Topics in Set Theory
Lent Term 2019
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #2

Course webpage: `https://www.math.uni-hamburg.de/home/loewe/Lent2019/TST_L19.html`

Example Classes.
#1. Monday 4 February, 3:30–5pm, MR5.
#2. Monday 18 February, 3:30–5pm, MR5.
#3. Monday 4 March, 3:30–5pm, MR5.
#4. TBD; probably Thursday 14 March.
You hand in your work at the beginning of the Example Class.

(13) As in the lectures, we let

$$\mathcal{D}(A) := \{X \subseteq A\,;\, \exists n \exists \vec{s} \in A^n \exists R \in \mathrm{Def}(A, n+1)(\forall x(x \in X \leftrightarrow (x, \vec{s}) \in R))\}$$

and $\mathbf{L}_0 := \varnothing$, $\mathbf{L}_{\alpha+1} := \mathcal{D}(\mathbf{L}_\alpha)$, and $\mathbf{L}_\lambda := \bigcup_{\alpha\lambda} \mathbf{L}_\alpha$. We have already seen that $\mathbf{L}_\alpha$ is transitive for all ordinals α. Show that

(a) if $\alpha \leq \beta$, then $\mathbf{L}_\alpha \subseteq \mathbf{L}_\beta$ and

(b) for all ordinals α, $\mathrm{Ord} \cap \mathbf{L}_\alpha = \alpha$.

(14) Let κ be an inaccessible cardinal. Show that

(a) $(\mathbf{L}_\kappa, \in) \models \mathsf{Union}$;

(b) $(\mathbf{L}_\kappa, \in) \models \mathsf{Replacement}$.

(15) Show the *Lévy Reflection Theorem* in ZF (i.e., without assuming the existence of an inaccessible cardinal): if Φ is a finite set of formulas and α is any ordinal, then there is some $\beta > \alpha$ such that all formulas in Φ are absolute for $\mathbf{V}_\beta$.

(*Hint.* Assume, without loss of generality that Φ is closed under subformulas and show a Tarski-Vaught style criterion for absoluteness by induction on the formula complexity.)

(16) What properties of $\mathbf{V}_\beta$ did you use in (15)? Can you generalise the Reflection Theorem to obtain absoluteness for other hierarchies than the von Neumann hierarchy?

(17) We define by transfinite recursion: $\beth_0 := \aleph_0$, $\beth_{\alpha+1} := 2^{\beth_\alpha}$, and $\beth_\lambda := \bigcup_{\alpha<\lambda} \beth_\alpha$. A cardinal is called a *beth fixed point* if $\kappa = \beth_\kappa$. Show that if κ is regular, then κ is a beth fixed point if and only if κ is inaccessible.

(18) Show in ZFC that for $\alpha > \omega$, $|\mathbf{V}_\alpha| = |\mathbf{L}_\alpha|$ if and only if α is a beth fixed point.

(19) Assume that $\mathsf{ZFC} + \mathsf{IC}$ is consistent and show that the following theory is consistent: $\mathsf{ZFC}+$ "there are ordinals $\alpha < \beta < \aleph_1$ such that $\mathbf{L}_\alpha \models \mathsf{ZFC}$, $\mathbf{L}_\beta \models \mathsf{ZFC}$ and $\mathbf{L}_\beta \models$ 'α is countable' ".

(20) Let x be any transitive set and define by transfinite recursion:

$$\begin{aligned}\mathbf{L}_0(x) &:= x,\\ \mathbf{L}_{\alpha+1}(x) &:= \mathcal{D}(\mathbf{L}_\alpha(x)),\\ \mathbf{L}_\lambda(x) &:= \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha(x) \text{ (for } \lambda \text{ limit).}\end{aligned}$$

As usual, we define $\mathbf{L}(x) := \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha(x)$ and write $\mathbf{V}{=}\mathbf{L}(x)$ for the formula $\forall x \exists \alpha (x \in \mathbf{L}_\alpha(x))$ (note that this is a formula with the parameter x). Show that

(a) for each α, $\mathbf{L}_\alpha(x)$ is transitive,

(b) if x is countable and $\alpha \geq \omega$, then $|\mathbf{L}_\alpha(x)| = |\alpha|$,

(21) Assume that κ is an inaccessible cardinal and $x \subseteq \mathbb{N}$; prove the *Condensation Lemma for* $\mathbf{L}(x)$:

Suppose $\mathbf{V}{=}\mathbf{L}(x)$ and that $A \subseteq \mathbb{N}$. Then there is $\lambda < \omega_1$ such that $A \in \mathbf{L}_\lambda(x)$.

Conclude that $\mathbf{L}(x)$ satisfies CH. If $x \subseteq \kappa$ for some uncountable cardinal κ, what bound does the proof of the condensation lemma give and what can you say about the size of $2^{\aleph_0}$ in $\mathbf{L}(x)$?

(22) Assume that for all $x \subseteq \mathbb{N}$, we have that $\aleph_1^{\mathbf{L}(x)}$ is countable. Suppose that κ is inaccessible and show that $\mathbf{L}_\kappa \models$ "$\aleph_1^{\mathbf{V}}$ is inaccessible" where $\aleph_1^{\mathbf{V}}$ refers to the first uncountable cardinal in the universe.