LECTURE IX CSST 2020

In lecture 5 we looked at CH.

In Andretta section 2.6 shows that G_δ [[CLOSED]] TRUE!
cannot violate CH.

Q3 "If X is Polish and $C \subseteq X$ then there are A and P s.t. A is countable, P is perfect, $A \cap P = \emptyset$, $C = A \cup P$."
(Andretta T3.32) FALSE!

Perfect Set Property

Q_{LS}: How simple can a set violating CH be?

Let Γ be a pointclass we define

$$PSP(\Gamma) := \forall A \in \Gamma \left(A \text{ is countable} \vee \exists P \subseteq A \; P \neq \emptyset \wedge P \text{ is perfect}\right)$$

WE HAVE A GLOBAL VERSION OF THIS DENOTED BY PSP.

$$PSP(P(\omega^\omega)).$$

LET Γ BE A POINTCLASS THEN WE DENOTE BY

$$DET(\Gamma) := \forall A \in \Gamma \ \ \underline{G(A)} \text{ IS DETERMINED.}$$

GAMES ON ω.

WE WILL SHOW THAT $AD \Rightarrow PSP$ AND EVEN MORE THAT FOR $\underset{\sim}{\Gamma}$ BOLDFACE POINTCLASSES

$$DET(\underset{\sim}{\Gamma}) \Rightarrow PSP(\underset{\sim}{\Gamma})$$

① WE WILL DEFINE A GAME CALLED $*$-GAME G^* [NOT ON ω]

② WE WILL SEE THAT THERE IS A CONTINUOUS CODING OF G^* INTO A GAME ON ω.

$DET(\underset{\sim}{\Gamma})$ a) IF A PLAYER HAS A WINNING STRATEGY IN ω THEN ONE

$\Downarrow$

DET OF $G^*(A)$ FOR $A \in \underset{\sim}{\Gamma}$ | HAS A WINNING STRATEGY IN G^*

b) IF $A \in \underset{\sim}{\Gamma}$ THEN THE "TRANSLATION" A^* OF A IN THE ω-GAME IS ALSO IN $\underset{\sim}{\Gamma}$.

③ PROVE THAT IF $G^*(A)$ IS DETERMINED THEN A IS EITHER COUNTABLE OR CONTAINS A CLOSED COPY OF 2^ω.

4) PUT 2 AND 3 TOGETHER TO GET

$$\mathrm{DET}(\underset{\sim}{\Gamma}) \Rightarrow \mathrm{PSP}(\underset{\sim}{\Gamma}).$$

WE KNOW THAT:

1) $AD \Rightarrow \neg AC$ (THERE IS NO WELL-ORD. OF $\mathbb{R}$)

2) $AD \Rightarrow PSP$.

LET Γ BE A POINTCLASS I will SAY THAT Γ IS <u>ADEQUATE</u> IFF Γ IS CLOSED UNDER FINITE UNIONS AND FINITE INTERSECTIONS.

<u>LEMMA</u> IF Γ IS ADEQUATE AND CONTAINS $\underset{\sim}{\Pi}^1_1$ THEN IF THERE IS A Γ-WELL-ORDERING OF $\mathbb{R}$ THEN THERE IS $B \in \forall^{\mathbb{R}}\Gamma$ VIOLATING PSP.

<u>PROOF</u> LET F BE A Γ WELL-ORDERING OF $\mathbb{R}$.

FOR $x \in WO$ I will DENOTE BY $\overset{\|x\|}{\alpha_x}$ THE ORDER TYPE OF x. $(x \in WO_{\alpha_x})$

LET B BE THE FOLLOWING SET:

$$B := \{ x \mid \underbrace{x \in WO}_{\underset{\sim}{\Pi}^1_1 \subseteq \Gamma} \wedge \forall y \in \overset{\omega^\omega}{\mathbb{R}} (\overset{\underset{\sim}{\Delta}^1_1}{\overbrace{y \in WO_{\alpha_x}}} \wedge \overset{\underset{\sim}{\Delta}^1_1}{\overbrace{y \neq x}} \Rightarrow \underbrace{(x,y) \in F}_{\Gamma}) \}$$

x IS F-MINIMAL IN WO_{α_x}

$$\underbrace{\forall y\in\mathbb{R}\ \underbrace{\Big(\underbrace{\underbrace{x\in WO}_{\mathbf{\Pi}^1_1}}_{\Gamma}\wedge\big(\underbrace{\underbrace{y\notin WO_{a_x}}_{\mathbf{\Delta}^1_1}\vee\underbrace{y=x}_{\mathbf{\Delta}^1_1}\vee\underbrace{(x,y)\in F}_{\Gamma}}_{\Gamma}\big)\Big)}_{\Gamma}}_{\Gamma}$$

$B\in\forall^{\mathbb{R}}\Gamma$.

<u>claim</u> B violates PSP.

note that B is not countable.

$\forall\alpha<\omega_1$, $|WO_\alpha\cap B|=1$

let $P\subseteq B$ non-empty and perfect. so P is closed. $P\in\mathbf{\Sigma}^1_1$ then by $\mathbf{\Sigma}^1_1$-boundedness there is α st

$P \subseteq WO_a$

But then $|WO_a \cap B| = 1$

So P would be countable. $\lightning$

∎

<u>Corollary</u> If $\mathbb{R}$ is well-ordered PSP fails.