LECTURE 6 CS:ST 2020

Borel Hierarchy (B.H.)

DEF LET $X \neq \emptyset$ THEN A σ-ALGEBRA S OVER X IS A SUBSET OF $P(X)$ WHICH IS CLOSED UNDER COMPLEMENTS AND COUNTABLE UNIONS.

NOTE: $\emptyset, X \in S$ AND S IS CLOSED UNDER COUNTABLE INTERSECTIONS.

DEF Let X be a topological space. Then we say that $A \subseteq X$ is **BOREL** iff it is contained in the smallest σ-alg. containing the open sets. We will denote the collection of **Borel sets** as $Bor(X)$.

DEF Let X be a top. space. Define:

$$\underset{\sim}{\Sigma}^0_1 \upharpoonright X = \{ A \subseteq X \mid A \text{ is open} \}$$

Boldface.

$$\underset{\sim}{\Pi}^0_1 \upharpoonright X = \{ A \subseteq X \mid A \text{ is closed} \}$$

For $\alpha > 1$:

$$\underset{\sim}{\Sigma}^0_\alpha \upharpoonright X = \{ \bigcup_{n \in \omega} A_n \mid \exists f \in \alpha^\omega \ \forall n \ A_n \in \underset{\sim}{\Pi}^0_{f(n)} \upharpoonright X \}$$

$$\underset{\sim}{\Pi}^0_\alpha \upharpoonright X = \{ A \mid X \setminus A \in \underset{\sim}{\Sigma}^0_\alpha \upharpoonright X \}$$

For $\beta \geq 1$:

$$\underset{\sim}{\Delta}^0_\beta \upharpoonright X = \underset{\sim}{\Sigma}^0_\beta \upharpoonright X \cap \underset{\sim}{\Pi}^0_\beta \upharpoonright X.$$

$$\underset{\sim}{\Delta}^0_1 = \underset{\sim}{\Sigma}^0_1 \cap \underset{\sim}{\Pi}^0_1$$

IF $X = \omega^\omega$ THEN WE DROP $\upharpoonright X$

$$\underset{\sim}{\Sigma}^0_1 \text{ FOR } \underset{\sim}{\Sigma}^0_1 \upharpoonright \omega^\omega$$

$$\underset{\sim}{\Delta}^0_1 \upharpoonright X = \text{CLOPEN SUBSETS.}$$

$$\underset{\sim}{\Sigma}^0_2 \upharpoonright X = F_\sigma$$ COUNTABLE UNIONS OF CLOSED SETS

$$\underset{\sim}{\Pi}^0_2 \upharpoonright X = G_\delta$$ COUNTABLE INTERSECTIONS OF OPEN SETS.

<u>LEMMA 1</u> FOR EVERY TOPOLOGICAL SPACE X AND $0 < \alpha$ THEN:

1. $\underset{\sim}{\Pi}^0_\alpha \upharpoonright X \subseteq \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X$
2. $\underset{\sim}{\Sigma}^0_\alpha \upharpoonright X \subseteq \underset{\sim}{\Pi}^0_{\alpha+1} \upharpoonright X.$

PROOF 1. $A \in \Pi^0_\alpha \upharpoonright X$ LET $f \in (\alpha+1)^\omega$ TO BE THE CONSTANT FUNCTION WITH VALUE α. AND LET $A_n = A$ $\forall n$.

$$A = \bigcup_{n \in \omega} A_n \qquad \forall n \quad A_n \in \underset{\sim}{\Pi}^0_\alpha \upharpoonright X \text{ SO}$$

$$\forall n \quad A_n \in \underset{\sim}{\Pi}^0_{f(n)} \upharpoonright X. \quad \text{SO} \quad A \in \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X$$

∎

$$
\begin{array}{ccccc}
 & & \boldsymbol{\Sigma}^0_1 & \subseteq & \boldsymbol{\Pi}^0_2 \cdots \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 \cdots \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \boldsymbol{\Pi}^0_1 & \subseteq & \boldsymbol{\Sigma}^0_2 \cdots
\end{array}
$$

LEMMA 2 (Ex 3.7) LET X BE METRIZABLE THEN:

1. $\forall \alpha > 0 \quad \boldsymbol{\Sigma}^0_\alpha \upharpoonright X \subseteq \boldsymbol{\Sigma}^0_{\alpha+1} \upharpoonright X$
2. $\forall \alpha > 0 \quad \boldsymbol{\Pi}^0_\alpha \upharpoonright X \subseteq \boldsymbol{\Pi}^0_{\alpha+1} \upharpoonright X$
3. $\forall \beta > \alpha > 0 \quad \boldsymbol{\Sigma}^0_\alpha \upharpoonright X \cup \boldsymbol{\Pi}^0_\alpha \upharpoonright X \subseteq \boldsymbol{\Delta}^0_\beta \upharpoonright X$

$$
\begin{array}{ccccc}
 & & \boldsymbol{\Sigma}^0_1 & \subseteq & \boldsymbol{\Sigma}^0_2 \cdots \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 \cdots \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \boldsymbol{\Pi}^0_1 & \subseteq & \boldsymbol{\Pi}^0_2 \cdots
\end{array}
$$

Q_1: IS THIS A STRATIFICATION OF $Bor(X)$?

