zeta_2)(s)\| \le c\big\|\big(\zeta_1-\zeta_2\big)|_{[t,t+\tau]}\big\|_\infty$$

for all $s \in [t,t+\tau]$ and this finishes the proof of the theorem. ∎

It is shown in [13] that the operator associated with the internal dynamics of a linearized model of a moving water tank system belongs to the class $\mathcal{T}_h^{\ell,q}$. In the subsequent section we consider another example which contains a transport equation.

IV. Example: The transport equation

We illustrate our results by considering the following system whose internal dynamics are described by a transport equation, that is

$$\begin{aligned}\dot{y}(t) &= T(y)(t) + \gamma u(t)\\ T(y)(t) &= z(t,0)\\ \frac{\partial z}{\partial t}(t,\xi) &= c\frac{\partial z}{\partial \xi}(t,\xi) + h(\xi)y(t),\\ z(0,\xi) &= 0,\end{aligned}\tag{6}$$

for $(t,\xi) \in (0,\infty)\times[0,\infty)$, where $c > 0$ and $h \in \mathrm{M}(\mathbb{R}_{\ge 0})$ is a Borel measure of bounded total variation. It is well-known that the third and fourth equations in (6) constitute a regular well-posed linear system (A,B,C) on $X = L^2(\mathbb{R}_{\ge 0};\mathbb{R})$, the so-called *shift-realization* of the Laplace transform $\mathcal{L}(h)$, see e.g. [27]–[29]. More precisely, the PDE is then considered on the abstract Sobolev space[2] X_{-1} to appropriately interpret the term $h(\xi)y(t)$ and the solutions are *mild solutions* in general.

Also note that the generated (left-) shift-semigroup is not exponentially stable. In particular, the Laplace transform $\mathcal{L}(h)$ of the measure h is defined on the closed right half-plane and bounded analytic on this domain. Moreover, the impulse response of the PDE equals h. More precisely, for sufficiently smooth y we have the representation

$$T(y)(t) = z(0,t) = (h * y)(t) = \int_0^t y(t-s)\,\mathrm{d}h(s).$$

As h is of bounded total variation, it follows that T is a bounded operator from $\mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$ to $L^\infty(\mathbb{R}_{\geq 0};\mathbb{R})$ and hence $T \in \mathcal{T}_0^{1,1}$. Therefore, the first equation in (6) formally reads

$$\dot{y}(t) = (h * y)(t) + \gamma u,$$

which is an integral-differential Volterra equation. Also note that for the following simple cases

- $h = \delta_0$, we obtain a finite-dimensional linear system:
$$\dot{y}(t) = y(t) + \gamma u(t);$$
- $h = \delta_{t_0}$, $t_0 > 0$, we obtain a delay differential equation:
$$\dot{y}(t) = \begin{cases} y(t-t_0) + \gamma u(t), & t \geq t_0, \\ \gamma u(t), & 0 \leq t < t_0. \end{cases}$$

Another typical case is that $h(\xi) = f(\xi)\mathrm{d}\xi$ with $f \in L^1(\mathbb{R}_{\geq 0};\mathbb{R})$, i.e., h is represented by its L^1-density with respect to the Lebesgue measure. If additionally $f \in L^2(\mathbb{R}_{\geq 0};\mathbb{R})$, then the input operator $B = h$ of the PDE is bounded.

For the simulation we have chosen $h(\xi) = f(\xi)\mathrm{d}\xi$ with $f(\xi) = e^{-\xi}/\sqrt{\xi}$, which is integrable but not square integrable on $\mathbb{R}_{\geq 0}$. Furthermore, we use the parameters $c = \gamma = 1$ and the reference signal

$$y_{\mathrm{ref}}(t) = \cos t, \quad t \geq 0.$$

The initial value is chosen as $y(0) = 0$ and for the controller (3) we chose the funnel function

$$\varphi(t) = \left(2e^{-2t} + 0.1\right)^{-1}, \quad t \geq 0.$$

Clearly, the initial error lies within the funnel boundaries as required in Theorem 1. Furthermore, by Theorem 2 the operator T satisfies (N4) and hence funnel control is feasible.

