Homework Sheet #5

Capita Selecta: Set Theory
2020/21: 1st Semester; block 1
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Homework. Homework is due on Fridays. Please submit your work as a single pdf file via Canvas. You will receive one point for each question that you attempt, independent of your performance on the question. The homework will not be formally marked, but the lecturers may give individual feedback in the case of problems.

Deadline. This homework set is due on **Friday, 9 October 2020, 1pm**.

15. Consider the following game on ω: if $z \in \omega^\omega$ is a play, we write $x := z_{\mathrm{I}}$ for the moves of player I and $y := z_{\mathrm{II}}$ for the moves of player II. If $x \notin \mathrm{WO}$, then player I loses. If $x \in \mathrm{WO}$, but $y \notin \mathrm{WO}$, then player II loses. If both are in WO, then player I wins if $\|x\| \geq \|y\|$. Show that player I cannot have a winning strategy in this game.

    [*Hint.* Use the Boundedness Lemma.]

16. Let $A \subseteq \omega^\omega \times \omega^\omega$. Define the *unfolded $*$-game*, denoted by $\mathrm{G}^*_{\mathrm{u}}(A)$ as follows: in each round of the game, player I plays triples $(y(n), s_{n,0}, s_{n,1})$ where $s_{n,0}$ and $s_{n,1}$ are as in the $*$-game and $y(n)$ is a natural number; player II plays an element i_n of $\{0,1\}$ as in the $*$-game.

    Let M be the move set for this game, i.e., the disjoint union of the two move sets of the two players. We write a play of this game as $z := \{(y(n), s_{n,0}, s_{n,1}, i_n)\,;\, n \in \mathbb{N}\}$. Suppose we are in round $n+1$ of the game in position $p = \{(y(k), s_{k,0}, s_{k,1}, i_k)\,;\, k \leq n\}$. A move (m, s, t) by player I is *legal in p* if

    (i) $\mathrm{lh}(s) \geq n$,
    (ii) $\mathrm{lh}(t) \geq n$,
    (iii) both s and t properly extend s_{n,i_n}, and
    (iv) s and t are incompatible

    Let T be the tree of all positions where all moves of player I are legal and fix a play z as above. If player I ever plays an illegal move in z, they lose. Otherwise, the play z is in $[T]$, and then (as in the $*$-game) $\bigcap_{n\in\mathbb{N}}[s_{n,i_n}]$ has a unique element, say, x. In that case, player I wins the game $\mathrm{G}^*_{\mathrm{u}}(A)$ if $(x, y) \in A$.

    (a) Define a map $g : [T] \to \omega^\omega \times \omega^\omega$ which sends a play of the game
    $$\{(y(n), s_{n,0}, s_{n,1}, i_n)\,;\, n \in \mathbb{N}\}$$
    to the pair $(y, x) \in \omega^\omega \times \omega^\omega$ where x is such that $\{x\} := \bigcap_{n\in\omega}[s_{n,i_n}]$.
    (b) Conclude that if A is closed, then $G^*_{\mathrm{u}}(A)$ is determined.
    (c) Show that if I has a winning strategy in $\mathrm{G}^*_{\mathrm{u}}(A)$, then the projection of A contains a closed copy of Cantor space.
    (d) Show that if II has a winning strategy in $\mathrm{G}^*_{\mathrm{u}}(A)$, then the projection of A is countable.
    (e) Conclude that $\mathrm{PSP}(\boldsymbol{\Sigma}^1_1)$ holds.

17. Read Theorem 22.2 in Andretta's draft book and write a short summary of the proof (maximum 250 words).