LEMMA (EX 3.6) FOR EVERY TOPOLOGICAL SPACE X:

$$\bigcup_{\alpha\in ORD} \underset{\sim}{\Pi}^0_{\alpha+1}\upharpoonright X \boxed{\overset{a}{=}} \bigcup_{\alpha\in ORD} \underset{\sim}{\Delta}^0_{\alpha+1}\upharpoonright X \boxed{\overset{b}{=}} \bigcup_{\alpha\in ORD}\underset{\sim}{\Sigma}^0_{\alpha+1}\upharpoonright X = Bor(X).$$

MOREOVER IF ω_1 IS REGULAR THEN.

$$\bigcup_{\alpha\in \omega_1} \underset{\sim}{\Pi}^0_{\alpha+1}\upharpoonright X = \bigcup_{\alpha\in \omega_1} \underset{\sim}{\Delta}^0_{\alpha+1}\upharpoonright X = \bigcup_{\alpha\in \omega_1}\underset{\sim}{\Sigma}^0_{\alpha+1}\upharpoonright X = Bor(X).$$

PROOF BY LEMMA 1 WE HAVE a AND b.

WE WILL PROVE

$$\bigcup_{\alpha\in\omega_1}\underset{\sim}{\Sigma}^0_{\alpha+1}\upharpoonright X = Bor(X)$$

$\subseteq$: WE PROVE $\forall\alpha$ $\underset{\sim}{\Sigma}^0_{\alpha+1}\upharpoonright X \subseteq Bor(X)$ AND
$\underset{\sim}{\Pi}^0_{\alpha+1}\upharpoonright X \subseteq Bor(X)$

$\alpha=0$ $\underset{\sim}{\Sigma}^0_1\upharpoonright X$ ARE THE OPEN SETS AND THEY ARE BOREL.

$\underset{\sim}{\Pi}^0_1\upharpoonright X$ ARE THE CLOSED SETS AND THEY ARE BOREL.

$\underline{\alpha > 0}$ IF $A \in \underset{\sim}{\Sigma}^0_\alpha \upharpoonright X$ THEN $A = \bigcup_{n \in \omega} A_n$
$\exists f \in \alpha^\omega$ $A_n \in \underset{\sim}{\Pi}^0_{f(n)} \upharpoonright X$ $\forall n$.

BY INDUCTIVE HP $\forall n$ $A_n \in Bor(X)$
$Bor(X)$ IS CLOSED UNDER COUNTABLE UNIONS
SO $A = \bigcup_{n \in \omega} A_n \in Bor(X)$.

SIMILARLY FOR $\underset{\sim}{\Pi}^0_{\alpha+1} \upharpoonright X$.

$\underline{2}$: WE PROVE THAT $\bigcup_{\alpha \in \omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X$ IS A σ-ALGEBRA CONTAINING THE OPEN SETS.

- $\underset{\sim}{\Sigma}^0_1 \upharpoonright X \subseteq \bigcup_{\alpha \in \omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X$ SO THE OPEN SETS ARE IN $\bigcup_{\alpha \in \omega_1} \Sigma^0_{\alpha+1} \upharpoonright X$

- LET $\langle A_n \mid n \in \omega \rangle$ OF SETS SUCH THAT $\forall n$ $A_n \in \bigcup_{\alpha \in \omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1}$.

FOR EACH $n \in \omega$ LET α_n TO BE THE SMALLEST s.t. $A_n \in \underset{\sim}{\Sigma}^0_{\alpha_n} \upharpoonright X$.
LET $\bar{\alpha} = \sup\{\alpha_n + 1 \mid n \in \omega\}$ SINCE $cof(\omega_1) = \omega_1$
$\bar{\alpha} < \omega_1$. THEN $\bigcup_{n \in \omega} A_n \in \underset{\sim}{\Sigma}^0_{\bar{\alpha}} \upharpoonright X$

AND THEREFORE $\bigcup_{n\in\omega} A_n \in \bigcup_{\alpha\in\omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1}$.

SO $\bigcup_{\alpha\in\omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X$ IS A σ-ALG. CONTAINING THE OPEN SETS AND BY MINIMALITY OF $Bor(X)$

$$Bor(X) \subseteq \bigcup_{\alpha\in\omega_1} \underset{\sim}{\Sigma}^0_{\alpha+1} \upharpoonright X. \qquad \blacksquare$$

WHAT DO WE KNOW?

