Descriptive aspects of large cardinals beyond HOD

Philipp Moritz Lücke
Universität Hamburg

8th Workshop on Generalized Descriptive Set Theory
Helsinki, 22. August 2025

Introduction

Following Woodin, seminal results of Jensen and Silver can be formulated in the following way:

Theorem (The L-Dichotomy, Jensen & Silver)

Exactly one of the following statements holds:

- If λ is a singular cardinal, then λ is singular in L and $(\lambda^+)^{\mathrm{L}} = \lambda^+$ holds ("L *is close to V*").
- Every regular cardinal is inaccessible in L ("L *is far from V*").

A surprising result of Woodin shows that strong large cardinal assumptions imply an analog of this dichotomy for the inner model HOD.

Theorem (Simplified HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds ("*HOD is close to V*").
- Every regular cardinal $\kappa \geq \delta$ is measurable in HOD ("*HOD is far from V*").

Questions

Are there canonical extensions of ZFC that prove that HOD is far from V? Are there such axioms that imply $\mathrm{V} \neq \mathrm{HOD}$?

All standard large cardinal axioms are compatible with the assumption that $\mathrm{V} = \mathrm{HOD}$ and therefore do not provide affirmative answers to these questions.

If we instead ask for extensions of ZF, then large cardinals beyond choice (e.g., Reinhardt cardinals) provide trivial affirmative answers to the second question.

In the following, we will observe that more interesting things can be said about the relationship between V and HOD in this setting.

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ with the property that $\alpha < \mathrm{crit}(j) < \lambda$ and λ is the first non-trivial fixed point of j.

Proposition (GB)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding, then the first non-trivial fixed point of j is a rank-Berkeley cardinal.

Proposition (ZF)

Rank-Berkeley cardinals are cardinals of countable cofinality that are regular in HOD.

Proof.

Assume, towards a contradiction, that a rank-Berkeley cardinal λ is singular in HOD.

Pick $\beta > \lambda$ such that V_β is sufficiently elementary in V.

Then there is an elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ such that $\mathrm{cof}(\lambda)^{\mathrm{HOD}} < \mathrm{crit}(j)$ and λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}} \longrightarrow \lambda$ be the least cofinal function in the canonical well-ordering of HOD.

Then c is definable from the parameter λ and hence $j(c) = c$.

Pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ with $c(\alpha) > \mathrm{crit}(j)$. Then

$$c(\alpha) \ < \ j(c(\alpha)) \ = \ j(c)(j(\alpha)) \ = \ c(\alpha),$$

a contradiction. □

Exacting cardinals

We now want to isolate canonical fragments of rank-Berkeleyness that are compatible with the Axiom of Choice and still allow us to carry out the above argument.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Note that, in order to prove that a cardinal λ is exacting, it suffices to find a single embedding $j : X \longrightarrow \mathrm{V}_\beta$ satisfying $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X \prec \mathrm{V}_\beta \prec_{\Sigma_2} \mathrm{V}$, $j(\lambda) = \lambda$ and $j \upharpoonright \lambda \neq \mathrm{id}_\lambda$.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (Aguilera–Bagaria–L.)

If λ is exacting, then λ is a singular cardinal that is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Corollary

If there is an exacting cardinal above an extendible cardinal, then eventually all regular cardinals are measurable in HOD.

Let λ be an exacting cardinal. Then there is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ and results of Kunen imply $\mathrm{cof}(\lambda) = \omega$.

Assume, towards a contradiction, that λ is singular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$. Then there is $z \in \mathrm{V}_\lambda$ such that λ is singular in $\mathrm{HOD}_{\{z\}}$.

Fix $\beta > \lambda$ such that V_β is sufficiently elementary in V. Pick $X \prec \mathrm{V}_\beta$ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \longrightarrow V_\beta$ with $\mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j(z) = z$.

Results of Kunen imply that λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} \longrightarrow \lambda$ be the least cofinal function with respect to the canonical well-ordering of $\mathrm{HOD}_{\{z\}}$. Then $c \in X$ with $j(c) = c$.

