Structure of the class of projective special real manifolds and their generalisations

Dissertation

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1 Introduction

In this thesis we study properties of projective special real manifolds and their generalisations. Projective special real manifolds are hyperbolic centro-affine hypersurfaces and thus they are objects of study in the fields of both Riemannian geometry and affine differential geometry. While Riemannian geometry is probably to some extent known to any mathematician, affine and centro-affine differential geometry is a little less common field of study. In most generality, affine differential geometry is the study of smooth manifolds M equipped with a torsion-free connection ∇ in $TM \to M$ together with a ∇-parallel volume form ω. Such a triple (M, ∇, ω) is called an equiaffine structure on M [NS]. One is then concerned with submanifolds of M and their induced geometric data. In particular, if the considered submanifold $N \subset M$ is of co-dimension one, it turns out that this study is closely related to non-vanishing transversal vector fields along N. The term centro-affine geometry is used when $M = \mathbb{R}^{n+1}$, equipped with the flat connection and standard parallel volume form $\det(\cdot)$, and $N \subset \mathbb{R}^{n+1}$ is a submanifold (embedded via the inclusion map), such that the position vector field $X \in \Gamma\left(T\mathbb{R}^{n+1}\right)$, $X_p = p$ for all $p \in \mathbb{R}^{n+1}$ with the usual identification $T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, is transversal along N. Another well studied subject in the field of affine differential geometry is the theory of Blaschke structures on hypersurfaces, named after Wilhelm Blaschke (1885–1962). This in particular includes the study of affine hyperspheres, see [CY] for a completeness theorem about locally strictly convex affine hyperspheres. For a history of the developments in the field of affine differential geometry (and also for an excelent textbook about affine differential geometry in general) we refer the reader to the book "Affine Differential Geometry" by Katsumi Nomizu and Takeshi Sasaki [NS], which contains a historical review in the introduction.

An n-dimensional projective special real manifold $\mathcal{H}$ is a hypersurface in $\mathbb{R}^{n+1}$ that is contained in the level set $\{h = 1\}$ of a cubic homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ with the property that the negative Hessian of h restricted to $\mathcal{H}$ is positive definite when viewed as bilinear form [CHM, Def. 1]. Another way to introduce projective special real manifolds is by defining them to be an open subset of $\{h = 1\} \cap \{\text{hyperbolic points of } h\}$, where $p \in \{h > 0\}$ is called a hyperbolic point of h if $-\partial^2 h_p$ has Lorentzian signature. Note that these two definitions of projective special real manifolds are equivalent. The aforementioned generalisations of projective special real manifolds that we will also study are defined analogously with the difference that the homogeneous polynomial h is also allowed to have degree greater than three, e.g. that h is a quartic or quintic homogeneous polynomial. We will call the manifolds obtained via this type of generalisation generalised projective special real manifolds. Both projective special real and generalised projective special real manifolds, equipped with the (automatically) transversal position vector field of $\mathbb{R}^{n+1}$ along them, are affine hypersurfaces of $\mathbb{R}^{n+1}$. Thus, such manifolds are centro-affine hypersurfaces of $\mathbb{R}^{n+1}$. It turns out that their induced centro-affine fundamental form [NS, Def. 3.2] is always positive definite. Hence, they naturally carry the structure of a Riemannian manifold.

Projective special real manifolds and our considered generalisation were studied under different points of view in the mathematics and physics literature. Projective special real curves and surfaces have been classified in [CHM, Thm. 7] and [CDL, Thm. 1, Thm. 2], respectively. Independently of their dimension, projective special real manifolds defined by a reducible cubic polynomial have also been classified, see [CDJL, Thm. 2, Prop. 8]. In [CNS, Thm. 2.5] it was shown that for all $n \geq 0$, an n-dimensional projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ equipped with its centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ is geodesically complete if and only if it is closed in the ambient space $\mathbb{R}^n$. In [CNS, Def. 2.2] projective special real manifolds were defined intrinsically as (intrinsic) centro-affine manifolds (M, ∇, g, ν) (cf. [CNS, Def. 1.5]) with the property that their respective cubic form $C := \nabla g$ fulfils

$$(\nabla_X C)(Y, Z, W) = g(X, Y)g(Z, W) + g(X, Z)g(W, Y) + g(X, W)g(Y, Z)$$

for all $X, Y, Z, W \in \Gamma(TM)$. It was then shown [CNS, Thm 2.3] that every projective special real manifold is also an intrinsic projective special real manifold in that sense, and on the other hand that every intrinsic projective special real manifold is isomorphic (as a centro-affine manifold) to a projective special real manifold. [CNS, Thm 2.3] is thus an analogue to the fundamental theorem of affine differential geometry [NS, Thm. 8.1, p. 73] in the setting of projective special real manifolds. Another connection to affine differential geometry is based upon the constructions of the supergravity r- and c-map which originate in the theory of supergravity [GST, FS, DV, CHM]. The supergravity r-map associates to a given n-dimensional projective special real manifold a projective special Kähler manifold of real dimension $2n+2$, and the supergravity c-map associates to such a Kähler manifold a quaternionic Kähler manifold of real dimension $4n+8$. In [CHM] it was proven that the r- and c-map preserver geodesic completeness. This fact was used in [CDJL, Thm. 3] to obtain an explicit series of inhomogeneous complete quaternionic Kähler manifolds with negative scalar curvature of real dimension $4n+8$ for $n \geq 1$. More precisely, manifolds in this series have the property that their respective isometry group acts with co-homogeneity one. Apart from the theory of supergravity, another connection of projective special real manifolds and their generalisations with physics is geometric scattering theory, see the discussion after [CNS, Thm. 1.18] and [Me]. Projective special real manifolds and related geometric objects have also been studied in the setting of affine differential geometry, which we will review now. In order to properly define projective special Kähler manifolds, we need the concept of an affine special Kähler manifold. An affine special Kähler manifold [F] is a (pseudo-)Kähler manifold (M, g, J, ∇) with Kähler metric $g = \omega(\cdot, J\cdot)$ equipped with a torsionfree, flat connection ∇, such that $d^\nabla J = 0$. The latter means that $d^\nabla J(X, Y) = (\nabla_X J)Y - (\nabla_Y)X = 0$ for all $X, Y \in \Gamma(TM)$. Note that g is allowed to be indefinite. Simply connected affine special Kähler manifolds have the property that they can be described by a holomorphic Lagrangian immersion [ACD, Thm. 4]. They can also be viewed as parabolic (also called improper) affine hyperspheres [NS, Def. 3.3], as it was shown in [BC1, Thm. 3.1] that for a given such manifold of real dimension $2n$ there exists a Blaschke immersion [NS, Def. 3.2] $\varphi : M \to \mathbb{R}^{2n+1}$ with induced Blaschke metric and Blaschke connection [NS, Def. 3.3] coinciding with the given metric g and connection ∇, such that $\varphi(M)$ is a parabolic affine hypersphere. In [ACD] a subclass of affine special Kähler manifolds is introduced and studied, namely conic affine special Kähler manifolds. A conic affine special Kähler manifold is an affine special Kähler manifold (M, g, J, ∇) equipped with a local holomorphic $\mathbb{C}^*$-action $\varphi_\lambda : M \to M$, $\lambda = re^{it} \in \mathbb{C}$, fulfilling $(\varphi_\lambda)_* X = r\cos(t)X + r\sin(t)JX$ for all ∇-parallel vector fields $X \in \Gamma(TM)$ (cf. [BC2, Sect. 1.2] and for the setting of the supergravity r-map also [CHM, Def. 3]). Under the assumption that the action lifts to a global $\mathbb{C}^*$-action on M, the orbit space $\overline{M} := M/\mathbb{C}^*$ equipped with the metric, almost complex structure, and connection induced by the projection $M \to \overline{M}$ is a Kähler manifold and will be called a projective special Kähler manifold. Under the additional assumption that M is a conic affine special Kähler domain, cf. [BC2, Sect. 2], the corresponding manifold $\overline{M}$ is called a projective special Kähler domain. A conical affine special Kähler domain M is by definition a subset of $\mathbb{C}^n$ and has a globally defined Kähler potential $k : M \to \mathbb{R}$ of the form

$$k = \frac{1}{2}\mathrm{Im}\left(\sum_{i=1}^{n} \frac{\partial F}{\partial z_i}\overline{z}_i\right)$$

for some holomorphic function $F : M \to \mathbb{C}$ which is homogeneous of degree two. The function F is called the holomorphic prepotential of the associated projective special Kähler domain $\overline{M}$ defined by M. With the knowledge of the Kähler potential $k : M \to \mathbb{R}$ one can study the level sets $M_c := \{z \in M \mid |k(z)| = c\}$, $c > 0$, which are hypersurfaces in $M \subset \mathbb{C}^n$. For any $c > 0$, M_c is an S^1-principle bundle over $\overline{M}$, and for $c = \frac{1}{2}$ the projection map $\pi_c : (M_c, g) \to \left(\overline{M}, \overline{g}\right)$ is a pseudo-Riemannian submersion [BC2, Prop. 1]. Here g denotes the restricted Kähler metric of M to M_c and g is the induced projective special Kähler metric on $\overline{M}$. Hypersurfaces of the form $M_{\frac{1}{2}}$ are also connected to affine differential geometry. In [BC2, Thm. 6] it is demonstrated that in certain special coordinates, one can view $M_{\frac{1}{2}} \subset M \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ as a proper affine hypersphere $M_{\frac{1}{2}} \subset \mathbb{R}^{2n}$ with affine mean curvature $\operatorname{sgn}(k)$. Summarising, for each projective special Kähler manifold $\overline{M}$, thus in particular for those obtained via the supergravity r-map applied to a projective special real manifold, we have its defining conic affine special Kähler manifold M which (under the assumption that it is simply connected) can be studied as a parabolic (or improper) affine sphere, and we also have an S^1-principle bundle over $\overline{M}$ (under the additional assumption that M is a conic affine Kähler domain), which can be understood as a proper affine hypersphere. The structure of projective real manifolds and their generalisations also appear in the study of the index cone of Kähler manifolds [Wi1, Wi2, Ma]. The index cone W of a M of a real $2n$-dimensional Kähler manifold is defined to be the subset of the positive cone $\{\omega \in H^{1,1}(M, \mathbb{R}) \mid \omega^n > 0\}$ that contains all elements ω, such that the induced quadratic form $H^{1,1}(M, \mathbb{R}) \ni \alpha \mapsto \omega^{2n-2} \cup \alpha^2 \in \mathbb{R}$ has signature $(1, h^{1,1} - 1)$. Here, ω^n denotes the n-fold cup product and $h^{1,1} = \dim H^{1,1}(M, \mathbb{R})$. In the case of complex 3-dimensional Kähler manifolds, e.g. complex 3-dimensional Calabi-Yau manifolds, the level set $\omega^3 = 1$ in the index cone can thus be interpreted as to be contained in some projective special real manifold of dimension $h^{1,1} - 1$. Historically, real plane cubic curves have already been studied by Newton [N], for a modern introduction see [BK]. The relation to projective special real surfaces $\mathcal{H}$ is that the boundary of their respective cone $\mathbb{R}_{>0} \cdot \mathcal{H} \subset \mathbb{R}^3$, intersected with an affine plane in $\mathbb{R}^3$ that does not contain the origin, is a real plane cubic curve.

Almost all of our studies in this thesis are from a mathematical point of view, although we will mention possible applications of our results to the theory of supergravity. Our main focus will be the study of projective special real manifolds and quartic generalised projective special real manifolds, the latter corresponding to quartic homogeneous polynomials, although some of our results hold for all generalised special real manifolds. Additionally, we will study examples and curvature properties of manifolds in the image of the (generalised) supergravity r-map. Before giving a summary of the contents of this thesis, we will highlight some of our main results and afterwards mention some of the open problems that we will discuss in this thesis.

Main results:

One of the subjects of this thesis is the scalar curvature of projective special real manifolds and their generalisations. Our first main result is Theorem 4.13. We prove that the scalar curvature of an $n \geq 2$-dimensional closed connected projective special real manifold is globally bounded by constants from above and from below. The corresponding bounds (see equation (4.15)) depend only on the dimension n and are independent of the specific considered closed connected projective special real manifold.

The second main result of this thesis, Theorem 5.6, is concerned with properties of the moduli space of closed connected projective special real manifolds. It says that given a maximal connected projective special real manifold $\mathcal{H} \subset \{h = 1\}$ in standard form (cf. Proposition 3.18, i.e. $h = x^3 - x\langle y, y\rangle + P_3(y)$, maximal means that $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ coincides with a connected component of hyperbolic points of the defining polynomial h), $\mathcal{H}$ is closed in the ambient space $\mathbb{R}^{n+1}$ if and only if the polynomial h fulfils the maximality condition $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$, independent of the dimension of $\mathcal{H}$. This implies in particular that the moduli space of closed connected projective special real manifolds in any dimension $n \in \mathbb{N}$ is generated by a convex compact subset of an affine subset of $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$, see Proposition 5.8. This allows us to define a deformation theory of closed connected projective special real manifolds as described in Section 5. Furthermore, Theorem 5.6 also has applications for curvature bounds of closed connected projective special real manifolds. We use it in Proposition 5.12 to calculate global bounds of the scalar curvature of closed connected special real surfaces which are sharp, meaning that they not only improve the bounds in Theorem 4.13 (which we do not expect to be sharp in any dimension) but are also the best possible choices for such bounds. The results of Proposition 5.8 provide a partial answer to Conjecture 5.14 which is a statement about possible sectional curvature bounds of level sets in the Kähler cones of Calabi-Yau three-folds formulated by P.M.H. Wilson in [Wi2].

Our third main result, Theorem 7.2, is about quartic generalisations of closed connected projective special real curves. We classify all quartic generalised projective special real curves $\mathcal{H} \subset \{h = 1\}$ up to linear equivalence and determine in each case the automorphism group of the corresponding polynomial h. In comparison with the classification of closed connected projective special real curves found in [CHM, Thm. 8 a),b)], which states that there are precisely two distinct such curves up to linear equivalence with one being homogeneous under the action of the respective linear automorphism group, it turned out that in the quartic case we have up to linear equivalence two homogeneous curves (Thm. 7.2 *a*) and *b*)), one inhomogeneous curve (Thm. 7.2 *c*)), and a one-parameter family of pairwise inequivalent inhomogeneous curves (Thm. 7.2 *d*)).

During the preparation of this thesis we encountered some interesting open problems that are related to our studies. One of them is Open problem 7.1, that is the question whether all quartic generalised projective special real manifolds $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ of arbitrary dimension $\dim(\mathcal{H}) = n$ are geodesically complete with respect to their centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{4}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ if and only if they are closed as a subset of the ambient space $\mathbb{R}^{n+1}$. During the preparation of this thesis, which was mainly motivated by the tasks to better understand global curvature properties of closed projective special real manifolds, to study properties of their moduli space, and to find possible generalisations of their properties to closed generalised projective special real manifolds, we also studied the latter open problem. Note that the completeness of closed projective special real manifolds has first been proven in [CNS, Thm. 2.5], and it is described therein after [CNS, Open problem 2.10] why their proof cannot easily be extended to quartic closed generalised projective special real manifolds. In Proposition 4.17 and Proposition 5.17 we find two different new ways to show that closed projective special real manifolds are complete, and in Section 7 we describe properties of quartic closed generalised projective special real manifolds that illustrate why these two new proofs also cannot be generalised in any obvious way to quartic closed generalised projective special real manifolds, see the related discussion in Section 9. Apart from this specific open problem we also discuss open questions for generalised projective special real manifolds independent of the corresponding homogeneity-degree $\tau \geq 3$ of their corresponding defining polynomial (Open problems 3.37 and 3.38), and in Open problem 8.20 we propose a way to study the curvature properties of manifolds in the image of the supergravity q-map, which is the composition of the r- and c-map, by employing our technical tools developed in Section 3.