① THE B.H. IS A STRATIFICATION OF $Bor(X)$
② IF $AC_\omega(\mathbb{R})$ THEN THE LENGHT OF B.H. IS AT MOST ω_1.

Q_2: CAN B.H. BE LONGER? YES!

LEMMA (MILLER) IF ZF IS CONSISTENT SO IS ZF + "THE B.H. ON 2^ω HAS SIZE ω_2"

UNDER LARGE CARDINAL ASSUMPTIONS ONE CAN PROVE (MILLER) THAT THE B.H. CAN BE ARBITRARILY LONG IN MODELS OF ZF.

Q_3: CAN B.H. COLLAPSE?

<u>CLAIM</u> (Miller) in ZF:

$$\underset{\sim}{\Sigma}^0_3 \neq \underset{\sim}{\Pi}^0_3 .$$

IF ZF IS CONSISTENT THEN SO IS
ZF + " B.H. has length 4 "

UNDER $AC_\omega(\mathbb{R})$ THE B.H. DOES NOT COLLAPSE!

WE will FOCUS on ω^ω!

<u>DEF</u> A POINTCLASS Γ IS A SUBSET OF $P(\omega^\omega)$ which contains $\emptyset, \omega^\omega$.

WE SAY THAT $\check{\Gamma}$ IS THE DUAL OF Γ
IFF

$$\check{\Gamma} = \{ A \mid \omega^\omega \setminus A \in \Gamma \}$$

AND A CLASS IS SELF-DUAL IF $\check{\Gamma} = \Gamma$.

FOR $\alpha > 0$ $\underset{\sim}{\Sigma}^0_\alpha$, $\underset{\sim}{\Pi}^0_\alpha$, $\underset{\sim}{\Delta}^0_\alpha$ ARE
POINTCLASSES. $\check{\underset{\sim}{\Sigma}}^0_\alpha = \underset{\sim}{\Pi}^0_\alpha$ AND $\check{\underset{\sim}{\Pi}}^0_\alpha = \underset{\sim}{\Sigma}^0_\alpha$

$\underset{\sim}{\Delta}^0_\alpha$ IS SELF-DUAL.

DEF A POINTCLASS $\underset{\sim}{\Gamma}$ IS BOLDFACE IF $\underset{\sim}{\Gamma}$ IS CLOSED UNDER CONTINUOUS PREIMAGES, i.e., $\forall f$ CONTINUOUS

$$\forall A \quad A \in \underset{\sim}{\Gamma} \Rightarrow f^{-1}[A] \in \underset{\sim}{\Gamma}.$$

LEMMA $\forall \alpha > 0$ $\underset{\sim}{\Sigma}^0_\alpha$, $\underset{\sim}{\Pi}^0_\alpha$, AND $\underset{\sim}{\Delta}^0_\alpha$ ARE BOLDFACE.

FIX FOR $0 < n \leq \omega$ AN HOMEOMORPHISM

$$\mathbb{h}^{n,\omega} : (\omega^\omega)^n \longrightarrow \omega^\omega$$

$$(-)_n : \omega^\omega \longrightarrow \omega^\omega$$

$$x = a_0 b_0 a_1 b_1 \ldots$$
$$(x)_0 = a_0 a_1 a_2 \ldots$$

$(x)_1 = b_0 b_1 \ldots$

DEF LET $0 < n \leq \omega$ AND Γ BE A POINTCLASS DEFINE

$$\Gamma \upharpoonright (\omega^\omega)^n = \{ A \mid \mathbb{h}^{n,1}[A] \in \Gamma \}$$

DEF A SET $\mathcal{U} \subseteq \omega^\omega \times \omega^\omega$ IS UNIVERSAL FOR A POINTCLASS Γ IFF

$$\forall A \subseteq \omega^\omega \left(A \in \Gamma \iff \exists a \in \omega^\omega \ A = \mathcal{U}_{(a)} \right)$$

$$\{ b \mid \langle a, b \rangle \in \mathcal{U} \}$$

MOREOVER $\mathcal{U}$ IS Γ-UNIVERSAL IFF $\mathcal{U} \in \Gamma$ AND $\mathcal{U}$ IS UNIVERSAL FOR Γ.

$$\mathcal{U} \in \underline{\Gamma \upharpoonright (\omega^\omega)^2}$$

PROP IF $\underset{\sim}{\Gamma}$ IS BOLDFACE AND SELF-DUAL THEN THERE IS NO $\underset{\sim}{\Gamma}$-UNIVERSAL SET.

PROOF SUPPOSE $\mathcal{U}$ IS $\underset{\sim}{\Gamma}$-UNIV.