If we pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}}$ with $c(\alpha) > \mathrm{crit}(j)$, then we have

$$c(\alpha) \ < \ j(c(\alpha)) \ = \ j(c)(j(\alpha)) \ = \ c(\alpha),$$

a contradiction.

We now discuss the naturalness of the notion of exactingness.

First, note that as a fragment of Reinhardtness, this property is phrased in a standard format for large cardinal axioms.

Next, we show that exactingness is equivalent to a natural model-theoretic reflection principle.

For this purpose, remember that a cardinal λ is Jónsson if every structure in a countable first-order language whose domain has cardinality λ has a proper elementary substructure of cardinality λ.

The next result shows that exactingness is equivalent to a strengthening of this property that incorporates external features of the given structure.

Theorem (Aguilera–Bagaria–L.)

The following are equivalent for each cardinal λ with $|\mathrm{V}_\lambda| = \lambda$:

- λ is an exacting cardinal.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ isomorphic to a structure in $\mathcal{C}$.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $\mathrm{V}_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ is isomorphic to a proper elementary substructure of a structure of cardinality λ in $\mathcal{C}$.

In another direction, exactingness can also be represented as a natural strengthening of the existence of I3-embeddings:

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- For every non-empty, ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding

$$j : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y).$$

Moreover, the consistency strength of exacting cardinal is surprisingly low:

Definition

- An I3-embedding is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An I2-embedding is a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{V}_\lambda \subseteq M$, where λ is the first non-trivial fixed point of j.

Theorem (Aguilera–Bagaria–Goldberg–L.)

Let $j : \mathrm{V} \longrightarrow M$ be an I2-embedding with critical point κ and let

$$U \ = \ \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

If G is generic over V for Prikry forcing with U, then κ is an exacting cardinal in $\mathrm{V}[G]$.

Finally, the consistency of the existence of an exacting cardinal above an extendible cardinal can be established from *large cardinals beyond Choice*.

Definition (GB)

A cardinal κ is *super Reinhardt* if for every ordinal α, there is an elementary embedding $j : \mathrm{V} \longrightarrow \mathrm{V}$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

Theorem (Aguilera–Bagaria–L., BG)

If there is a super Reinhardt cardinal, then there is a model of ZFC with an exacting cardinal above an extendible cardinal.

Ultraexacting cardinals

We now consider the possibility of further strengthening the notion of exacting cardinals.

Our motivation for the formulation of stronger notions comes from the observation that certain elements of $\mathrm{H}(\lambda^+)$ have to be missing from the domains of embeddings witnessing the exactingness of a cardinal λ.

The proof of the following result uses ideas from Woodin's proof of the Kunen Inconsistency:

Proposition

If λ is a cardinal, $\zeta > \lambda$ is an ordinal with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$, then $\lambda^+ \nsubseteq X$ and $[\lambda]^\omega \nsubseteq X$.

The above proposition shows that we can strengthen the notion of exacting cardinals by demanding that certain sets are contained in the domains of the elementary embeddings witnessing the given property.

The consistency proof for exacting cardinals from I2-embeddings shows that initial segments of the given elementary embeddings are canonical examples of sets that are, in general, not contained in their domains.

This motivates the following definition:

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

As before, this notion is equivalent to a natural strengthening of a rank-into-rank axioms:

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ:

- λ is an ultraexacting cardinal.
- For every ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exists an elementary embedding

$$j : (\mathrm{V}_{\lambda+1}, \in, A) \longrightarrow (\mathrm{V}_{\lambda+1}, \in, A).$$

This equivalence can be used to show that ultraexactingness is equivalent to an assumption considered by Woodin in his *Suitable Extender Models I.* paper as a candidate for an axiom that, together with the existence of an extendible cardinal, causes HOD to be far from V.

Moreover, we can determine the exact consistency strength of ultraexactingness:

Definition

An I0-embedding is a non-trivial elementary embedding

$$j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1}),$$

where λ is the first non-trivial fixed point of j.

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following statements are equiconsistent over ZFC:

- There is an ultraexacting cardinal.
- There is an I0-embedding.

Ultraexacting cardinals and ordinal definability

We now discuss results showing that analogs of results about ordinal definable sets in determinacy models can be proven for ordinal definable sets in the presence of ultraexacting cardinals.