Summary of this thesis:
In the preliminaries, that is Section 2, we explain the notation used in this work and give a short overview of pseudo-Riemannian and centro-affine geometry. We will then introduce hyperbolic centro-affine hypersurfaces of which projective special real manifolds are a special case and review some known results about them which we will use later.

In Section 3 we will develop the mathematical machinery that is needed for our study of (generalised) projective special real manifolds. We will in particular find a "standard form" for homogeneous polynomials corresponding to such manifolds and use this result to find formulas for their different curvature tensors. The main results of Section 3 are:

- Proposition 3.18, which allows us to find for any chosen point $p \in \mathcal{H}$ in a connected (generalised) projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, h of homogeneity-degree $\tau \geq 3$, a linear transformation $A \in \mathrm{GL}(n+1)$ of the ambient space $\mathbb{R}^{n+1}$, which maps $(1, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$ to p and fulfils
$$h\left(A \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum_{i=3}^{\tau} x^{\tau-i} P_i(y).$$
Here $y = (y_1, \ldots, y_n)^T$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean standard scalar product induced on $\mathbb{R}^n$ via the choice of the y-coordinates, and $P_i : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous polynomial of degree i for all $3 \leq i \leq \tau$. This result in particular gives a mathematical proof for the concept of "canonical parametrisation" of h in the context of supergravity theory where h is a cubic homogeneous polynomial, see the discussion in Remark 3.20. Our result however holds for all possible degrees $\tau \geq 3$ of h and furthermore tells us explicitly how the polynomials P_i, $3 \leq i \leq \tau$, depend on the choice of the reference point $p \in \mathcal{H}$.

- Propositions 3.29, 3.30, and Lemma 3.31, which are formulas for the scalar curvature, the first derivative of the scalar curvature, and the Riemannian, Ricci, and sectional curvature tensors of (generalised) projective special real manifolds at one particular point. While having a formula at one point might not appear to be too useful at first, when combined with the aforementioned Proposition 3.18 and under the assumption that the considered (generalised) projective special real manifold is closed this will allow us to find curvature bounds for these manifolds in the next section.

- Proposition 3.34, which yields a necessary and sufficient condition for a (generalised) projective special real manifold to be a Riemannian homogeneous space under the action of its linear isometry group and allows us to avoid calculating the said linear isometry group when we want to show that some (generalised) projective special real manifold fulfils that condition. To obtain this result we have to study the infinitesimal changes of the polynomials P_i, $3 \leq i \leq \tau$, as defined in Proposition 3.18, see Definition 3.27.

In Section 4 we restrict our studies to projective special real manifolds. We are concerned with the scalar and sectional curvature and will determine upper and lower bounds for them that hold for all closed projective special real manifold of fixed dimension. It turns out that the technicalities that are needed for these results can also be used to find an alternative proof (in comparison with [CNS, Thm. 2.5]) that a projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ equipped with its centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H} \times T\mathcal{H}}$ is geodesically complete if and only if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed. The main results of Section 4 are:

- Theorem 4.13, where we show that the scalar curvature of an $n \geq 2$-dimensional closed projective special real manifold $\mathcal{H}$ is always bounded from above and from below, where the upper and lower bound depend only on the dimension $\dim(\mathcal{H}) = n$ of $\mathcal{H}$.

- Proposition 4.15, which is an analogous result for the sectional curvature instead of the scalar curvature.

- Proposition 4.17, in which we give a proof that closed projective special real manifolds are geodesically complete. This new proof might be useful when studying the still open question whether a closed generalised projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, h of homogeneity degree $\tau \geq 4$, is automatically geodesically complete.

In the next section, that is Section 5, we are again concerned with projective special real manifolds and develop a deformation theory of closed connected projective special real manifolds. The results characterise the moduli space of n-dimensional closed connected projective special real manifolds under the action of $\mathrm{GL}(n+1)$ for all $n \geq 1$ and allow us to find sharp lower and upper bounds for the scalar curvature of closed projective special real surfaces (for a discussion why the bounds constructed in Theorem 4.13 are not expected to be sharp see Remark 4.14). In order to obtain these results we study regularity of closed projective special real manifolds in the sense of [CNS, Def. 1.7], respectively Definition 5.1. Altogether, this allows us to find a second alternative proof of the statement that closed projective special real manifolds are complete. The main results of Section 5 are:

- Theorem 5.3, in which we prove that a closed connected projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is not singular at infinity (cf. Definition 3.16), that is there exists no point $p \in \partial U \setminus \{0\}$, where $U = \mathbb{R}_{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ denotes the cone spanned by $\mathcal{H}$, such that $dh_p = 0$, if and only if $\mathcal{H}$ has regular boundary behaviour in the sense of Definition 5.1.

- Theorem 5.6, where we show that the connected component $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ that contains the point $(1, 0, \ldots, 0)^T \in \{h = 1\} \subset \mathbb{R}^{n+1}$, h of the form (3.12) found in Proposition 3.18, that is
$$h = x^3 - x\langle y, y\rangle + P_3(y),$$
is a closed connected projective special real manifold if and only if the cubic homogeneous polynomial $P_3 : \mathbb{R}^n \to \mathbb{R}$ fulfils $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$. Thus, we do not need to check by hand that every point $p \in \mathcal{H}$ is a hyperbolic point of h, but instead it suffices to study the maxima of P_3 on $S^{n-1} = \{z \in \mathbb{R}^n \mid \langle z, z\rangle = 1\}$.

- Proposition 5.8, which states that the moduli space of n-dimensional closed connected projective special real manifolds is generated by the convex compact uniformly bounded subset
$$\mathcal{C}_n = \left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; \max_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}} \right\}$$
which is affinely embedded in $\mathrm{Sym}^3 \left(\mathbb{R}^{n+1}\right)^*$ (when equipped with the topology induced by the real vector space structure). Furthermore, we find that closed connected projective special real manifolds which are singular at infinity correspond precisely to the $\mathrm{GL}(n+1)$-orbits of $\partial\mathcal{C}_n$. Here, a closed connected projective special real manifold $\mathcal{H} \subset \{h = 1\}$ being singular at infinity means that there exists a point $p \in \partial\left(\mathbb{R}_{>0} \cdot \mathcal{H}\right)$ such that $dh_p = 0$.

- Together, Theorem 5.6 and Proposition 5.8 can be interpreted as a deformation theory of closed connected projective special real manifolds in the following sense. Whenever $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is a closed connected projective special real manifold, h is of the form (3.12), that is $h = x^3 - x\langle y, y\rangle + P_3(y)$, $(1, 0, \ldots, 0)^T \in \mathcal{H}$ (note: this is not a restriction of generality, cf. Proposition 3.18), and $V : \mathbb{R}^n \to \mathbb{R}$ is any given cubic homogeneous polynomial, we now have a precise answer to the question when the connected component

$$\mathcal{H}_\varepsilon \subset \left\{ h_\varepsilon := x^3 - x\langle y, y\rangle + P_3(y) + \varepsilon V(y) = 1 \right\}, \quad (1, 0, \ldots, 0)^T \in \mathcal{H}_\varepsilon,$$

is also a closed connected projective special real manifold, namely if and only if

$$\max_{\|z\|=1} (P_3(z) + \varepsilon V(z)) \leq \frac{2}{3\sqrt{3}}.$$

Furthermore, we have found a way to connect two closed connected projective special real manifolds with a curve consisting pointwise of closed connected projective special real manifolds since $\mathcal{C}_n$ is a convex and thus in particular path-connected set.

- Proposition 5.12. Here we derive a sharp estimate for the scalar curvature of closed connected projective special real surfaces. More specifically, we will show that the scalar curvature $S_\mathcal{H}$ of a closed connected projective special real surface $\mathcal{H}$ equipped with its centro-affine fundamental form $g_\mathcal{H}$ is globally bounded by

$$-\frac{9}{4} \leq S_\mathcal{H} \leq 0,$$

independently of which closed connected projective special real surface $\mathcal{H}$ is considered. This estimate being sharp means in this case that there exists precisely one homogeneous closed connected projective special real surface with constant scalar curvature equal to $-\frac{9}{4}$ (Thm. 2.45 b)), and another homogeneous closed connected projective special real surface with constant scalar curvature equal to 0 (Thm. 2.45 a)). Recall that we do not expect the bounds found in Theorem 4.13 to be sharp, and we will indeed see that they are not sharp for dimension two. This is an application of Theorem 5.6 to a low-dimensional question and the proof makes use of the already known classification of closed connected projective special real surfaces found in [CDL, Thm. 1] (see also Theorem 2.45 a)-f) for the statement of this classification).

- An application of Proposition 5.12 is Corollary 5.15, where we give a partial answer to Conjecture 5.14, which is a statement for bounds of the sectional curvatures of level sets in the Kähler cone of complex 3-dimensional Calabi-Yau manifolds stated by P.M.H. Wilson in [Wi2].

- Finally, we will use the result of Proposition 5.8 to find another alternative proof of the statement that closed projective special real manifolds are complete, see Proposition 5.17. This approach might be extendable to generalised projective special real manifolds with corresponding polynomial h of homogeneity-degree $\tau \geq 4$, see Section 9 for a discussion on how such a generalisation might look like (and why it is most likely worth a try at least for quartic closed connected generalised projective special real manifolds).

In Section 6 we will study two examples of $(n-2)$-parameter families of pairwise inequivalent n-dimensional closed connected projective special real manifolds for each $n \geq 3$. Pairwise inequivalent means that two distinct elements of one of these families are not related by a linear transformation of the ambient space $\mathbb{R}^{n+1}$. Some of the results of this section are part of [CDJL], namely Theorem 6.1, Corollary 6.5, and in part Corollary 6.7. A one-parameter family of pairwise inequivalent closed connected projective special real surfaces corresponding to the Weierstraß cubics has been studied in [CDL], but until the results in

[CDJL] no pairwise inequivalent multi-parameter family of complete projective special real manifolds has been known, albeit the existence of such a family in high enough dimension was expected from the fact that the dimension of the vector space of cubic homogeneous polynomials in $n+1$ variables grows cubically in n, while the dimension of $\mathrm{GL}(n+1)$ grows only quadratically in n. This was the initial motivation for finding such a multi-parameter family. The main results of Section 6 are:

- Theorem 6.1, the existence of two $(n-2)$-parameter families of pairwise inequivalent $n \geq 3$-dimensional closed connected projective special real manifolds. The corresponding cubic homogeneous polynomials are given in $\mathcal{F}$ (6.1) and $\mathcal{G}$ (6.2), respectively. (This result is a part of [CDJL].)

- Corollary 6.5, in which we list the possible automorphism groups for all $h \in \mathcal{F} \cup \mathcal{G}$. (This result is a part of [CDJL].)

- Proposition 6.6, where we show that each closed connected projective special real manifold $\mathcal{H}(h)$ corresponding to $h \in \mathcal{F} \cup \mathcal{G}$ as in equation (6.3) and equation (6.4), respectively, is singular at infinity in the sense that the boundary of the cone $U = \mathbb{R}_{>0}\cdot\mathcal{H}(h) \subset \mathbb{R}^{n+1}$ excluding the origin contains a point $p \in \partial U \setminus \{0\}$, such that $dh_p = 0$ (cf. Definition 3.16).

- Proposition 6.9, where we show that each closed connected projective special real manifold $\mathcal{H}(h)$ is inhomogeneous for all $h \in \mathcal{F} \cup \mathcal{G}$.

- Lemma 6.10, in which we calculate the scalar curvature of the two homogeneous projective special real manifolds $\mathcal{H}_{1,n} \cong \mathbb{R}_{>0} \ltimes \mathbb{R}^{n-1}$ (6.44) and $\mathcal{H}_{2,n} \cong \frac{\mathbb{R}_{>0}\times \mathrm{SO}^+(n-1,1)}{\mathrm{SO}(n-1)}$ (6.45) for $n \geq 3$.

Next, in Section 7 we will switch our focus from projective special real manifolds to quartic generalised projective special real manifolds. We will give a classification of quartic closed connected generalised projective special real curves and we will find analogues to some results from Section 4 to quartic generalised projective special real manifolds. We will also discuss explicit examples of closed connected generalised projective special real manifolds. The main results of Section 7 are:

- Theorem 7.2 in which we classify all quartic closed connected generalised projective special real curves $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^2$ up to linear equivalence. Furthermore, we determine the hyperbolic closed connected components of $\{h > 0\} \subset \mathbb{R}^2$ and the automorphism group of h in each case.

- Proposition 7.8, which can be understood as a quartic analogue to Corollary 4.5. We show that the Euclidean length of points in the boundary of the set $\mathrm{dom}(\mathcal{H})$ as in Definition 3.22 is bounded from above by $\sqrt{6}$ for all quartic closed connected generalised projective special real manifolds $\mathcal{H}$.

- Lemma 7.9, which is the quartic analogue to Lemma 4.8. We formulate a necessary and sufficient condition for a certain connected component of a quartic homogeneous polynomial of the form (3.12) to be a closed connected quartic generalised projective special real manifold. The analogous construction for projective special real manifolds in Lemma 4.8 was a key component in one of the new proofs that closed projective special real manifolds are geodesically complete (cf. Proposition 4.17).

In Section 8 we will be concerned with manifolds in the image of the (generalised) supergravity r-map. We will derive a formula for their scalar curvature using our technical tools from Section 3 and find that it has some properties analogous to the properties of the scalar curvature of closed connected projective special real manifolds that we have studied in Section 4. As examples, we will study r-map images of the elements in the two multi-parameter families of closed connected projective special real manifolds that were studied in Section 6 and we will in particular show that all manifolds that are obtained in this way are inhomogeneous. The main results of Section 8 are:

- Proposition 8.8, where we derive a formula for the scalar curvature of manifolds in the image of the (generalised) supergravity r-map at one point, analogous to Proposition 3.29 in which we found a formula for the scalar curvature of (generalised) projective special real manifolds at one point.

- Proposition 8.9, in which we determine (not necessarily sharp) upper and lower bounds for manifolds in the image of the supergravity r-map where the initial projective special real manifold is assumed to be closed and connected.

- Lemma 8.11, where we find sharp upper and lower bounds for manifolds in the image of the supergravity r-map under the assumption that the initial projective special real manifold is closed, connected, and one-dimensional.

- Proposition 8.14, where we determine a formula for the first derivative of the scalar curvature of manifolds in the image of the (generalised) supergravity r-map at one point, analogous to Proposition 3.30 which contains a similar formula for (generalised) projective special real manifolds.