CONSIDER THE SET:

$$A = \{ x \mid \underline{\langle x,x\rangle \notin \mathcal{U}} \}$$

NOTE THAT $f(x) = \langle x,x\rangle$ IS CONTINUOUS.
[CHECK!]

MOREOVER $f^{-1}[\mathcal{U}] = \omega^\omega \setminus A$

$\mathcal{U} \in \underset{\sim}{\Gamma}$ AND $\underset{\sim}{\Gamma}$ IS BOLDFACE SO $f^{-1}[\mathcal{U}] \in \underset{\sim}{\Gamma}$ BUT SINCE $\underset{\sim}{\Gamma}$ IS

$\omega^\omega \setminus A$ "

SELF-DUAL $A \in \underset{\sim}{\Gamma}$. LET $a \in \omega^\omega$ BE SUCH THAT $A = \mathcal{U}_{(a)}$.

$$a \in A \iff \langle a,a\rangle \notin \mathcal{U} \iff a \notin \mathcal{U}_{(a)} = A$$

$\lightning$ ■

COR $\forall \alpha > 0$ $\underset{\sim}{\Delta}^0_\alpha$ has no $\underset{\sim}{\Delta}^0_\alpha$-UNIVERSAL SET.

WE WANT TO SHOW THAT $\forall 0 < \alpha < \omega_1$

$$\underset{\sim}{\Sigma}^0_\alpha \neq \underset{\sim}{\Pi}^0_\alpha$$

SO WE will ACTUALLY PROVE THAT $\forall 0 < \alpha < \omega_1$

THERE IS $\underset{\sim}{\Sigma}^0_\alpha$-UNIVERSAL SET.

This is PROVED BY INDUCTION ON α.

PROP 3 THERE IS A $\underset{\sim}{\Sigma}^0_1$-UNIVERSAL SET AND A $\underset{\sim}{\Pi}^0_1$-UNIVERSAL SET.

Proof FIX AN ENUMERATION f OF $\omega^{<\omega}$
DEFINE $\mathcal{U}$:

$$\mathcal{U} = \{ \langle x, y \rangle \mid y \in \bigcup_{n \in \omega} N_{f((x(n))} \}$$

$\blacksquare$

DEF GIVEN Γ BE A POINTCLASS AND α BE AN ORDINAL DEF

$$\bigcap(\alpha;\Gamma) = \left\{ A \;\middle|\; \exists f \in \Gamma^{\alpha}\ A = \bigcap_{\beta\in\alpha} f(\beta) \right\}$$

$$\bigcup(\alpha;\Gamma) = \left\{ A \;\middle|\; \exists f \in \Gamma^{\alpha}\ A = \bigcup_{\beta\in\alpha} f(\beta) \right\}$$

LEMMA (3.18) ASSUME $AC_{\omega}(\mathbb{R})$ AND LET $\underset{\sim}{\Gamma}$ BE B.F.. IF $\mathcal{U}$ IS $\underset{\sim}{\Gamma}$-UNI.

THEN

$$V = \{ \langle x,y\rangle \mid \forall n\, ((x)_n, y) \in \mathcal{U} \}$$

$$W = \{ \langle x,y\rangle \mid \exists n\, ((x)_n, y) \in \mathcal{U} \}$$

ARE $\bigcap(\omega;\underset{\sim}{\Gamma})$-UNIVERSAL AND $\bigcup(\omega;\underset{\sim}{\Gamma})$-UNIVERSAL, RESPECTIVELY.

PROP ASSUME $\Delta C_\omega(\mathbb{R})$ AND LET $0 < \alpha < \omega_1$ THEN

NOT USED SO FAR!

$$\underset{\sim}{\Sigma}^0_\alpha \neq \underset{\sim}{\Pi}^0_\alpha$$

Proof INDUCTION ON α. TO show

$\underset{\sim}{\Sigma}^0_\alpha$ - UNIVERSAL SETS AND
$\underset{\sim}{\Pi}^0_\alpha$ - UNIVERSAL SETS EXIST.

BASE CASE PROP. 3.

STEP $\alpha+1$ LEMMA 3.18.

λ LIMIT? ← WE NEED $\lambda < \omega_1$

FIX A COUNTABLE SUBSEQUENCE $(y_n \mid n \in \omega)$ COFINAL IN λ.

USING $\Delta C_\omega(\mathbb{R})$

$\forall n$ choos $\mathcal{U}_{\eta_n}$ to be $\underset{\sim}{\Sigma}^0_{\eta_n}$ - universal then

$$\mathcal{U} = \{ \langle x,y \rangle \mid \exists n \, ((x)_n, y) \in \mathcal{U}_{\eta_n} \}$$

is $\underset{\sim}{\Sigma}^0_{\lambda}$ - universal ∎