We start with analog of the classical result that the ω_1 of a determinacy model is a measurable cardinal in its HOD.

Definition (Woodin)

An uncountable regular cardinal κ is *ω-strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\mathrm{HOD}} < \kappa$ and HOD contains no partition of the set E^κ_ω of all elements of κ with cofinality ω into δ-many sets that are all stationary in V.

Lemma (Woodin)

If a cardinal κ is ω-strongly measurable in HOD, then κ is a measurable cardinal in HOD.

Theorem (Aguilera–Bagaria–L.)

If λ is an ultraexacting cardinal, then λ^+ is ω-strongly measurable in HOD.

The proof of this result relies on arguments showing that, given an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ witnessing the ultraexactness of a cardinal $\lambda < \zeta$, for certain ordinals

$$\lambda \ < \ \alpha \ \in \ X \cap \zeta,$$

we can find functions

$$f : \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}_{j(\alpha)}(\mathrm{V}_{\lambda+1})$$

in X with the property that

$$f \restriction (X \cap \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1})) \ = \ j \restriction (X \cap \mathrm{L}_\alpha(\mathrm{V}_{\lambda+1})).$$

In the following, fix

- a cardinal λ,
- an ordinal $\zeta > \lambda$ such that V_ζ is sufficiently elementary in V,
- an elementary submodel X of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \upharpoonright \lambda \neq \mathrm{id}_\lambda$ and $j \upharpoonright \mathrm{V}_\lambda \in X$.

Extensions to $\mathrm{V}_{\lambda+1}$

Remember that, if ξ is an ordinal of countable cofinality and $k : \mathrm{V}_\xi \longrightarrow \mathrm{V}_\xi$ is an elementary embedding, then the map

$$k_+ : \mathrm{V}_{\xi+1} \longrightarrow \mathrm{V}_{\xi+1};\ A \longmapsto \bigcup\{k(A \cap V_\alpha) \mid \alpha < \xi\}$$

is the unique Σ_0-elementary function from $\mathrm{V}_{\xi+1}$ to $\mathrm{V}_{\xi+1}$ extending k.

If $i : \mathrm{V} \longrightarrow M$ is an I2-embedding with least non-trivial fixed point λ, then

$$(i \restriction \mathrm{V}_\lambda)_+ = i \restriction \mathrm{V}_{\lambda+1} : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is Σ_1-elementary. Conversely, every non-trivial Σ_1-elementary function from $\mathrm{V}_{\lambda+1}$ to itself can be extended to an I2-embedding.

Lemma

The map

$$(j \restriction \mathrm{V}_\lambda)_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is an elementary embedding that is contained in X and satisfies

$$(j \restriction \mathrm{V}_\lambda)_+ \restriction (X \cap \mathrm{V}_{\lambda+1}) \ = \ j \restriction (X \cap \mathrm{V}_{\lambda+1}).$$

Proof.

Since $j \restriction \mathrm{V}_\lambda \in X$, elementarity implies that $(j \restriction \mathrm{V}_\lambda)_+ \in X$.

Moreover, the equality

$$(j \restriction \mathrm{V}_\lambda)_+ \restriction (X \cap \mathrm{V}_{\lambda+1}) \ = \ j \restriction (X \cap \mathrm{V}_{\lambda+1}).$$

holds by the definition of $(j \restriction \mathrm{V}_\lambda)_+$.

The elementarity of j and the above equality then imply that $(j \restriction \mathrm{V}_\lambda)_+$ is an elementary embedding of $\mathrm{V}_{\lambda+1}$ into itself in X. Finally, the correctness properties of X ensure that $(j \restriction \mathrm{V}_\lambda)_+$ also has this property in V. □

Extensions below $\Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$

Lemma

Let γ be an ordinal in X such that there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$. Then there exists a unique function $j_\gamma : \gamma \longrightarrow j(\gamma)$ that is an element of X and satisfies

$$j \restriction (X \cap \gamma) \;=\; j_\gamma \restriction (X \cap \gamma).$$

Proof.