- Proposition 8.15, where we prove that r-map-images of closed connected projective special real manifolds of the form $\mathcal{H}(h)$ for all $h \in \mathcal{F} \cup \mathcal{G}$, cf. Theorem 6.1, are inhomogeneous. Recall that $\mathcal{H}(h)$ itself was shown to be inhomogeneous for all $h \in \mathcal{F} \cup \mathcal{G}$ in Proposition 6.9.

- Lemma 8.16, in which we calculate for each dimension $n \geq 3$ the (constant) scalar curvature of the image under the r-map of the two homogeneous projective special real manifolds $\mathcal{H}_{1,n} \cong \mathbb{R}_{>0} \ltimes \mathbb{R}^{n-1}$ (6.44) and $\mathcal{H}_{2,n} \cong \frac{\mathbb{R}_{>0} \times \mathrm{SO}^+(n-1,1)}{\mathrm{SO}(n-1)}$ (6.45).

We will conclude this thesis with an outlook in Section 9. We will discuss the still open question if every quartic closed generalised projective special real manifold is automatically geodesically complete, and we will also present ideas for a possible proof that have neither been fully tried nor excluded by our research yet. Another interesting problem we will discuss is the construction of possible ways to map (generalised) projective special real manifolds $\mathcal{H}_\tau \subset \{h_\tau = 1\}$, h_τ of homogeneity degree τ, to generalised projective special real manifolds $\mathcal{H}_{\tau+1} \subset \{h_{\tau+1} = 1\}$, $h_{\tau+1}$ of homogeneity degree $\tau + 1$, for all $\tau \geq 3$. This question has been motivated by the proof of Theorem 7.2 and an analogue for projective special real curves (cf. Remark 7.4), which turned out to provide possibilities for such constructions to map closed connected projective special real curves to quartic closed connected generalised projective special real curves.

2 Preliminaries

2.1 Notation

We will give an overview of the notations and conventions used in this thesis that are either not frequently used or not standardised.

- Unless stated otherwise, we will always assume that manifolds and maps are smooth.
- For a vector bundle over a manifold $E \to M$ we denote its sections by $\Gamma(E)$. We omit specifying the corresponding projection map if it is clear from the context.
- In order to omit special notations for vector fields on a manifold M, we will denote the set of vector fields by $\Gamma(TM)$ instead of the also commonly used notation $\mathfrak{X}(M)$. The term $\Gamma(TM)|_U$ for a subset $U \subset M$ denotes the set of vector fields along U that are obtained by restricting vector fields on M.
- We consider elements in the vector space, respectively manifold, $\mathbb{R}^{n+1}$ as column vectors.
- We will not use the Einstein sum convention. We will, however, frequently omit summation ranges if they are clear from the context, e.g. we will write $\sum\limits_k$ instead of $\sum\limits_{k=1}^n$. This usually makes formulas a little easier to read while still indicating the summation and the corresponding indices.
- For local coordinates $(x_1, \ldots, x_n)$ on a manifold M we will often abbreviate the induced local frame fields $\frac{\partial}{\partial x_i}$ of TM by either ∂_i or ∂_{x_i}.
- For the positions vector field $\xi \in \Gamma(T\mathbb{R}^{n+1})$ we will frequently omit the symbol ξ and canonically identity p and ξ_p. This makes many equations a lot easier to read.
- We identify homogeneous polynomials of degree $\tau \geq 1$ in $\mathbb{R}[x_1, \ldots, x_n]$ with symmetric tensors in $\mathrm{Sym}^\tau\left(\mathbb{R}^{n+1}\right)^*$ in the sense that for every homogeneous polynomial $h : \mathbb{R}^n \to \mathbb{R}$ there exists precisely one symmetric $(0,\tau)$-tensor H, such that $h(x) = H(x, \ldots, x)$. Also, instead of writing "$h \in \mathbb{R}[x_1, \ldots, x_n]$ is homogeneous of degree τ" we will write $h \in \mathrm{Sym}^\tau(\mathbb{R}^{n+1})^*$.
- Whenever $x = (x_1, \ldots, x_n)^T$ denotes linear coordinates of $\mathbb{R}^n$, we will identify $dx = (dx_1, \ldots, dx_n)^T$. This means for example that for a bilinear form $Q(x,x)$, we will write $dQ_x = 2Q(x, dx)$.
- Empty spaces in matrices are always supposed to be zeros. Writing down zeros and dots would make the corresponding equations more difficult to read.
- The natural numbers $\mathbb{N}$ are given by $\mathbb{N} = \{1, 2, 3, \ldots\}$. In particular $0 \notin \mathbb{N}$.

We start with some remarks about vector bundles and restriction of corresponding sections to images of immersions.

Definition 2.1 (Sections along immersions)**.** *Let $E \to M$ be a vector bundle over a manifold. For an immersion $f : \widetilde{M} \to M$ and an open subset $U \subset \widetilde{M}$, such that $f|_U$ is an embedding or equivalently $f(U)$ is a submanifold of M, we denote by $\Gamma_{f(U)}(E)$ the* sections of $E \to M$ along $f(U)$*. These are precisely the sections of the pullback bundle $\iota^*_{f(U)}E \to f(U)$, which can be identified with the sections of the corresponding pullback bundle $f|^*_U E \to U$. Here $\iota_{f(U)}$ denotes the inclusion map of the submanifold $f(U)$ into M.*

In order to talk about properties of sections along immersions, one has to be careful whenever f is not an embedding, i.e. whenever f is an immersion but not a homeomorphism onto its image with the induced subspace topology.

Remark 2.2 (Terminology for sections along immersions)**.** Let $E \to M$ be a vector bundle and let $f : \widetilde{M} \to M$ be an immersion. We can restrict any section $s \in \Gamma(E)$ to the subset $f(\widetilde{M}) \subset M$. Since f need not be an embedding, we say that $s|_{f(\widetilde{M})}$ has some property, e.g. is nowhere vanishing, if that property holds locally around each point. This means that for all $p \in f(\widetilde{M})$ and all open sets $U \subset \widetilde{M}$ with $p \in U$, such that $f|_U$ is an embedding and, hence, $f(U) \subset M$ a submanifold, $s|_{f(U)}$ has that property.

Remark 2.3 (Induced connection on pullback bundle)**.** If a vector bundle over a manifold $E \to M$ is endowed with a connection ∇ and we consider the (at least locally defined) associated pullback bundle along an immersion $f : \widetilde{M} \to M$, then we will use the same symbol ∇ for the induced connection in $\iota^*_{f(U)}E \to f(U)$, respectively $f|^*_U E \to U$. An example would be the induced connection along a curve in a manifold with nowhere vanishing velocity where ∇ is a connection in $TM \to M$.

2.2 Pseudo-Riemannian geometry and completeness theorems for Riemannian manifolds

In the following we will quickly review definitions and results from pseudo-Riemannian geometry, in particular completeness theorems that are used in this thesis.

We start with the most basic definitions.

Definition 2.4 (Pseudo-Riemannian manifold)**.** *Let M be a manifold and g a symmetric $(0,2)$-tensor field on M, that is $g \in \Gamma(\mathrm{Sym}^2 T^*M)$. The tupel (M, g) is called a* pseudo-Riemannian manifold *if $g_p = g|_{T_pM \times T_pM}$ is a non-degenerate bilinear form for all $p \in M$. If $g_p > 0$ for all $p \in M$, (M, g) is called a* Riemannian manifold*. The symmetric tensor field g is then called* pseudo-Riemannian metric*, respectively* Riemannian metric*.*

Definition 2.5 (Signature of a pseudo-Riemannian metric)**.** *Let (M, g) be a connected pseudo-Riemannian manifold. The* signature of *g is defined as the signature (i, j) of the bilinear form g_p for some some $p \in M$, i denoting the number of positive eigenvalues of g_p and j denoting the number of negative eigenvalues of g_p. Global non-degeneracy of g and M being connected implies that the signature is well-defined, that is, independent of $p \in M$.*

Riemannian manifolds (M, g) of dimension n have signature $(n, 0)$. Another class of pseudo-Riemannian manifolds are Lorentz manifolds, that is $(n + 1)$-dimensional pseudo-Riemannian manifolds with signature $(n, 1)$. Lorentz manifolds are of particular interest in the theory of general relativity, see for example [O] for an introduction.

Definition 2.6 (Isometry)**.** *Two pseudo-Riemannian manifolds (M, g) and $(\overline{M}, \overline{g})$ are called* isometric *if there exists a diffeomorphism $F : M \to \overline{M}$, such that $F^*\overline{g} = g$.*

Note that every manifold admits a Riemannian metric. This can be proven with the help of a partition of unity and adding up locally defined Riemannian metrics. For every pseudo-Riemannian manifold (M, g) there exists a unique, torsion free connection, such that g is parallel, called the Levi-Civita connection.

Definition 2.7 (Levi-Civita connection)**.** *Let (M, g) be a pseudo-Riemannian manifold. Then there exists a unique connection ∇ in $TM \to M$, called the* Levi-Civita connection*, such that* *(i)* $\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \Gamma(TM)$,

(ii) $\nabla g = 0$,

where in (ii) ∇ *denotes the induced connection in* $\mathrm{Sym}^2 T^*M \to M$.

An important formula used to calculate the components of the local 1-forms of the Levi-Civita connection is the following.

Lemma 2.8 (Koszul formula)**.** *The Levi-Civita connection of a pseudo-Riemannian manifold* (M, g) *is uniquely determined by the so-called Koszul formula*

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(Y, [X, Z]) - g(X, [Y, Z]) + g(Z, [X, Y])$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof. See for example [O, p. 61, Thm. 11]. □

For an n-dimensional pseudo-Riemannian manifold (M, g) and $(x_1, \ldots, x_n)$ local coordinates of M, the Koszul formula shows that in the induced local frame $(\partial_1, \ldots, \partial_n)$ of TM we have

$$\nabla_{\partial_i}\partial_j = \frac{1}{2}\sum_{k,\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})g^{\ell k}\partial_k \quad \forall 1 \leq i, j \leq n,$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $g^{\ell k} = g^{-1}(dx_i, dx_j)$. This leads to the following definition.

Definition 2.9 (Christoffel symbols)**.** *Let* (M, g) *be an* n*-dimensional pseudo-Riemannian manifold and* $(x_1, \ldots, x_n)$ *local coordinates of* M *with induced local frame* $(\partial_1, \ldots, \partial_n)$ *of* TM. *We define the Christoffel symbols* Γ_{ij}^k, $1 \leq i, j, k \leq n$, *of* (M, g) *in the given local coordinates to be*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})g^{\ell k}.$$

We will now present the most important invariants of pseudo-Riemannian manifolds, namely their different curvature tensors. For a reference on this topic see e.g. [KN, O].

Definition 2.10 (Curvature tensor)**.** *The pseudo-Riemannian curvature tensor of a pseudo-Riemannian manifold* (M, g) *with Levi-Civita connection* ∇ *is defined as*

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \forall X, Y, Z \in \Gamma(TM).$$

The Ricci curvature is defined as follows.

Definition 2.11 (Ricci curvature)**.** *Let* (M, g) *be a pseudo-Riemannian manifold and* R *its pseudo-Riemannian curvature tensor. The Ricci curvature* $\mathrm{Ric} \in \Gamma(\mathrm{Sym}^2 T^*M)$ *(also called Ricci tensor) of* (M, g) *is defined as*

$$\mathrm{Ric}(X, Y) := \mathrm{tr}(R(\cdot, X)Y) \quad \forall X, Y \in \Gamma(TM).$$

In the above formula, $R(\cdot, X)Y \in \Gamma(\mathrm{End}(TM))$ *for each pair* $X, Y \in \Gamma(TM)$ *and* $\mathrm{tr} : \Gamma(\mathrm{End}(TM)) \to C^\infty(M)$ *denotes the trace. In local coordinates* $(x_1, \ldots, x_n)$ *of* M *with induced local frame* $(\partial_1, \ldots, \partial_n)$ *of* TM, *the components of* Ric *are of the form*

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = \sum_a \left(\partial_a \Gamma_{ji}^a - \partial_j \Gamma_{ia}^a + \sum_k \left(\Gamma_{ij}^k \Gamma_{ak}^a - \Gamma_{ia}^k \Gamma_{jk}^a\right)\right).$$

Remark 2.12. The Ricci curvature (also called Ricci tensor) is central in the study of Einstein manifolds where one is concerned with pseudo-Riemannian manifolds (M, g), such that $\mathrm{Ric} = \lambda g$ for some constant λ. For a reference see [B].

Next we will define the scalar curvature, which is in practice an important tool to check whether two pseudo-Riemannian manifolds can be isometric or not by studying extremal points of their respective scalar curvature.

Definition 2.13 (Scalar curvature)**.** *The* scalar curvature $S \in C^\infty(M)$ *of a pseudo-Riemannian manifold* (M, g) *is defined as*

$$S := \mathrm{tr}_g(\mathrm{Ric}) = \mathrm{tr}(g^{-1} \circ \mathrm{Ric}),$$

where $g^{-1} : T^*M \to TM$ *is understood as a vector bundle isomorphism and* $\mathrm{Ric} : TM \to T^*M$ *is viewed as a vector bundle homomorphism. In local coordinates* $(x_1, \dots, x_n)$ *of* M *with induced local frame* $(\partial_1, \dots, \partial_n)$ *of* TM,

$$S = \sum_{i,j} \mathrm{Ric}_{ij} g^{ij} = \sum_{a,i,j} \left(\partial_a \Gamma^a_{ji} - \partial_j \Gamma^a_{ia} + \sum_k \left(\Gamma^k_{ij} \Gamma^a_{ak} - \Gamma^k_{ia} \Gamma^a_{jk} \right) \right) g^{ij}.$$

Another important curvature of pseudo-Riemannian manifolds is the sectional curvature.

Definition 2.14 (Sectional curvature)**.** *Let* (M, g) *be a Riemmanian manifold of dimension at least two. Let* $p \in M$ *be arbitrary,* $v, w \in T_pM$ *two linearly independent vectors, and denote* $E = \mathrm{span}\{v, w\} \subset T_pM$*. Then the* sectional curvature *of the* 2*-dimensional vector subspace* $E \subset T_pM$ *is defined as*

$$K(E) = K(v, w) := \frac{g(R(v, w)w, v)}{g(v, v)g(w, w) - g(v, w)^2}.$$

This definition is independent of the choice of the basis $\{v, w\}$ *of* E *which justifies the identification* $K(E) = K(v, w)$*.*

Remark 2.15. One can show that in any local orthogonal frame $(e_1, \dots, e_n)$ of TM,

$$S = \sum_{i \neq j} K(e_i, e_j).$$

Definition 2.16 (Length and velocity of a curve)**.** *For a Riemannian manifold* (M, g) *we define the* length of a curve $\gamma : I \to M$, I *a possibly unbounded interval, as*

$$\mathcal{L}(\gamma) := \int_I \sqrt{g_\gamma(\dot\gamma, \dot\gamma)} dt.$$

Notice that $\mathcal{L}(\gamma)$ *might be an improper integral and need not converge, so that* $\mathcal{L}(\gamma) = \infty$ *is allowed. In the latter case we say that* γ *has infinite length. The term* $\sqrt{g_\gamma(\dot\gamma, \dot\gamma)}$ *is called the velocity of* γ*.*

One main interest in the study of Riemannian manifolds is the question of geodesic completeness. We will present the necessary definitions to study this subject.

