Homework Sheet #2

Capita Selecta: Set Theory
2020/21: 1st Semester; block 1
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Homework. Homework is due on Fridays. Please submit your work as a single pdf file via Canvas. You will receive one point for each question that you attempt, independent of your performance on the question. The homework will not be formally marked, but the lecturers may give individual feedback in the case of problems.

Deadline. This homework set is due on **Friday, 18 September 2020, 1pm**.

Let X be a non-empty set and $a \in X$. We shall define a game on X where player I produces a finite sequence of elements of X without repetitions and then *challenges* player II to produce an element of X different from all those already produced. The element a serves the purpose of being a *marker* to indicate that the challenge is now complete. We now give formal definitions. A position $p \in X^{<\omega}$ is called a *challenge* if

(i) $\mathrm{lh}(p) = 2n+1$ for some n, i.e., $p = (p(0), p(1), ..., p(2n))$,

(ii) for each $k < n$, we have that $s(2k) \neq a$,

(iii) the sequence $(s(2k)\,;\, k \leq n)$ has no repetitions, and

(iv) $s(2n) = a$.

The finite sequence produced by player I in the challenge is the sequence $(p(2k)\,;\, k < n)$. If p is a challenge of length $2n+1$, this sequence consists of n distinct elements. For any position p, we write $S_p := \{p(2k)\,;\, 2k < \mathrm{lh}(p)\}$ for the set of elements played by player I.

We call p a *successful challenge* if $\mathrm{lh}(p) = 2n+2$, the predecessor position $p{\restriction}2n+1$ is a challenge, and $p(2n+1) \in S_p$. We call p a *failed challenge* if $\mathrm{lh}(p) = 2n+2$, the predecessor position $p{\restriction}2n+1$ is a challenge, and $p(2n+1) \notin S_p$. We call p a *mischallenge* if there is some $2m < \mathrm{lh}(p)$ with $p(2m) = a$ and for the least such m, there are $k < \ell < m$ with $p(2k) = p(2\ell)$. Let

$$C := \{z \in X^\omega\,;\ \text{there is an } n \text{ such that } z{\restriction}n \text{ is a successful challenge}\}.$$

4. We call a set $A \subseteq X^\omega$ *clopen* if both it and its complement are closed. Remember that a set is *closed* if it is the set of branches through a tree $T \subseteq X^{<\omega}$.

   Show that the complement of C is closed. Show that if X is Dedekind-finite, then C is closed as well.

   [*Hint.* For the complement of C, consider the tree consisting of all sequences that do not contain a in even digits at all and all sequences that are compatible with a failed challenge or a mischallenge. For C, consider the tree consisting of all sequences that are compatible with a successful challenge. Give precise definitions and prove all required inclusions.]

5. Show that X is finite if and only if player I has a winning strategy in $\mathrm{G}_X(C)$ and that X is Dedekind-infinite if and only if player II has a winning strategy in $\mathrm{G}_X(C)$. Conclude that if X is infinite and Dedekind-finite, then AD_X is inconsistent.

6. Suppose that X is Dedekind-infinite. Show that "all clopen subsets of X^ω are determined" implies $\mathsf{AC}_{X^{<\omega}}(X)$.

7. Combine 5. & 6. to prove in ZF that for arbitrary sets X the following are equivalent:

   (a) "all clopen subsets of X^ω are determined" and

   (b) $\mathsf{AC}_{X^{<\omega}}(X)$.

   [Hint. Consider the three cases "X Dedekind-infinite", "X finite", and "X infinite Dedekind-finite" separately.]