The PDE is solved using explicit finite differences with a grid in t with $M = 1000$ points for the interval $[0,T]$, where $T = 15$, and a grid in ξ with $N = \lfloor M(b-a)/(\alpha T)\rfloor$ points for $\alpha = 0.4$ and $a = 0$, $b = 10$. The method has been implemented in Python and the simulation results are shown in Fig. 3.

It can be seen that even in the presence of infinite-dimensional internal dynamics which are not exponentially stable a prescribed performance of the tracking error can be

[2]This space is sometimes referred to as *rigged Hilbert space*.

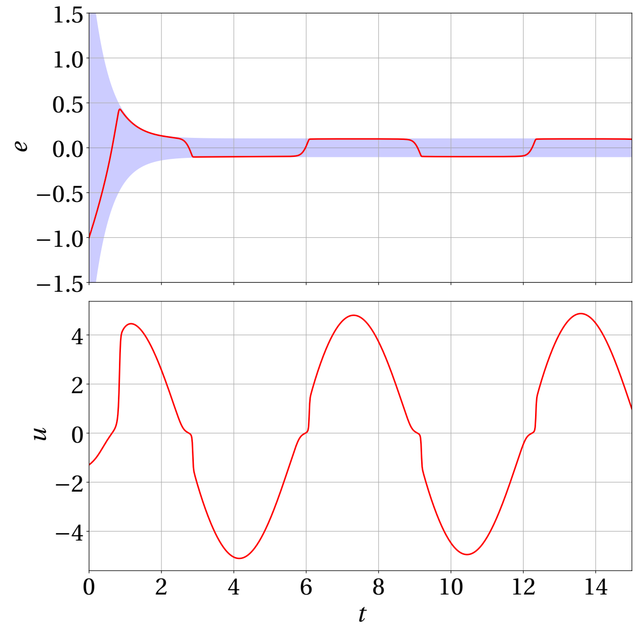

Fig. 3a: Performance funnel with tracking error e and generated input function u.

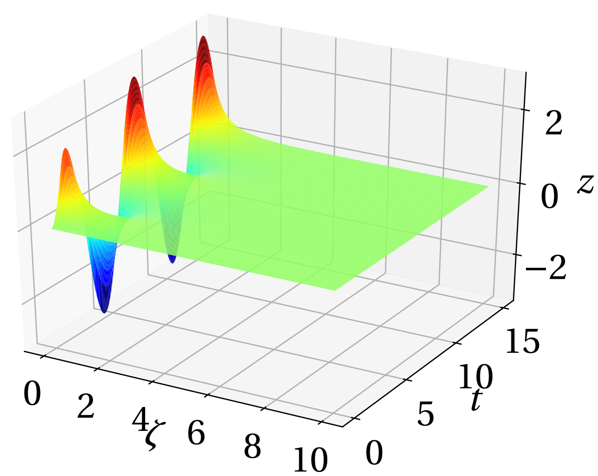

Fig. 3b: State z of the PDE.

Fig. 3: Simulation of the funnel controller (3) for the system (6).

achieved with the funnel controller (3). At the same time the input generated by the controller is bounded with a very good performance.

V. Conclusion

In the present paper we considered the question which classes of systems with infinite-dimensional internal dynamics are encompassed by the abstract system class (1) for which funnel control is feasible by Theorem 1. We have defined a class of operators $\mathcal{T}_h^{\ell,q}$, which model the internal dynamics of the system, that encompass BIBO stable linear

PDEs. These PDEs may either be coupled with a nonlinear, but polynomially bounded observation operator, or with a linear observation operator which may be unbounded. For the latter we additionally assumed that the resulting system is regular well-posed such that the inverse Laplace transform of its transfer function defines a measure with bounded total variation. In Theorem 2 we have proved that any operator belonging to $\mathcal{T}_h^{\ell,q}$ satisfies the conditions of the system class (1).

Several extensions of the operator class $\mathcal{T}_h^{\ell,q}$ and Theorem 1 may be investigated in future research. In particular, extensions to nonlinear PDE systems with unbounded observation operators are of interest as well as systems with infinite-dimensional input and output spaces which do not have an integer-valued relative degree.

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