Definition 2.17 (Geodesic)**.** *Let* $\gamma : I \to M$ *be a curve in a pseudo-Riemannian manifold* (M, g) *defined on an open interval* I *and let* ∇ *denote the Levi-Civita connection of* (M, g)*. Then* γ *is called a* geodesic *of* (M, g) *if* $\nabla_{\dot\gamma}\dot\gamma = 0$.

Remark 2.18. Geodesics have many interesting properties. One can show that the velocity of a geodesic is constant and for a reparametrisation of the domain I of a geodesic, $f : I' \to I$, one can show that $\gamma \circ f$ is a geodesic if and only if f is affine-linear. In local coordinates $(x_1, \ldots, x_n)$ of a Riemannian manifold (M, g) the geodesic equation $\nabla_{\dot\gamma}\dot\gamma$ takes the form $\ddot\gamma_k + \sum\limits_{i,j} \Gamma^k_{ij}\dot\gamma_i\dot\gamma_j = 0$ for all $1 \le k \le n$, where $\gamma_i = x_i(\gamma)$ for $1 \le i \le n$. For a reference on classical results for geodesics see e.g. [O].

Theorem 2.19 (Hopf-Rinow)**.** *Let (M, g) be a Riemannian manifold. Then the following are equivalent:*

(i) M is complete as a metric space.

(ii) M is geodesically complete, i.e. all geodesics are defined for all times.

(iii) Closed and bounded subsets of M are compact.

Proof. See for example [Jo, Thm. 1.7.1, p. 35]. □

Theorem 2.19 justifies to talk simply about completeness of a Riemannian manifold (M, g) instead of always writing geodesic or metric completeness, respectively. Whenever there are other connections involved in the discussion of a Riemannian, completeness will always mean completeness with respect to the Levi-Civita connection.

Completeness is, in general, hard to prove or disprove. One very useful fact in Riemannian geometry is the following.

Lemma 2.20. *A Riemannian manifold (M, g) is complete if and only if every curve in M that leaves every compact subset of M has infinite length.*

Proof. [CHM, Lem. 1]. □

Lemma 2.20 yields another way to describe complete Riemannian manifolds.

Lemma 2.21. *A Riemannian manifold (M, g) is complete if and only if there exists $r > 0$, such that for all $p \in M$ the closure of the geodesic ball of radius r around p with respect to g, i.e. the set $\overline{B^g_r(p)}$, is a compact subset of M.*

Proof. Assume that (M, g) is complete. Then Thm. 2.19 (ii) implies that for all $r > 0$ and all $p \in M$, the geodesic ball $B^g_r(p) \subset M$ is bounded. Thm. 2.19 (iii) now implies that $\overline{B^g_r(p)}$ is compact for all $r > 0$ and all $p \in M$.

For the other direction of the proof assume that (M, g) is incomplete. Then there exists a geodesic $\gamma : (0, 1) \to M$ of finite length, such that γ leaves every compactum in M. Without loss of generality assume that $\gamma(t)$ converges to some $p \in M$ as $t \to 0$, and let $\mathcal{L}(\gamma) < \infty$ denote the length of γ. Then $\overline{B^g_{\mathcal{L}(\gamma)}(p)} \subset M$ is not compact, since otherwise it would be contained in some compactum in M which is excluded by the assumption that γ leaves every compactum in M. □

2.3 Centro-affine geometry

Now we will give a short introduction to affine differential geometry and specifically centro-affine differential geometry. In most generality, one considers the following, cf. [NS, p. 27].

Definition 2.22 (Distribution along an immersion)**.** *For an immersion $f : M \to \overline{M}$ between two manifolds $M, \overline{M}$ with $\dim \overline{M} > \dim M$ and $k \in \mathbb{N}$, a k-dimensional distribution along f is an assignment $M \ni x \to N_x \subset T_{f(x)}\overline{M}$, such that around each point $p \in M$ we can find an open neighbourhood $U \subset M$, such that $f|_U : U \to \overline{M}$ is an embedding, and k pointwise linearly independent vector fields $\{X_1, \dots, X_k\}$, $X_i \in \Gamma\left(T\overline{M}\right)\Big|_{f(U)}$ for all $1 \leq i \leq k$, with the property that for all $x \in U$ we have that $N_x = \operatorname{span}\{X_1, \dots, X_k\}$.*

Definition 2.23 (Affine immersion)**.** *Let (M, ∇) and $(\overline{M}, \overline{\nabla})$ be two manifolds of dimension $\dim(M) = m$ and $\dim\left(\overline{M}\right) = n$ with torsion-free covariant derivatives ∇ in $TM \to M$ and $\overline{\nabla}$ in $T\overline{M} \to \overline{M}$. Assume that $n > m$. An immersion $f : M \to \overline{M}$ is called affine immersion if there exists a $k = (n-m)$-dimensional distribution N along f and a N-valued $(0,2)$-tensor field $\alpha \in \Gamma(T^*M \otimes T^*M \otimes N)$, that is $\alpha(X,Y)|_p \in N_p$ for all $X, Y \in \Gamma(TM)$ and all $p \in M$, such that*

(i) $T_{f(p)}M = df_p(T_pM) \oplus N_p$,

(ii) $\overline{\nabla}_X(df(Y)) = df(\nabla_X Y) + \alpha(X,Y)$

for all $X, Y \in \Gamma(TM)$ and all $p \in M$.

Note that in [NS], f is only assumed to be differentiable. In thesis all considered immersions are smooth. A special case of affine immersions are affine hypersurface immersions, i.e. affine immersions of co-dimension 1. We are interested in the case where the ambient manifold $\overline{M}$ is $\mathbb{R}^{n+1}$ endowed with the standard flat connection.

Definition 2.24 (Affine hypersurface immersions)**.** *An affine hypersurface immersion in $\mathbb{R}^{n+1}$ is an affine immersion $f : M \to \mathbb{R}^{n+1}$ of an n-dimensional manifold M into $\mathbb{R}^{n+1}$. The corresponding 1-dimensional distribution is locally spanned by a non-vanishing vector field ξ along f that is transversal to $f(M)$ at each point.*

On the other hand, one might consider a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$ with a given transversal 1-dimensional distribution along f and ask for a torsion-free connection in TM, such that f is an affine immersion. This is the content of the following proposition, cf. [NS, p. 29].

Proposition 2.25 (Gauß formula for hypersurface immersions)**.** *Let $f : M \to \mathbb{R}^{n+1}$ be a hypersurface immersion, $\{U_i \mid i \in I\}$ an open covering of M, and $\xi_i \in \Gamma_{f(U_i)}(T\mathbb{R}^{n+1})$ locally defined transversal vector fields along f that generate a 1-dimensional distribution along f. Let $\overline{\nabla}$ denote the standard flat connection in $T\mathbb{R}^{n+1}$. Then there exists a torsion-free connection ∇ in TM and for each $i \in I$ a symmetric $(0,2)$-tensor field $h_i \in \Gamma(S^2T^*M)_{U_i}$, such that*

$$\overline{\nabla}_X(df(Y)) = df(\nabla_X Y) + h_i(X,Y)\xi_i \quad \forall X, Y \in \Gamma(TM)|_{U_i}, \forall i \in I. \tag{2.1}$$

Equation (2.1) is called (affine) Gauß equation. With this choice of ∇, $f : M \to \mathbb{R}^{n+1}$ is an affine hypersurface immersion as defined in Definition 2.24. ∇ is called the induced affine connection. Note that ∇ is independent of $i, j \in I$ whenever $U_i \cap U_j \neq \emptyset$.

Next we will see how to differentiate the transversal part of (local) sections in $f^*T\mathbb{R}^{n+1}$, cf. [NS, p. 30].

Proposition 2.26 (Weingarten equation for affine hypersurface immersions)**.** *Let $f : M \to \mathbb{R}^{n+1}$ be an affine hypersurface immersion as in Definition 2.24. Then there is a uniquely defined $(1,1)$-tensor $S \in \Gamma(\mathrm{End}(TM))$ and a collection of 1-forms $\tau_i \in \Gamma(T^*M)|_{U_i}$, $i \in I$, satisfying*

$$\overline{\nabla}_X \xi_i = -df(SX) + \tau_i(X)\xi_i \quad \forall X \in \Gamma(TM), \forall i \in I. \tag{2.2}$$

Equation (2.2) is called the (affine) Weingarten equation, the tensor S is called (affine) shape operator (or affine Weingarten map), and each τ_i (local) transversal connection 1-form.

Now that we have introduced general concepts of affine differential geometry, we will consider the special case of centro-affine hypersurface immersions. The main part of this thesis considers hypersurface immersions or, more precisely, hypersurface embeddings of that type.

Definition 2.27 (Centro-affine hypersurface immersion)**.** *Let $f = (f_1, \dots, f_{n+1})^T : M \to \mathbb{R}^{n+1}$ be a hypersurface immersion. It is called a centro-affine hypersurface immersion if the position vector field $\xi \in \Gamma(T\mathbb{R}^{n+1})$, $\xi_p = p$ for all $p \in \mathbb{R}^{n+1}$ under the canonical identification, is transversal along f, that is*

$$df(T_pM) \oplus \mathbb{R}\xi_{f(p)} = T_{f(p)}\mathbb{R}^{n+1} \quad \forall p \in M,$$

where $\mathbb{R}\xi_{f(p)}$ denotes the 1-dimensional vector subspace spanned by $\xi_{f(p)}$ of $T_{f(p)}\mathbb{R}^{n+1}$. Whenever f is clear from the context, we will call M a centro-affine hypersurface.

If f is additionally an embedding, it will be called a centro-affine hypersurface embedding. In the case of centro-affine hypersurface immersions, the Weingarten equation (2.2) takes a particularly simple form

Lemma 2.28 (Weingarten for centro-affine hypersurface immersions)**.** *Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface immersion. Then the affine shape operator fulfils $S = -\mathrm{Id}$ and all local transversal connection 1-forms vanish identically.*

Proof. For any locally defined position vector field ξ defined on $f(U) \subset \mathbb{R}^{n+1}$ and all $X \in \Gamma_U(T\mathbb{R}^{n+1})$ we obtain

$$\overline{\nabla}_X \xi = \overline{\nabla}_X(f) = df(X).$$

Comparing this result with the Weingarten equation (2.2) in Proposition 2.26 proves our claim. □

The Gauß equation (2.1) in Definition 2.25 for centro-affine hypersurface immersions $f : M \to \mathbb{R}^{n+1}$ is of the form

$$\overline{\nabla}_X(df(Y)) = df(\nabla_X Y) + g(X,Y)\xi_f, \tag{2.3}$$

where ξ_f denotes the position vector field along f. This leads to the following definition.

Definition 2.29 (Centro-affine connection and centro-affine fundamental form)**.** *Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersuface immersion. The induced connection ∇ in TM (2.3) is called the centro-affine connection, the symmetric $(0,2)$-tensor $g \in \Gamma\left(\mathrm{Sym}^2 T^*M\right)$ is called the centro-affine fundamental form.*

Depending on the signature of the centro-affine fundamental form, centro-affine hypersurfaces are classified as follows.

Definition 2.30 (Types of centro-affine hypersurface immersions). *A centro-affine hypersurface immersion $f : M \to \mathbb{R}^{n+1}$ with centro-affine fundamental form g is called*

- non-degenerate, *if g is non-degenerate,*
- definite, *if g is definite, i.e. either positive or negative definite,*
- elliptic, *if $g < 0$, i.e. negative definite,*
- hyperbolic, *if $g > 0$, i.e. positive definite.*

In this thesis we are interested in certain hyperbolic cases which we will introduce next.

2.4 Projective special real manifolds and other examples of centro-affine manifolds

After the introduction centro-affine geometry we will present examples of centro-affine hypersurface immersions. We will discuss examples of hyperbolic centro-affine hypersurface immersions and related questions from Riemannian geometry. In particular, we will introduce projective special real manifolds, which are one of the main objects of our studies in this thesis.

Proposition 2.31. *Let $U \subset \mathbb{R}^{n+1}$, $n \in \mathbb{N} \cup \{0\}$, be an open set invariant under positive rescaling, i.e. the $\mathbb{R}_{>0}$-action $(r, p) \mapsto rp$ for all $r \in \mathbb{R}_{>0}$ and $p \in U$. Let $h : U \to \mathbb{R}$ be a homogeneous function of degree $k > 1$, i.e. $h(rp) = r^k h(p)$. Assume that the level set $\{p \in U \mid h(p) = 1\}$ is not empty and let $\mathcal{H} \subset \{p \in U \mid h(p) = 1\}$ be an open subset. Then the inclusion map $\iota : \mathcal{H} \to \mathbb{R}^{n+1}$ is a centro-affine hypersurface embedding with centro-affine fundamental form $g = -\frac{1}{k}\iota^*(\overline{\nabla}^2 h)$, where $\overline{\nabla}$ denotes the canonical flat connection in $T\mathbb{R}^{n+1}$ and $\overline{\nabla}^2$ its Hessian.*

Proof. For a proof of this statement in a slightly more general setting see [CNS, Prop. 1.3]. □

If $\mathbb{R}^{n+1}$ is equipped with linear coordinates, we will write ∂^2 instead of $\overline{\nabla}^2$. We will also omit writing down the map ι for an embedding $\iota : M \to \mathbb{R}^{n+1}$, that is we will write $M \subset \mathbb{R}^{n+1}$ instead of $\iota(M) \subset \mathbb{R}^{n+1}$, if the context is clear. In this thesis we are interested in hypersurface embeddings as above where the centro-affine fundamental form g is a Riemannian metric on an open subset $\mathcal{H} \subset \{h = 1\}$ and h is a homogeneous polynomial of degree $\tau \geq 3$. We will now introduce concepts needed for our studies of said hypersurfaces.

Remark 2.32 (Euler identity for homogeneous functions). Let U be an open subset of $\mathbb{R}^{n+1}$ invariant under multiplication with positive real numbers and let $h : U \to \mathbb{R}$ be a homogeneous function of homogeneity-degree $\tau \in \mathbb{R}$. Then

$$dh_x(x) = \tau h(x) \quad \forall x \in U. \tag{2.4}$$

Equation (2.4) is called the Euler identity for homogeneous functions.

Definition 2.33 (Hyperbolic point). *Let $U \subset \mathbb{R}^{n+1}$ be an open subset that is invariant under multiplication with positive real numbers, and let $h : U \to \mathbb{R}$ be a homogeneous function of degree $\tau > 1$. Then a point $p \in \{h > 0\}$ is called a* hyperbolic point (of h) *if $-\partial^2 h_p$ has signature $(n, 1)$, i.e. it is of Lorentz type. A function h that has at least one hyperbolic point is called a* hyperbolic homogeneous function.

Note that this implies that for a hyperbolic point p of h, $-\partial^2 h_p|_{\ker(dh_p)\times\ker(dh_p)} > 0$, which follows from $-\partial^2 h_p(p,p) = -\tau(\tau-1)h(p) < 0$ and $-\partial h_p(p,\cdot) = -(\tau-1)dh_p$.

