Universiteit van Amsterdam
Institute for Logic, Language and Computation

Recursion Theory

2003/2004; 1st Semester
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Written Exam: February 4th, 9:30-12:30

The following are the rules (I) to (V) defining the closure properties of the class of primitive recursive functions:

(I) The successor function $x \mapsto x + 1$ is in $\mathcal{C}$.

(II) The constant functions $\langle x_1, \ldots, x_n \rangle \mapsto m$ (for all $n, m \geq 0$) are in $\mathcal{C}$.

(III) The identity functions (or projections) $\langle x_1, \ldots, x_n \rangle \mapsto x_i$ (for $n \geq 0$ and $1 \leq i \leq n$) are in $\mathcal{C}$.

(IV) If $g_1, \ldots, g_m \in \mathcal{C}$ and $h \in \mathcal{C}$, then the function f defined by

$$f(x_1, \ldots, x_n) := h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)$$

is in $\mathcal{C}$.

(V) If $g \in \mathcal{C}$ and $h \in \mathcal{C}$, and $n \geq 1$, then the function f defined by

$$f(0, x_2, \ldots, x_n) := g(x_2, \ldots, x_n), \text{ and}$$
$$f(x_1 + 1, x_2, \ldots, x_n) := h(x_1, f(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n)$$

is in $\mathcal{C}$.

The class of primitive recursive function is the smallest class of functions closed under the rules (I) to (V). Please use this definition in Questions #1 and #2.

Question # 1.
Is the function f defined by $f(n, m, k) = k \cdot (10^n)^m$ primitive recursive? (Prove or refute.)

Question # 2.
Let $g(n, m)$ be the remainder when dividing n by m, *i.e.*, the unique $r < m$ such that $n = \ell \cdot m + r$ for some nonnegative ℓ. Is the function g primitive recursive? (Prove or refute.)

Question # 3.
State the *s*-*m*-*n* Theorem.

Question # 4.
Prove that there is a recursive function f such that for all n, we have

$$\varphi_{f(x)}(n) = \begin{cases} 0 & \text{if } \varphi_x(n)\downarrow, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Question # 5.
Let A and B be sets of natural numbers. Prove that $A \leq_{\mathrm{m}} B$ implies $A \leq_{\mathrm{T}} B$. (In other words, many-one reducibility implies Turing reducibility.)

Question # 6.
Give an example of sets A and B such that $A \leq_{\mathrm{T}} B$ but $A \not\leq_{\mathrm{m}} B$. (Give an argument why your sets A and B satisfy this.)

Question # 7.
Give an example of sets A and B such that $A \leq_{\mathrm{m}} B$ but $A \not\leq_{1} B$. (Give an argument why your sets A and B satisfy this.)

Question # 8.
Give an example of a family of sets $\{A_n\,;\, n \in \mathbb{N}\}$ and sets A_∞ and $A_{\infty*}$ such that all of the following conditions are satisfied:

- for all $n \in \mathbb{N}$ and all $k < n$, we have $A_k <_{\mathrm{T}} A_n$,
- for all $k \in \mathbb{N}$, we have $A_k <_{\mathrm{T}} A_\infty$, and
- $A_\infty <_{\mathrm{T}} A_{\infty*}$.

(Defining a family of sets with these properties and giving an informal argument that they have these properties referring to results of the lecture course is enough.)

Question # 9.
Let A be a set of natural numbers. A family $\{A_n\,;\, n \in \mathbb{N}\}$ is called an **approximation of** A if

(1) all sets A_n are finite, and
(2) $x \in A$ if and only if there is some s such that for all $t \geq s$, we have $x \in A_t$.

We say that A **has a recursive approximation** if there is a recursive function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the family $\{A_n\,;\, n \in \mathbb{N}\}$ with $A_n := \{x\,;\, f(n,x) \neq 0\}$ is an approximation of A.
Show that if A has a recursive approximation, then $A \leq_{\mathrm{T}} \varnothing'$.