Define $j_\gamma : \gamma \longrightarrow j(\gamma)$ to be the unique function satisfying

$$j_\gamma(s(x)) \;=\; j(s)((j \restriction \mathrm{V}_\lambda)_+(x))$$

for all $x \in \mathrm{V}_{\lambda+1}$. Then j_γ possesses all of the listed properties. □

Remember that, given $E \subseteq \mathrm{V}_{\lambda+1}$, we let $\Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ denote the least ordinal γ such that $\mathrm{L}(\mathrm{V}_{\lambda+1},E)$ does not contain a surjection from $\mathrm{V}_{\lambda+1}$ onto γ.

The basic structure theory of $\mathrm{L}(\mathrm{V}_{\lambda+1},E)$ now yields the following statement:

Lemma

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ with $j(\gamma) \in X$, then there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$.

Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1},E)}$ with $j(\gamma) = \gamma$, then there is a function $j_\gamma : \gamma \longrightarrow \gamma$ in X with

$$j \restriction (X \cap \gamma) \ = \ j_\gamma \restriction (X \cap \gamma).$$

We now use the above results to prove that successors of ultraexacting cardinals are measurable in HOD.

Definition (Woodin)

An uncountable regular cardinal κ is *ω-strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\mathrm{HOD}} < \kappa$ and HOD contains no partition of the set E^κ_ω of all elements of κ with cofinality ω into δ-many sets that are all stationary in V.

Theorem (Aguilera–Bagaria–L.)

If λ is an ultraexacting cardinal, then λ^+ is ω-strongly measurable in HOD.

Let λ be an ultraexacting cardinal. Fix an ordinal $\zeta > \lambda$ with $\mathrm{V}_\zeta \prec_{\Sigma_3} \mathrm{V}$, an elementary submodel X of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{id}_\lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Assume, towards a contradiction, that there is a partition of $E^{\lambda^+}_\omega$ into $\mathrm{crit}(j)$-many stationary sets that is an element of HOD. Let

$$\vec{S} \ = \ \langle S_\alpha \mid \alpha < \mathrm{crit}(j) \rangle$$

denote the least such partition in the canonical well-ordering of HOD. Set

$$j(\vec{S}) \ = \ \langle S'_\beta \mid \beta < j(\mathrm{crit}(j)) \rangle.$$

Then $j(\vec{S})$ is the minimal partition of $E^{\lambda^+}_\omega$ into $j(\mathrm{crit}(j))$-many stationary sets in HOD. In particular, the sequence $j(\vec{S})$ is an element of X.

By an earlier lemma, there is a function $f : \lambda^+ \longrightarrow \lambda^+$ in X with

$$f \restriction (X \cap \lambda^+) \;=\; j \restriction (X \cap \lambda^+).$$

Elementarity then implies that f is strictly increasing and continuous at limit ordinals of countable cofinality.

This implies that

$$C \;=\; \{\gamma < \lambda^+ \mid f(\gamma) = \gamma\}$$

is an ω-club in λ^+ that is an element of X and we can find

$$\gamma \;\in\; C \;\cap\; S'_{\mathrm{crit}(j)} \;\cap\; X.$$

Then there exists $\alpha < \mathrm{crit}(j)$ with $\gamma \in S_\alpha$ and the fact that $j(\gamma) = f(\gamma) = \gamma$ implies that

$$\gamma \;\in\; S'_\alpha \;\cap\; S'_{\mathrm{crit}(j)} \;=\; \emptyset,$$

a contradiction. $\square$

Using the above techniques, it is possible to derive other properties of HOD in the presence of ultraexacting cardinals that resemble results from the determinacy setting. For example:

Definition

Given an inner model N and ordinals $\lambda < \vartheta$, we say that the ordinal λ is *N-ϑ-Berkeley* if for every $\alpha < \lambda$ and every transitive set M in N that contains λ as an element and has cardinality less than ϑ in N, there exists a non-trivial elementary embedding $j : M \longrightarrow M$ with $\alpha < \mathrm{crit}(j) < \lambda$.

Theorem (Aguilera–Bagaria–L.)

Every ultraexacting cardinal λ is HOD-$\Theta^{L(\mathrm{V}_{\lambda+1})}$-Berkeley.

Thank you for listening!