Definition 2.34 (Hyperbolic centro-affine hypersurface)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a centro-affine hypersurface as in Proposition 2.31. Then $\mathcal{H}$ is called a* hyperbolic centro-affine hypersurface *if it consists only of hyperbolic points.*

Note that the above definition of hyperbolic centro-affine hypersurface coincides with Definition 2.30 for f the inclusion map $\iota : \mathcal{H} \to \mathbb{R}^{n+1}$. Hyperbolic centro-affine hypersurfaces equipped with their respective centro-affine fundamental form $(\mathcal{H}, g)$ are Riemannian manifolds. Continuity of the determinant implies that a connected non-degenerate centro-affine hypersurface $\mathcal{H}$ is hyperbolic if and only if it contains one hyperbolic point. Note that hyperbolicity at a certain point is an open condition in the sense every homogeneous function $h : U \to \mathbb{R}$ as in Definition 2.33 with a hyperbolic point p is hyperbolic on some open neighbourhood $V \subset U$ of p. This follows from the continuity of the determinant of $-\partial^2 h$. Hence, for every hyperbolic homogeneous function h of degree $\tau > 1$ we can choose an open subset $\mathcal{H} \subset \{h = 1\}$ that is a hyperbolic centro-affine hypersurface.

We are in particular interested in the case where h is additionally assumed to be a polynomial. We define the following.

Definition 2.35 (Hyperbolic polynomial)**.** *A homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $\tau \geq 2$ is called a* hyperbolic homogeneous polynomial *if there exists a $p \in \{h > 0\}$, such that p is a hyperbolic point of h.*

Note that Definition 2.35 in comparison with the more general Definition 2.33 does not depend on a chosen domain for a given polynomial h. We will now discuss the easiest example for a hyperbolic centro-affine hypersurface defined by a hyperbolic polynomial.

Example 2.36 (Two-sheeted hyperboloid)**.** *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$, $h = x_{n+1}^2 - \sum\limits_{i=1}^{n} x_i^2$. Then every point in $\{h > 0\}$ is a hyperbolic point, which follows from*

$$-\partial^2 h = 2\left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -1 \end{array}\right).$$

Each of the two components of $\{h = 1\}$, namely $\{h = 1,\ x_{n+1} > 0\}$ and $\{h = 1,\ x_{n+1} < 0\}$, are hyperbolic centro-affine hypersurfaces. For $n = 2$, the set $\{h = 1\}$ is called the two-sheeted hyperboloid *(see Figure 1).*

A question that might come to mind in this setting is whether there are other hyperbolic polynomials of degree 2 that define a hyperbolic centro-affine hypersurface. To deal with this question, we need a notion of when two hyperbolic hypersurfaces contained in the level set of hyperbolic polynomials are considered equivalent.

Definition 2.37 (Equivalence of hyperbolic polynomials)**.** *Two hyperbolic homogeneous polynomials $h, \overline{h} : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $\tau \geq 2$ are called* equivalent *if there exists a linear transformation $A \in \mathrm{GL}(n+1)$, such that $h \circ A = \overline{h}$. Two connected hyperbolic centro-affine hypersurfaces $\mathcal{H}$ and $\overline{\mathcal{H}}$ contained in a level set of h, respectively $\overline{h}$, are called equivalent if h and $\overline{h}$ are equivalent and $A(\overline{\mathcal{H}}) \subset \mathcal{H}$ or $A(\overline{\mathcal{H}}) \supset \mathcal{H}$.*

We usually consider the following type of hyperbolic centro-affine hypersurfaces.

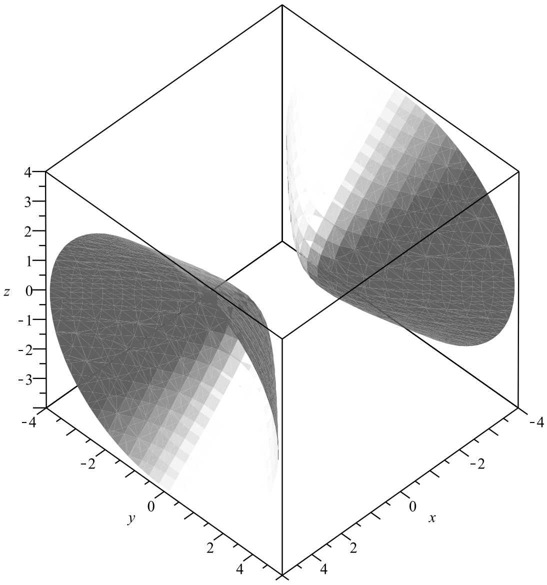


Figure 1: A rendering of a part of the two-sheeted hyperboloid embedded in $\mathbb{R}^3$.

Definition 2.38 (Maximal connected hyperbolic centro-affine hypersurfaces)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be a connected hyperbolic centro-affine hypersurface as in Definition 2.34. Then $\mathcal{H}$ is called a* maximal *(or* maximally extended*)* connected hyperbolic centro-affine hypersurface *if it coincides with a maximal subset of $\{h = 1\}$ that consists only of hyperbolic points, i.e. if it is a connected component of the set*

$$\left\{p \in \mathbb{R}^{n+1} \mid h(p) = 1,\ p \textit{ is a hyperbolic point of } h\right\}.$$

Note that the continuity of $\det(-\partial^2 h)$ and Proposition 2.31 ensure that connected component of $\{p \in \mathbb{R}^{n+1} \mid h(p) = 1,\ p \text{ is a hyperbolic point of } h\}$ are always open in $\{h = 1\}$ with respect to the induced subspace topology of $\{h = 1\} \subset \mathbb{R}^{n+1}$. For maximal connected hyperbolic centro-affine hypersurfaces, the terms $A\left(\overline{\mathcal{H}}\right) \subset \mathcal{H}$ and $A\left(\overline{\mathcal{H}}\right) \supset \mathcal{H}$ in Definition 2.37 simply become $A\left(\overline{\mathcal{H}}\right) = \mathcal{H}$. Furthermore, we obtain the following lemma.

Lemma 2.39 (Isometry of equivalent hypersurfaces)**.** *Any two equivalent maximal connected centro-affine hyperbolic hypersurfaces $\mathcal{H}$ and $\overline{\mathcal{H}}$ defined by hyperbolic polynomials $h, \overline{h} : \mathbb{R}^{n+1} \to \mathbb{R}$, respectively, are isometric.*

Proof. Let $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a linear transformation, such that $h \circ A = \overline{h}$. Then the linearity of A implies

$$-\partial^2 \overline{h}_p(\cdot, \cdot) = -\partial^2 h_{Ap}(A\cdot, A\cdot) = A^*(-\partial^2 h)_p.$$

In particular, this hold for the restrictions to $T\overline{\mathcal{H}}$, respectively $T\mathcal{H}$, that is for their respective centro-affine fundamental forms. This shows that $\mathcal{H}$ and $\overline{\mathcal{H}}$ are isometric and one isometry is given by the respective linear transformation A relating their respective defining polynomial h and $\overline{h}$. $\square$

Equivalence classes of bilinear forms on $\mathbb{R}^{n+1}$ are determined by their signature. Hence, one easily obtains the following.

Lemma 2.40. *Let $\mathcal{H} \subset \{h = 1\}$ be a connected maximal hyperbolic centro-affine hypersurface and $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a hyperbolic polynomial of degree 2. Then $\mathcal{H}$ is equivalent to*

$\left\{x_{n+1}^2 - \sum\limits_{i=1}^{n} x_i^2 = 1 \;\middle|\; x_{n+1} > 0\right\}$, *that is to one sheet of the two-sheeted hyperboloid defined in Example 2.36.*

Note that in the case of the two-sheeted hyperboloid, the centro-affine metric g and the Riemannian metric of the two-sheeted hyperboloid induced by the embedding into $(n+1)$-dimensional Minkowski space via the inclusion map, that is the second fundamental form $\mathrm{II} \in \Gamma(\mathrm{Sym}^2 T^*\mathcal{H})$ with respect to a unit normal, coincide for all $n \geq 1$.

One central interest of this thesis are so-called projective special real manifolds, which we will define now.

Definition 2.41 (Projective special real manifold)**.** *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a cubic hyperbolic homogeneous polynomial. An open subset $\mathcal{H} \subset \{h = 1\}$ that consists only of hyperbolic points is called a* projective special real manifold*, or* PSR manifold *for short.*

We immediately obtain the following properties of PSR manifolds.

Lemma 2.42 (PSR manifolds are hyperbolic centro-affine hypersurfaces)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be a PSR manifold. Then $\mathcal{H}$ is a hyperbolic centro-affine hypersurface as defined in Definition 2.34 and their centro-affine fundamental form as in Definition 2.29 in chosen linear coordinates of the ambient space $\mathbb{R}^{n+1}$ is given by*

$$g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}.$$

Proof. This follows from Proposition 2.31. □

Two connected PSR manifolds $\mathcal{H} \subset \{h = 1\}$ and $\overline{\mathcal{H}} \subset \{\overline{h} = 1\}$ are called equivalent if they are equivalent as in Definition 2.37. A connected PSR manifold $\mathcal{H} \subset \{h = 1\}$ is called maximal (or maximally extended) if it is maximal in the sense of Definition 2.38. In particular, equivalent maximal connected PSR manifolds are isometric.

We will now discuss known results in the study of PSR manifolds. Since PSR manifolds are Riemannian manifolds, it is a natural question whether they are always complete or not, where completeness means geodesically complete with respect to the Levi-Civita connection of the centro-affine fundamental form. Note that completeness of a given PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ automatically implies that $\mathcal{H}$ needs to be closed as a subset of $\mathbb{R}^{n+1}$ since otherwise one can extend its centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ smoothly to its boundary points. This would imply that there are curves leaving each compact set in $\mathcal{H}$ with finite length which contradicts completeness, cf. Lemma 2.20. Hence, a necessary condition for completeness of $(\mathcal{H}, g_{\mathcal{H}})$ is that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed. We will call such a PSR manifold a closed PSR manifold. It has recently been shown in [CNS] that closed PSR manifolds are always complete.

Theorem 2.43. *An n-dimensional PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is complete with respect to its centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ if and only if $\mathcal{H}$ is closed as a subset of $\mathbb{R}^{n+1}$.*

Proof. [CNS, Thm. 2.5]. □

In Propositions 4.17 and 5.17 we give two alternative proofs of Theorem 2.43.

Remark 2.44 (Difficulties in classifying closed connected PSR manifolds)**.** One interesting questions is to ask if it is possible to classify all closed connected PSR manifolds. In general, it turns out to be a very difficult question. This problem is equivalent to classifying all cubic hyperbolic homogeneous polynomials up to equivalence. One of the encounterd difficulties is that being hyperbolic as a cubic homogeneous polynomial is an open condition in the sense that if $h \in \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$ is hyperbolic and $H \in \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$ is any cubic polynomial, then there exists an $\varepsilon > 0$, such that for all $0 \leq k \leq \varepsilon$ the polynomial $h + kH$ is hyperbolic. This follows easily from Sylvester's law of inertia. Furthermore, the dimension of $\mathrm{Sym}^3(\mathbb{R}^{n+1})^*$ grows cubically in n while the dimension of $\mathrm{GL}(n+1)$ grows quadratically in n, so we can not expect to have only finitely many examples as n grows large. In dimensions $n = 1$ and $n = 2$ however, cubic hyperbolic homogeneous polynomials in 2 and 3 variables, respectively, and the corresponding closed connected PSR manifolds have been classified up to equivalence, see [CHM] for 1-dimensional PSR manifolds and [CDL] for 2-dimensional PSR manifolds.

The known classification results for projective special real curves and projective special real surfaces are as follows.

Theorem 2.45 (Classification of closed connected PSR curves and surfaces)**.** *Every closed connected PSR curve $\mathcal{H}$, that is closed connected PSR manifold of dimension one, is equivalent to exactly one of the following closed connected PSR curves:*

A) $\{x^2 y = 1,\ x > 0,\ y > 0\}$,

B) $\{x\,(x^2 - y^2) = 1,\ x > 0\}$,

where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ *denote linear coordinates of the ambient space* $\mathbb{R}^2$.

For closed connected PSR surfaces $\mathcal{H} \subset \mathbb{R}^3$, that is closed connected PSR manifold of dimension two, let $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ *denote the linear coordinates of the ambient space $\mathbb{R}^3$. Each such $\mathcal{H}$ is equivalent to exactly one of the following closed connected PSR surfaces:*

a) $\{xyz = 1,\ x > 0,\ y > 0\}$,

b) $\{x(xy - z^2) = 1,\ x > 0\}$,

c) $\{x(yz + x^2) = 1,\ x < 0,\ y > 0\}$,

d) $\{z(x^2 + y^2 - z^2) = 1,\ z < 0\}$,

e) $\{x(y^2 - z^2) + y^3 = 1,\ y < 0,\ x > 0\}$,

f) $\{y^2 z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0,\ 2x > z\}$ *for precisely one* $b \in (-1, 1)$.

Proof. See [CHM, Thm. 8] for curves, [CDL, Thm. 1] for surfaces. □

Aside from the low-dimensional restriction, another restriction to PSR manifolds is to consider only those that are contained in the level set of a reducible cubic hyperbolic homogeneous polynomial. In this case, PSR manifolds are classified in any dimension, cf. [CDJL]. Since 1- and 2-dimensional PSR manifolds are completely classified, only $n \geq 3$-dimensional PSR manifolds with reducible polynomial are considered in the following Theorem.

Theorem 2.46 (Classification of closed connected PSR manifolds corresponding to reducible polynomials)**.** *Every closed connected PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ of dimension $n \geq 3$ for which h is reducible is linearly equivalent to exactly one of the following closed connected PSR manifolds*

a) $\left\{x_{n+1}\left(\sum\limits_{i=1}^{n-1} x_i^2 - x_n^2\right) = 1,\ x_{n+1} < 0,\ x_n > 0\right\}$, *b)* $\left\{(x_1+x_{n+1})\left(\sum\limits_{i=1}^{n} x_i^2 - x_{n+1}^2\right)=1,\ x_1+x_{n+1}<0\right\}$,

c) $\left\{x_1\left(\sum\limits_{i=1}^{n} x_i^2 - x_{n+1}^2\right)=1,\ x_1<0,\ x_{n+1}>0\right\}$,

d) $\left\{x_1\left(x_1^2-\sum\limits_{i=2}^{n+1} x_i^2\right)=1,\ x_1>0\right\}$.

Proof. [CDJL, Thm. 2]. □

Remark 2.47. Theorem 2.46 is a combined result of [Ju] and [Li]. Results of these two works are also part of [CDJL, Thm. 2, Prop. 8].

Lastly, there is a classification of PSR manifolds that are homogeneous spaces under the action of their respective automorphism groups, cf. Definition 3.13, for which we refer the reader to [DV].

3 Standard form and curvature of generalized projective special real manifolds

In this section we will study hyperbolic homogeneous polynomials of degree $\tau \geq 3$ and the corresponding hypersurfaces contained in their level sets. These geometric object can be viewed as a generalisation of PSR manifolds. The "machinery" and results of this section will be used extensively in the following sections.

