UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2003/2004; 2nd Trimester
dr Benedikt Löwe

Homework Set # 7 Deadline: Thursday, March 4th, 2004

Exercise 7.1 (Ordinal Exponentiation).
Show that:

- $3^\omega = \omega$,
- $\omega^\omega \cdot \omega^\omega = \omega^{\omega \cdot 2}$, and
- $(\omega^\omega)^\omega = \omega^{(\omega^2)}$.

Exercise 7.2 (Different Ordinals)
Consider the following list of (terms for) ordinals. Some of them are equal (*e.g.*, ω and $7 \cdot \omega$) some aren't (*e.g.*, ω and $\omega + 7$).
Sort them into blocks of equal ordinals and sort the blocks according to the size of the ordinals in them (*i.e.*, the block containing ω before the block containing $\omega + 7$):

ω, $7 \cdot \omega$, $\omega + 7$, $7 \cdot (\omega^7 + \omega)$, $7 \cdot (\omega \cdot 7) + 7$, $7 + \omega$, $\omega^7 \cdot 7$, $\aleph_7$, $7 + 7 + 7 + (7 \cdot 7 \cdot \omega)$, $\omega^{\omega^7} + \omega^7$, $\omega^7 + \aleph_7$, 7, $\omega \cdot 7$, $\omega \cdot (\omega + 7)$, $(\omega + 7) \cdot \omega$, ω, $\omega + 7 + \omega^7$, ω^7, $(7 \cdot \omega) \cdot 7 + 7$, $7 + \omega + \omega^7$, $\omega + 7$, $7 + \omega^7 + \omega$, $\omega \cdot 7 + 7$, $7 + 7 \cdot \omega$, $\omega^7 + \omega + 7$, $\omega^7 + \omega^{\omega^7}$.

Exercise 7.3 (Fixed points).
Prove the following:

(1) ξ is a $\boldsymbol{\gamma}$-number if and only if for all $\eta < \xi$, we have $\eta + \xi = \xi$.
(2) ξ is a $\boldsymbol{\delta}$-number if and only if for all $0 < \eta < \xi$, we have $\eta \cdot \xi = \xi$.

Reminder. An ordinal ξ is called a $\boldsymbol{\gamma}$-number if it is a fixed point of the ordinal addition, *i.e.*, if $\alpha, \beta \in \xi$, then $\alpha + \beta \in \xi$. It is called a $\boldsymbol{\delta}$-number if it is a fixed point of the ordinal multiplication, *i.e.*, if $\alpha, \beta \in \xi$, then $\alpha \cdot \beta \in \xi$.