Definition 3.1 (GPSR and CCGPSR manifolds)**.** *Let $n \in \mathbb{N} \cup \{0\}$ and $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an n-dimensional hyperbolic centro-affine hypersurface as in Definition 2.34, contained in the level set of a hyperbolic homogeneous polynomial of degree $\tau \geq 3$. Then $\mathcal{H}$ will be called GPSR manifold (for Generalised Projective Special Real manifold). If we further assume that $\mathcal{H}$ is closed and connected as a subset of $\mathbb{R}^{n+1}$, we will call $\mathcal{H}$ a CCGPSR manifold (for Closed Connected Generalised Projective Special Real manifold) of degree τ. For $\tau = 3$, GPSR manifolds coincide with PSR manifolds defined in Definition 2.41. If $\mathcal{H}$ is a closed connected PSR manifold and we will call it a CCPSR manifold. As a convention we regard the set of CCPSR manifolds as a subset of the set of CCGPSR manifolds.*

If the degree $\tau \geq 3$ of a GPSR manifold is not of particular importance, we will omit the phrase "of degree τ". Recall that according to Definition 2.37, two CCGPSR manifolds of the same degree are called equivalent if they are related by a linear change of coordinates of the ambient space.

Definition 3.2 (Moduli space of CCGPSR manifolds)**.** *Let $n \in \mathbb{N}$. We define the moduli space of n-dimensional CCGPSR manifolds of degree τ to be the set of equivalence classes*

$$\{[\mathcal{H}] \mid \mathcal{H} \text{ is a } CCGPSR \text{ manifold of degree } \tau,\ \dim(\mathcal{H}) = n\},$$

where $[\widetilde{\mathcal{H}}] = [\mathcal{H}]$ if and only if $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ are equivalent. For $\tau = 3$, we will call the above set the moduli space of n-dimensional CCPSR manifolds.

Note that for $n = 0$, there is for each degree $\tau \geq 3$ precisely one CCGPSR manifold up to equivalence, which is simply a point.

Remark 3.3. We will consider the moduli space of n-dimensional CCGPSR manifolds in general without the assumption of any topological data and view it simply as a set. For a discussion why it is difficult to find a meaningful topology for that space, see Remark 6.14 later in this thesis.

Lemma 3.4 (Centro-affine fundamental form of GPSR manifolds)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an n-dimensional GPSR manifold of degree $\tau \geq 3$. Then its centro-affine fundamental form $g_{\mathcal{H}}$ is given by*

$$g_{\mathcal{H}} = -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}, \tag{3.1}$$

where ∂^2 is determined by the chosen linear coordinates on the ambient space $\mathbb{R}^{n+1}$.

Proof. This follows immediately from Proposition 2.31. □

Definition 3.5 (Maximal connected GPSR manifold)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a connected GPSR manifold. We will call $\mathcal{H}$ a maximal (or maximally extended) connected GPSR manifold if it is a maximal connected hyperbolic centro-affine hypersurface in the sense of Definition 2.38. If $\mathcal{H} \subset \mathbb{R}^{n+1}$ is furthermore closed, we will call it a maximal (or maximally extended) CCGPSR manifold.*

Remark 3.6 (CCGPSR manifolds are maximal)**.** Any CCGPSR manifold $\mathcal{H} \subset \{h = 1\}$ coincides by definition with a connected component of $\{h = 1\}$ and is thus automatically maximal in the sense of Definition 3.5.

Note that for $\tau = 3$, Theorem 2.43 shows that n-dimensional CCPSR manifolds are precisely n-dimensional complete connected PSR manfolds. We will now show that we can, after a possible linear coordinate change of the ambient space $\mathbb{R}^{n+1}$, assume that the defining polynomial is of a certain form. To do so we first review two results from [CNS] that apply in particular to the geometry of CCGPSR manifolds.

Proposition 3.7 (Convexity of the cone spanned by CCGPSR manifolds)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an n-dimensional CCGPSR manifold. Then*

$$U = \mathbb{R}_{>0} \cdot \mathcal{H} = \left\{ rp \in \mathbb{R}^{n+1} \;\middle|\; r > 0,\ p \in \mathcal{H} \right\} \subset \mathbb{R}^{n+1}$$

is a convex cone and the map

$$\mathbb{R}_{>0} \times \mathcal{H} \ni (r, p) \mapsto r \cdot p \in U$$

is a diffeomorphism.

Proof. [CNS, Prop. 1.10] for the special case of CCGPSR manifolds. □

Lemma 3.8. *Let $\mathcal{H}$ be a CCGPSR manifold and let $U = \mathbb{R}_{>0} \cdot \mathcal{H}$. Then for every $p \in \mathcal{H}$, the intersection*

$$(p + T_p\mathcal{H}) \cap U \subset p + T_p\mathcal{H}$$

is open, precompact, and convex. Here $(p + T_p\mathcal{H}) \subset \mathbb{R}^{n+1}$ denotes the affinely embedded tangent space $T_p\mathcal{H}$ in the ambient vector space $\mathbb{R}^{n+1}$ equipped with the induced subspace topology.

Proof. [CNS, Lem. 1.14]. □

Definition 3.9 (Homogeneous connected GPSR manifolds)**.** *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a connected GPSR manifold. We call $\mathcal{H}$ a* homogeneous connected GPSR manifold *if there exists a Lie group G acting transitively on $(\mathcal{H}, g_{\mathcal{H}})$ via isometries.*

Note that in Definition 3.9 we do not require the action to be linear. In fact, we consider connected GPSR manifolds to be homogeneous if they are homogeneous as Riemannian manifolds.

Remark 3.10 (Completeness of homogeneous connected GPSR manifolds)**.** Recall that homogeneous Riemannian manifolds are always complete, cf. [KN, Thm. 4.5] or [BEE, Lem. 5.4]. Thus, in particular homogeneous connected GPSR manifolds are complete.

An immediate consequence of Remark 3.10 is the following corollary.

Corollary 3.11 (Homogeneous connected GPSR manifolds are CCGPSR manifolds)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be a homogeneous connected GPSR manifold. Then $\mathcal{H}$ is a CCGPSR manifold.*

Proof. $(\mathcal{H}, g_{\mathcal{H}})$ is complete and, hence, closed in $\mathbb{R}^{n+1}$, cf. [CNS, Prop. 1.8]. □

Homogeneous CCGPSR manifolds provide interesting examples of CCGPSR manifolds of homogeneity degree $\tau \geq 4$, since in general as of now it is unknown if CCGPSR manifolds of homogeneity degree $\tau \geq 4$ are always complete. It has however been shown that every one-dimensional CCGPSR manifold is complete, hence in particular every homogeneous one-dimensional CCGPSR manifold is complete [CNS, Thm. 2.9].

Proposition 3.12. *Let $\mathcal{H} \subset \{h = 1\}$ be an n-dimensional connected GPSR manifold with h of homogeneity-degree $\tau \geq 3$. Then*

$$G^h := \left\{ M \in \mathrm{Mat}(n \times n, \mathbb{R}) \;\middle|\; h(p) = h(Mp)\ \forall p \in \mathbb{R}^{n+1} \right\}$$

is a Lie subgroup of $\mathrm{GL}(n+1)$ *with Lie algebra*

$$T_{\mathbb{1}} G^h = \left\{ m \in \mathfrak{gl}(n+1) \;\middle|\; dh_p(mp) = 0\ \forall p \in \mathbb{R}^{n+1} \right\}. \tag{3.2}$$

Proof. The set G^h contains $\mathbb{1} \in \mathrm{GL}(n+1)$, and $M, N \in G^h$ implies $h(MNp) = h(Np) = h(p)$ for all $p \in \mathbb{R}^{n+1}$. For G^h to be a subgroup of $\mathrm{GL}(n+1)$ it thus suffices to show that G^h is a subset of $\mathrm{GL}(n+1)$. Suppose that it is not. Then there exists an element $M \in G^h$, such that $\mathrm{rk}(M) < n+1$. By assumption the level set $\{h = 1\}$ contains the connected GPSR manifold $\mathcal{H}$, h is a hyperbolic polynomial and there exist a point $q \in \mathbb{R}^{n+1}$ with $\det\left(-\partial^2 h_q\right) < 0$. Since $h(p) = h(Mp)$, we obtain the identity

$$-\partial^2 h_q = -M^T \partial^2 h_{Mq} M.$$

But then $\det\left(-\partial^2 h_q\right) = \det(M)^2 \det\left(-\partial^2 h_q\right) = 0$, which is a contradiction to q being a hyperbolic point of h. Thus, G^h is a subset of $\mathrm{GL}(n+1)$ and we conclude that $G^h \subset \mathrm{GL}(n+1)$ is a subgroup. In order to show that G^h is also a Lie subgroup of $\mathrm{GL}(n+1)$, we will use the closed subgroup theorem, which was first proven in [Ca] (for a modern reference see [Le, Thm. 20.12]). With $p = (p_1, \ldots, p_{n+1})^T$ and $M = (M_{ij}) \in \mathrm{Mat}(n \times n, \mathbb{R})$, the equation $h(p) - h(Mp) = 0$ is of the form

$$\sum_{|I|=\tau} f_I \prod_{i=1}^{n+1} p_i^{I_i} = 0,$$

where $I = (I_1, \ldots, I_{n+1})$ denotes a multi-index with $I_i \geq 0$ for all $1 \leq i \leq n+1$ of length $|I| = \sum\limits_{i=1}^{n+1} I_i = \tau$, and f_I denotes a polynomial in the variables M_{ij}, $1 \leq i, j \leq n+1$ for all such multi-indices I. Using this notation, G^h can be written as

$$G^h = \bigcap_{|I|=\tau} \{f_I = 0\}.$$

For each considered multi-index I the set $\{f_I = 0\} \subset \mathrm{Mat}(n \times n, \mathbb{R})$ is closed since each f_I and, hence, continuous. Hence, $G^h = \bigcap\limits_{|I|=\tau} \{f_I = 0\}$ is also closed in $\mathrm{Mat}(n \times n, \mathbb{R})$. But since we have already shown that G^h is a subset of $\mathrm{GL}(n+1)$, and $\mathrm{GL}(n+1)$ is an open subset of $\mathrm{Mat}(n \times n, \mathbb{R})$ and equipped with the subspace topology, we deduce that G^h is also closed as a subset of $\mathrm{GL}(n+1)$. The closed subgroup theorem now implies that $G^h \subset \mathrm{GL}(n+1)$ is indeed a Lie subgroup. The identity (3.2) now easily follows via differentiating both sides of $h(p) = h(\exp(tm)p)$ with respect to the variable t at $t = 0$. □

Using Proposition 3.12, we now define the automorphism group of a hyperbolic homogeneous polynomial h corresponding to a connected GPSR manifolds $\mathcal{H} \subset \{h = 1\}$.

Definition 3.13 (Automorphism group of h)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be an n-dimensional connected GPSR manifold. The Lie subgroup*

$$G^h = \left\{M \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid h(p) = h(Mp)\ \forall p \in \mathbb{R}^{n+1}\right\} \subset \mathrm{GL}(n+1) \tag{3.3}$$

is called the automorphism group of *h. We denote by*

$$G_0^h \subset G^h \tag{3.4}$$

the connected component of G^h that contains the neutral element $\mathbb{1} \in G^h$. The Lie algebra of G^h is given by

$$T_{\mathbb{1}} G^h = \left\{m \in \mathfrak{gl}(n+1) \;\middle|\; dh_x(mx) = 0\ \forall x \in \mathbb{R}^{n+1}\right\}$$

with Lie bracket $[\cdot,\cdot]$ induced by the Lie subalgebra structure $T_{\mathbb{1}} G^h \subset \mathfrak{gl}(n+1)$.

Lemma 3.14 (Action of G_0^h on $\mathcal{H}$)**.** *The Lie group G_0^h (3.4) corresponding to a maximal connected GPSR manifold $\mathcal{H} \subset \{h = 1\}$ acts on $(\mathcal{H}, g_{\mathcal{H}})$ via isometries.*

Proof. The action of an element $M \in G_0^h$ on $\mathcal{H} \subset \mathbb{R}^{n+1}$ is given by the corresponding linear transformation of coordinates of the ambient space $\mathbb{R}^{n+1}$. The action of G_0^h on $\mathcal{H}$ is well-defined for the following reasons. Both G_0^h and $\mathcal{H}$ are (path-)connected and G_0^h contains the identity $\mathbb{1}$. Linear transformations of $\mathbb{R}^{n+1}$ are by definition of $g_{\mathcal{H}}$ (3.1) isometries of $(\mathcal{H}, g_{\mathcal{H}})$ as long as they map $\mathcal{H}$ into itself. This is ensured by the path-connectedness of G_0^h, by the fact that all points in $G_0^h \cdot p$ are by definition of G_0^h automatically hyperbolic points of h for all $p \in \mathcal{H}$, and by the assumption that $\mathcal{H}$ is maximally extended that $G_0^h \cdot p$ is a connected component of $\{h = 1\} \cap \{p \text{ hyperbolic point of } h\}$. □

Note that independent of whether a connected GPSR manifold $\mathcal{H} \subset \{h = 1\}$ is maximally extended or not, there is at least always a well-defined local action of G_0^h on $\mathcal{H}$, i.e. there exists an open neighbourhood U of $\mathbb{1} \in G_0^h$, such that there is a well defined action $U \times \mathcal{H} \to \mathcal{H}$ via linear transformations of the ambient space $\mathbb{R}^{n+1}$. This can be shown by considering the unique maximal connected GPSR manifold $\widetilde{\mathcal{H}}$ that contains $\mathcal{H}$ and the corresponding action $G_0^h \times \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ and observing that the action is continuous.

Remark 3.15 (Classification of homogeneous PSR manifolds with transitive G_0^h-action)**.** Recall at this point that there is a classification of PSR manifolds $\mathcal{H} \subset \{h = 1\}$ that are homogeneous spaces under the action of G_0^h, see [DV]. As of now there is no analogous classification of GPSR manifolds of any homogeneity-degree $\tau \geq 4$ that are homogeneous spaces under the action of the respective identity-component of their automorphism group, that is G_0^h. We will classify all such GPSR manifolds that are of homogeneity-degree $\tau = 4$ and of dimension one, i.e. quartic GPSR curves, this is one result of Theorem 7.2.

One important class of CCGPSR manifolds is characterised as follows.

Definition 3.16 (Singular at infinity)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be a CCGPSR manifold and let $U = \mathbb{R}_{>0} \cdot \mathcal{H}$ be the corresponding convex cone. We will call $\mathcal{H}$* singular at infinity *if there exists a point $p \in \partial U \setminus \{0\}$, such that $dh_p = 0$.*

Remark 3.17. Note that there is another meaning of the term "singular" which might be used in the setting of CCGPSR manifolds. Consider for a CCGPSR manifold $\mathcal{H} \subset \{h = 1\}$ the projective variety $\{h = 0\}$ for which the term singular is defined as the existence of a point $p \in \{h = 0\} \setminus \{0\}$, such that $dh_p = 0$. In comparison with Definition 3.16, this is a priori a weaker definition of "singular" since it is not clear if singular in the projective variety sense automatically implies that one such singular point is contained in the respective $\partial\mathrm{dom}(\mathcal{H})$. If, however, $\mathcal{H} \subset \{h = 1\}$ is singular at infinity in the sense of Definition 3.16, then the projective variety $\{h = 0\}$ will also always be singular.

Proposition 3.18 (Standard form). *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an $n \geq 1$-dimensional connected GPSR manifold and h of homogeneity-degree $\tau \geq 3$. Then for each $p \in \mathcal{H}$ there exists a linear change of coordinates on $\mathbb{R}^{n+1}$ described by $A(p) \in \mathrm{GL}(n+1)$, such that*

(i) $(h \circ A(p))\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum\limits_{k=3}^{\tau} x^{\tau-k} P_k(y)$,

(ii) $A(p)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = p$,

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ *denotes the standard linear coordinates of* $\mathbb{R}^n$, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ *denotes the corresponding coordinates of* $\mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *denotes the point* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^{n+1}$, *and* $\langle\cdot,\cdot\rangle$ *denotes the standard Euclidean scalar product on* $\mathbb{R}^n$ *induced by the* y*-coordinates. Furthermore, if* $\mathcal{H}$ *is a CCGPSR manifold then the transformations* $A(p)$ *can be chosen in such a way that* $A : \mathcal{H} \to \mathrm{GL}(n+1)$ *is smooth. If* $\mathcal{H}$ *is not closed as a subset of* $\mathbb{R}^{n+1}$, *we can still find for each* $p \in \mathcal{H}$ *a subset* $V \subset \mathcal{H}$ *that contains* p *and is open in the subspace-topology of* $\mathcal{H} \subset \mathbb{R}^{n+1}$, *such that* $A : V \to \mathrm{GL}(n+1)$ *can be chosen so that it is a smooth map.*

Proof. First we will show that (i) and (ii) hold for all connected GPSR manifolds. Then we will prove that in the case of CCGPSR manifolds, $A : \mathcal{H} \to \mathrm{GL}(n+1)$ can be chosen to be smooth. In the case of connected GPSR manifolds which are not necessarily closed we will show that for all $p \in \mathcal{H}$ there always exists an open neighbourhood $V \subset \mathcal{H}$ of p, and that $A : V \to \mathrm{GL}(n+1)$ can be chosen so that it is a smooth map.

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a connected GPSR manifold and denote by $\langle\cdot,\cdot\rangle$ the standard Euclidean scalar product on $\mathbb{R}^{n+1}$ induced by the choice of the linear coordinates on $\mathbb{R}^{n+1}$. Let $p \in \mathcal{H}$ be arbitrary. We will differentiate between two cases.

Case 1: $dh_p = r\langle p, \cdot\rangle$ **for some** $r \neq 0$**.**
Note that the property $dh_p \in (\mathbb{R} \setminus \{0\}) \cdot \langle r, \cdot\rangle$ is preserved by changing the linear coordinates of the ambient $\mathbb{R}^{n+1}$ by rotations in $\mathrm{SO}(n+1)$ and by positive rescaling of the linear coordinates. We can thus without loss of generality assume that $p = (1, 0, \ldots, 0)^T$, and denote the linear coordinates on $\mathbb{R}^{n+1}$ by $(x, y_1, \ldots, y_n)^T$. Since $h(p) = 1$ is a necessary condition for $p \in \mathcal{H}$, we find that h must be of the form

$$h = x^\tau + x^{\tau-1}L(y) + x^{\tau-2}Q(y, y) + (\text{terms of lower order in } x),$$

where $L \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ is linear in y and $Q \in \mathrm{Sym}^2(\mathbb{R}^n)^*$ is a symmetric bilinear form. We can now check that $dh_p \in (\mathbb{R} \setminus \{0\}) \cdot \langle r, \cdot\rangle$ implies $L \equiv 0$. By assumption, p is a hyperbolic point of $\mathcal{H}$. We calculate

$$-\partial^2 h_p = \left(\begin{array}{c|c} -\tau(\tau-1) & \\ \hline & -2Q(\cdot,\cdot) \end{array}\right).$$

The hyperbolicity of the point p thus shows that Q must be negative definite, i.e. $Q < 0$. Hence, after a suitable transformation of the y-coordinates, we find that h is of the desired form

$$h = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum_{k=3}^{\tau} x^{\tau-k} P_k(y).$$

Case 2: $dh_p \neq r\langle p, \cdot\rangle$ **for all** $r \neq 0$.
Note that in this case, $r = 0$ is automatically excluded by $dh_p(p) = \tau \neq 0$. We will find a linear coordinate-transformation $B \in \mathrm{GL}(n+1)$ of the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H}$, such that $Bq = p$ and

$$dh_{Bq}(B\cdot) = r\langle q, \cdot\rangle, \tag{3.5}$$

which will take us to the setting of the first case since $d(h \circ B)_q = dh_{Bq}(B\cdot)$. Note that in the above equation (3.5), $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product induced by the new coordinates, that is the standard linear coordinates in the domain of $B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. In order to prove the existence of such a transformation B, let

$$\langle\langle\cdot,\cdot\rangle\rangle := \langle p, p\rangle\langle\cdot,\cdot\rangle - \langle p, \cdot\rangle^2 + dh_p^2.$$

We claim that $\langle\langle\cdot,\cdot\rangle\rangle > 0$, i.e. that $\langle\langle\cdot,\cdot\rangle\rangle \in \mathrm{Sym}^2\left(\mathbb{R}^{n+1}\right)^*$ is a positive definite bilinear form on. To show this, write $v \in \mathbb{R}^{n+1} \setminus \{0\}$ as

$$v = ap + w, \quad w \in p^{\perp_{\langle\cdot,\cdot\rangle}}.$$

Note that a and w are uniquely determined since $\mathbb{R}^{n+1} = \mathbb{R}p \oplus p^{\perp_{\langle\cdot,\cdot\rangle}}$. We obtain

$$\langle\langle v, v\rangle\rangle = \langle p, p\rangle\langle w, w\rangle + \left(dh_p(ap + w)\right)^2.$$

For $w \neq 0$ we immediately see that $\langle\langle v, v\rangle\rangle > 0$. For $w = 0$, $v \neq 0$ implies $a \neq 0$. In that case $\langle\langle v, v\rangle\rangle = a^2\tau^2 > 0$. Summarising, this shows that $\langle\langle\cdot,\cdot\rangle\rangle$ is indeed positive definite. Now let $B \in \mathrm{GL}(n+1)$ be an orthonormal basis[1] of $\langle\langle\cdot,\cdot\rangle\rangle$,

$$B^*\langle\langle\cdot,\cdot\rangle\rangle = \langle\cdot,\cdot\rangle.$$

Denote by $\widetilde{h} = h \circ B$ the transformed polynomial h and let $q = B^{-1}p$. Then

$$d\widetilde{h}_q = dh_{Bq}(B\cdot) = dh_p(B\cdot)$$

and

$$\begin{aligned}
\langle q, \cdot\rangle &= \langle\langle Bq, B\cdot\rangle\rangle \\
&= \langle\langle p, B\cdot\rangle\rangle \\
&= \langle p, p\rangle\langle p, B\cdot\rangle - \langle p, p\rangle\langle p, B\cdot\rangle + dh_p(p)dh_p(B\cdot) \\
&= \tau dh_p(B\cdot) \\
&= \tau d\widetilde{h}_q
\end{aligned}$$

Hence, B fulfils (3.5) with $r = \frac{1}{\tau}$, and we have $d\widetilde{h}_q = \frac{1}{\tau}\langle q, \cdot\rangle$ with $q \in B^{-1}\mathcal{H}$. We are now in the setting of the first case and can proceed as described therein.

Summarising up to this point, we have shown that for any $n \geq 1$-dimensional connected GPSR manifold $\mathcal{H} \subset \{h = 1\}$ and all $p \in \mathcal{H}$ we can find $A \in \mathrm{GL}(n+1)$, such that the conditions (i) and (ii) are fulfilled. Now we will describe how to construct A explicitly.

We will start with the case where $\mathcal{H}$ is a CCGPSR manifold, and first construct the transformation $A(p)$ explicitly for one arbitrarily chosen point $p \in \mathcal{H}$, so that $A(p)$ fulfils (i) and (ii). We start by choosing initial linear coordinates $(x, y_1, \ldots, y_n)^T$ of $\mathbb{R}^{n+1}$ and a point

[1] We interpret the columns of B as the basis vectors.

$p = \left(\begin{smallmatrix} p_x \\ p_y \end{smallmatrix}\right) \in \mathcal{H}$. After a possible reordering of the coordinates we can assume that $\partial_x h(p) \neq 0$. This follows from $dh_p \neq 0$, since otherwise $\tau h(p) = dh_p(p) = 0$. Let

$$\widetilde{A} = \left(\begin{array}{c|c} p_x & -\frac{\partial_y h}{\partial_x h}\Big|_p \\ \hline p_y & \mathbb{1} \end{array}\right)$$

where $\partial_x h := \frac{\partial h}{\partial x}$ and $\partial_y h := \sum\limits_{i=1}^{n} dh(\partial_{y_i}) dy_i$. $\widetilde{A} \in \mathrm{GL}(n+1)$ follows from

$$\begin{aligned}
\det\left(\widetilde{A}\right) &= \det \left(\begin{array}{c|c} p_x + \frac{\partial_y h}{\partial_x h}\Big|_p (p_y) & -\frac{\partial_y h}{\partial_x h}\Big|_p \\ \hline 0 & \mathbb{1} \end{array}\right) \\
&= p_x + \left.\frac{\partial_y h}{\partial_x h}\right|_p (p_y) \\
&= \frac{1}{\partial_x h} \left(\partial_x h_p \cdot p_x + \partial_y h_p(p_y) \right) \\
&= \frac{\tau}{\partial_x h_p} \neq 0.
\end{aligned}$$

In the above formula we have used the Euler identity for homogeneous functions, and p_y is viewed as an n-vector. This shows that $\widetilde{A}$ describes a linear change of coordinates. Furthermore, $\widetilde{A}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = p$. We obtain

$$\begin{aligned}
h\left(\widetilde{A}\begin{pmatrix} x \\ y \end{pmatrix}\right) &= x^\tau h(p) \\
&\quad + x^{\tau-1} dh_p \begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix} \\
&\quad + x^{\tau-2} \frac{1}{2} \partial^2 h_p \left(\begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix} \right) \\
&\quad + (\text{terms of lower order in } x) \\
&= x^\tau \\
&\quad + x^{\tau-2} \frac{1}{2} \partial^2 h_p \left(\begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix} \right) \\
&\quad + (\text{terms of lower order in } x).
\end{aligned}$$

The vanishing of the $x^{\tau-1}$-term follows from $dh_p \begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix} = 0$ for all $y \in \mathbb{R}^n$. This is equivalent to $\begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (y) \\ y \end{pmatrix} \in T_p\mathcal{H}$ for all $y \in \mathbb{R}^n$. Hence,

$$\mathbb{R}^n \times \mathbb{R}^n \ni (v, w) \mapsto -\frac{1}{2} \partial^2 h_p \left(\begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (v) \\ v \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h}{\partial_x h}\Big|_p (w) \\ w \end{pmatrix} \right) \tag{3.6}$$

is a positive definite bilinear form since p is, by assumption, a hyperbolic point of h. This implies that there exists a linear transformation $\widetilde{E} \in \mathrm{GL}(n)$, such that

$$h\left(\widetilde{A} \cdot \left(\begin{array}{c|c} 1 & \\ \hline & \widetilde{E} \end{array}\right) \begin{pmatrix} x \\ y \end{pmatrix} \right) = x^\tau - x^{\tau-2} \langle y, y \rangle + \sum_{k=3}^{\tau} x^{\tau-k} P_k(y).$$

Since $\widetilde{A}\cdot\left(\begin{array}{c|c}1 & \\ \hline & \widetilde{E}\end{array}\right)\begin{pmatrix}1\\0\end{pmatrix}=p$, we have shown that for one choice of $p\in\mathcal{H}$ we can find a linear transformation fulfilling both (i) and (ii), namely $\widetilde{A}\cdot\left(\begin{array}{c|c}1 & \\ \hline & \widetilde{E}\end{array}\right)$.

In order to prove the statement of this proposition for all $p\in\mathcal{H}$, we have shown that we can assume without loss of generality that h is of the form $h=x^\tau-x^{\tau-2}\langle y,y\rangle+\sum\limits_{k=3}^{\tau}x^{\tau-k}P_k(y)$ and that $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}\subset\{h=1\}$. For $p=\left(\begin{smallmatrix}p_x\\p_y\end{smallmatrix}\right)\in\mathcal{H}$ and $E(p)\in\mathrm{GL}(n)$ consider the matrix

$$A(p):=\left(\begin{array}{c|c}p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p\circ E(p)\\ \hline p_y & E(p)\end{array}\right).\tag{3.7}$$

Firstly we need to ensure that $A(p)$ is well-defined for all $p\in\mathcal{H}$ and all choices for $E(p)\in\mathrm{GL}(n)$. This follows from

$$\partial_x h|_{\mathcal{H}}>0,\tag{3.8}$$

which we will prove next. In order to show that (3.8) holds for all $n\geq 1$-dimensional CCGPSR manifolds, it in facts suffices to prove it for all 1-dimensional CCGPSR manifolds. To see this, suppose that $\dim(\mathcal{H})>1$ and that there exists a point $\overline{p}=\left(\begin{smallmatrix}\overline{p}_x\\\overline{p}_y\end{smallmatrix}\right)\in\mathcal{H}$, such that $\partial_x h|_{\overline{p}}=0$. Then the set

$$\widetilde{\mathcal{H}}:=\mathcal{H}\cap\operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\overline{p}\right\}$$

is a 1-dimensional CCGPSR manifold which coincides with the connected component of the level set

$$\left\{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\in\mathbb{R}^2\;\middle|\;h\left(x\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+y\left(\begin{smallmatrix}0\\v\end{smallmatrix}\right)\right)=1\right\}\tag{3.9}$$

containing the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathbb{R}^2$. In (3.9), $v\in\mathbb{R}^n$ is chosen to fulfil $\operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\overline{p}\right\}=\operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\v\end{smallmatrix}\right)\right\}$ and $\langle v,v\rangle=1$. Note that $\widetilde{h}:=h\left(x\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+y\left(\begin{smallmatrix}0\\v\end{smallmatrix}\right)\right)$ is then automatically of the form (i). Denote by $\widetilde{p}=\left(\begin{smallmatrix}\widetilde{p}_x\\\widetilde{p}_y\end{smallmatrix}\right)\in\mathbb{R}^2$ the point fulfilling $\widetilde{p}_x\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+\widetilde{p}_y\left(\begin{smallmatrix}0\\v\end{smallmatrix}\right)=\overline{p}$ and note that $\widetilde{p}_y\neq 0$. Then $\widetilde{p}\in\widetilde{\mathcal{H}}$ by construction and $\partial_x\widetilde{h}|_{\widetilde{p}}=0$. It now follows from Lemma 3.8 that there exists $R>0$, such that $\widetilde{h}\left(\widetilde{p}+R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=0$, since $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in T_{\widetilde{p}}\widetilde{\mathcal{H}}$ by assumption. The convexity of the cone $\widetilde{U}:=\mathbb{R}_{>0}\cdot\widetilde{\mathcal{H}}\subset\mathbb{R}^2$ (cf. Proposition 3.7) implies that

$$\widetilde{U}\subset\left(\mathbb{R}_{>0}\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)+\mathbb{R}\cdot\left(\widetilde{p}+R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\right)=:\widetilde{V}.$$

But $\widetilde{p}\notin\widetilde{V}$, and we conclude with $\widetilde{\mathcal{H}}\subset\widetilde{U}$ that $\widetilde{p}\notin\widetilde{\mathcal{H}}$, which is a contradiction. We have thus shown that (3.8) holds for every $n\geq 1$-dimensional CCGPSR manifold $\mathcal{H}$.

We now show that for all $p\in\mathcal{H}$ and all choices for $E(p)\in\mathrm{GL}(n)$, $A(p)\in\mathrm{GL}(n+1)$. The calculation is similar to calculating $\det(\widetilde{A})$. Since $E(p)$ is invertible, we have with (3.7)

$$\begin{aligned}\det A(p)&=\left(\begin{array}{c|c}p_x+\left.\frac{\partial_y h}{\partial_x h}\right|_p(p_y) & -\left.\frac{\partial_y h}{\partial_x h}\right|_p\circ E(p)\\ \hline 0 & E(p)\end{array}\right)\\&=\left(p_x+\left.\frac{\partial_y h}{\partial_x h}\right|_p(p_y)\right)\det(E(p))\\&=\frac{1}{\partial_x h}\left(\partial_x h_p\cdot p_x+\partial_y h_p(p_y)\right)\det E(p)\\&=\frac{\tau}{\partial_x h_p}\det E(p)\neq 0.\end{aligned}$$

In order to obtain the conditions for $E(p)$ so that $A(p)$ fulfils condition (i), we calculate

$$\begin{aligned} h\left(A(p)\begin{pmatrix} x \\ y \end{pmatrix}\right) &= x^\tau h(p) \\ &\quad + x^{\tau-1} dh_p \begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (E(p)y) \\ E(p)y \end{pmatrix} \\ &\quad + x^{\tau-2}\frac{1}{2}\partial^2 h_p \left(\begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (E(p)y) \\ E(p)y \end{pmatrix}, \begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (E(p)y) \\ E(p)y \end{pmatrix} \right) \\ &\quad + (\text{terms of lower order in } x). \end{aligned}$$

By definition, $dh_p \begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (E(p)y) \\ E(p)y \end{pmatrix} = 0$ for all $y \in \mathbb{R}^n$ and all $E(p) \in \mathrm{GL}(n)$, which is equivalent to $\begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (E(p)y) \\ E(p)y \end{pmatrix} \in T_p\mathcal{H}$ for all $y \in \mathbb{R}^n$ and all choices $E(p) \in \mathrm{GL}(n)$. Thus,

$$\mathbb{R}^n \times \mathbb{R}^n \ni (v,w) \mapsto -\frac{1}{2}\partial^2 h_p \left(\begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (v) \\ v \end{pmatrix}, \begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (w) \\ w \end{pmatrix} \right) \tag{3.10}$$

is a positive definite bilinear form since, by definition, $\mathcal{H} \subset \{h = 1\}$ consists only of hyperbolic points of the defining polynomial h. We conclude that for all $p \in \mathcal{H}$, $E(p) \in \mathrm{GL}(n)$ can be chosen in such a way that

$$-\frac{1}{2}\partial^2 h_p \left(\begin{pmatrix} -\frac{\partial_y h(E(p)y)}{\partial_x h} \\ E(p)y \end{pmatrix}, \begin{pmatrix} -\frac{\partial_y h(E(p)y)}{\partial_x h} \\ E(p)y \end{pmatrix} \right) = \langle y, y \rangle \tag{3.11}$$

for all $y \in \mathbb{R}^n$.

Summarising, we have shown for each $p \in \mathcal{H}$ how to explicitly construct a linear change of coordinates $A(p) \in \mathrm{GL}(n+1)$ which fulfils (i) and (ii). It remains to show that the assignment $A : \mathcal{H} \to \mathrm{GL}(n+1)$ can be chosen so that it is a smooth map. To see this observe that

$$A(p) = \left(\begin{array}{c|c} p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p \\ \hline p_y & \mathbb{1} \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & \\ \hline & E(p) \end{array}\right).$$

The matrix $\left(\begin{array}{c|c} p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p \\ \hline p_y & \mathbb{1} \end{array}\right)$ in the above equation depends smoothly on $p \in \mathcal{H}$. Hence, it suffices to show that $E : \mathcal{H} \to \mathrm{GL}(n)$ can be chosen so that it is a smooth map and fulfils equation (3.11). This follows from the fact that, as we have seen above,

$$-\frac{1}{2}\partial^2 h_p \left(\begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (\cdot) \\ \cdot \end{pmatrix}, \begin{pmatrix} -\left.\frac{\partial_y h}{\partial_x h}\right|_p (\cdot) \\ \cdot \end{pmatrix} \right) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

understood as in (3.10) is positive definite for all $p \in \mathcal{H}$, cf. [Le, Lem. 8.13].

It remains to deal with the cases where $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is a connected GPSR manifold, but is <u>not</u> closed in $\mathbb{R}^{n+1}$. For $p \in \mathcal{H}$ arbitrary and fixed, we want to show that there exists a neighbourhood $V \subset \mathcal{H}$ of p in $\mathcal{H}$, such that $A : V \to \mathrm{GL}(n+1)$ can be chosen to fulfil (i) and (ii) and to be a smooth map. We have already seen in the beginning of the proof that we can, after a possible linear transformation of the coordinates of $\mathbb{R}^{n+1}$, assume without loss of generality that $p = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, that h is of the form $h = x^\tau - x^{\tau-2}\langle y,y\rangle + \sum\limits_{k=3}^{\tau} x^{\tau-k}P_k(y)$, and that $\mathcal{H}$ is the contained in the connected component of $\{h=1\}$ that contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$. Since $\partial_x h|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \tau > 0$, it immediately follows that we can find a neighbourhood V of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ in $\mathcal{H}$, such that $\partial_x h|_q > 0$ for all $q \in V$. We can now define A as in equation (3.7) and proceed as for the case when $\mathcal{H}$ was supposed to be closed. □

Proposition 3.18 shows in particular that for any CCGPSR manifold $\mathcal{H} \subset \{h=1\}$ we can assume without loss of generality that h is of the form

$$h = x^\tau - x^{\tau-2}\langle y,y\rangle + \sum_{k=3}^{\tau} x^{\tau-k}P_k(y) \tag{3.12}$$

and that $\mathcal{H}$ is the precisely the connected component of $\{h=1\}$ which contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{R}^{n+1}$. If $\mathcal{H}$ is just assumed to be a connected an not necessarily closed GPSR manifold, we can still assume without loss of generality that $\mathcal{H}$ is a connected open subset of $\{h=1\}$ with h of the form (3.12), and that $\mathcal{H}$ contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{R}^{n+1}$. Also note that whenever $\mathcal{H}$ is a CCGPSR manifold, then the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$ is the unique point in $\mathcal{H}$ that minimises the Euclidean distance of $\mathcal{H} \subset \mathbb{R}^{n+1}$ and the origin $0 \in \mathbb{R}^{n+1}$ (in the chosen linear coordinates $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ of $\mathbb{R}^{n+1}$).

Remark 3.19. The polynomials P_i, $3 \leq i \leq \tau$, in equation (3.12) are in general not uniquely determined for the respective connected GPSR manifold $\mathcal{H} \subset \{h=1\}$. For example, for PSR manifolds the P_i's are never uniquely determined, see Lemma 4.1 in Section 4.

Remark 3.20. The form (3.12) (up to a constant prefactor of the y-coordinates) of h corresponding to a PSR manifold $\mathcal{H} \subset \{h=1\}$ has already been used in physics literature under the name "canonical parametrization", see [GST, Eqn. (3.31)] and [DV, Eqn. (1.5)]. However, the motivation for studying h of the form (3.12) has been of physical origin. We have verified that we can in fact always assume that h is of said form.

If h is already of the form (3.12), we obtain the following result.

Lemma 3.21. *With the assumptions of Proposition 3.18 and the additional assumption that h is of the form (3.12), we can assume that $A\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = \mathbb{1}$.*

Proof. It is clear that for every open neighbourhood $V \subset \mathcal{H}$ of the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$ and every smooth map $F: V \to \mathrm{O}(n)$, all linear transformations of the form

$$A'(p) := A(p) \cdot \left(\begin{array}{c|c} 1 & \\ \hline & F(p) \end{array}\right)$$

fulfil conditions (i) and (ii) of Proposition 3.18. Furthermore, $\partial_y h|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = 0$, which implies $E\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) \in \mathrm{O}(n)$. Hence, choosing any smooth map $F: V \to \mathrm{O}(n)$ with $F\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = \left(E\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\right)^{-1}$ and considering A' instead of A proves our claim. □

For the following considerations it is helpful to consider a certain parametrisation of connected GPSR manifolds which we will introduce now.

Definition 3.22 (dom($\mathcal{H}$)). *Let $\mathcal{H} \subset \{h=1\} \subset \mathbb{R}^{n+1}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$, be a connected GPSR manifold and assume that h is of the form (3.12) for the chosen linear coordinates $(x, y_1, \ldots, y_n)^T$ on $\mathbb{R}^{n+1}$. We define*

$$\mathrm{dom}(\mathcal{H}) := \mathrm{pr}_{\mathbb{R}^n}\left((\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{ \begin{pmatrix}1\\y\end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; y \in \mathbb{R}^n \right\} \right) \subset \mathbb{R}^n \tag{3.13}$$

where $\mathrm{pr}_{\mathbb{R}^n}: \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto y$. *The set* $\mathrm{dom}(\mathcal{H})$ *is precisely the section of the cone spanned by* $\mathcal{H}$, *that is* $\mathbb{R}_{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$, *and* $T_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\mathcal{H}$ *embedded affinely in* $\mathbb{R}^{n+1}$ *via* $v \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right)$.

Independent of whether the connected GPSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is closed or not, $\mathrm{dom}(\mathcal{H}) \subset \mathbb{R}^n$ is well-defined, open in $\mathbb{R}^n$, and always contains an open ball $B_\varepsilon(0) \subset \mathbb{R}^n$ with respect to the standard scalar product $\langle dy, dy\rangle$ on $\mathbb{R}^n$ for $\varepsilon > 0$ small enough. In order to check that these claims are true, one uses the following facts. Firstly, every ray $\mathbb{R}_{>0} \cdot p$ for $p \in \mathcal{H}$ meets $\mathcal{H}$ precisely once. This follows from the homogeneity of degree $\tau \geq 3$ of the corresponding polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$. Secondly, $\mathcal{H} \subset \{x \geq 1\} \subset \mathbb{R}^{n+1}$ and $\mathcal{H} \cap \{x = 1\} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. This follows from the fact that $\mathcal{H}$ is locally around each point in $\mathcal{H}$ contained in the boundary of a strictly convex domain of in $\mathbb{R}^{n+1}$, which in turn follows from the Sacksteder-van Heijenoort Theorem[2] [Wu]. Note that if $\mathcal{H}$ is a CCGPSR manifold, then $\mathcal{H}$ is (globally) the boundary of the strictly convex domain $\mathbb{R}_{>1} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$. Thus, every ray $\mathbb{R}_{>0} \cdot p$ for $p \in \mathcal{H}$ has a unique intersection-point with the set $\mathrm{dom}(\mathcal{H})$. We see that $\mathrm{dom}(\mathcal{H})$ is bijective to $\mathcal{H}$ via

$$\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}, \quad \Phi(z) = \frac{1}{\sqrt[\tau]{h\left(\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)\right)}} \begin{pmatrix} 1 \\ z \end{pmatrix}. \tag{3.14}$$

One can check that Φ is everywhere a local diffeomorphism. This and $\mathcal{H}$ being a hypersurface of $\mathbb{R}^{n+1}$ also show that $\mathrm{dom}(\mathcal{H}) \subset \mathbb{R}^n$ is open and, hence, that Φ is a diffeomorphism. Note, however, that the set $\mathrm{dom}(\mathcal{H})$ <u>does depend</u> on the chosen linear coordinates of the ambient space $\mathbb{R}^{n+1}$.

Lemma 3.8 implies the following property of $\mathrm{dom}(\mathcal{H})$ if $\mathcal{H}$ is a CCGPSR manifold.

Corollary 3.23 (Properties of $\mathrm{dom}(\mathcal{H})$ for CCGPSR manifolds)**.** *Let* $\mathcal{H}$ *be a CCGPSR manifold. Then* $\mathrm{dom}(\mathcal{H}) \subset \mathbb{R}^n$ *is open, precompact, and convex.*

Note that the statement of Corollary 3.23 is independent of the linear coordinates of the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H}$.

Now we will demonstrate how to explicitly calculate the standard form (3.12) of a cubic polynomial h corresponding to a CCPSR manifold $\mathcal{H} \subset \{h = 1\}$ as in Proposition 3.18 (i) with the example of CCPSR surfaces.

Example 3.24 (Standard form of cubics for CCPSR surfaces)**.** *Let* $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ *denote the standard linear coordinates on* $\mathbb{R}^3$. *Recall that CCPSR surfaces* $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^3$ *have been classified up to equivalence in [CDL, Thm. 1], cf. Theorem 2.45 a)–f). In the following we will for each* h *corresponding to the cases a)–f) give a choice of* $A = A(p) \in \mathrm{GL}(3)$ *corresponding to a given point* $p \in \mathcal{H}$, *such that* $A \cdot \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) = p$, $h\left(A \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right)$ *is of the form (3.12), and* $A^{-1}(\mathcal{H}) \subset \{h \circ A = 1\}$ *is precisely the connected component of* $\{h \circ A = 1\} \subset \mathbb{R}^3$ *that contains the point* $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$.

a) $\mathcal{H} = \{h = xyz = 1,\ x > 0,\ y > 0\}$**.**
It is clear that $p = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \in \mathcal{H}$. *One choice for the corresponding linear transformation of the form (3.7) is*

$$A = \begin{pmatrix} 1 & -\frac{2}{\sqrt{3}} & 0 \\ 1 & \frac{1}{\sqrt{3}} & -1 \\ 1 & \frac{1}{\sqrt{3}} & 1 \end{pmatrix},$$

[2] To apply said theorem, one first needs to extend the considered local neighbourhood of $\mathcal{H}$ to a Euclidean complete convex hypersurface.

which brings h to the form

$$h\left(A\cdot\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2)-\frac{2}{3\sqrt{3}}y^3+\frac{2}{\sqrt{3}}yz^2, \tag{3.15}$$

with corresponding $P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=-\frac{2}{3\sqrt{3}}y^3+\frac{2}{\sqrt{3}}yz^2$.

b) $\mathcal{H}=\{h=x(xy-z^2)=1,\ x>0\}$.
*Similar to the surface in **a)**, consider the point* $p=\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)\in\mathcal{H}$ *and*

$$A=\begin{pmatrix}1 & -\frac{1}{\sqrt{3}} & 0\\ 1 & \frac{2}{\sqrt{3}} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then

$$h\left(A\cdot\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2)+\frac{2}{3\sqrt{3}}y^3+\frac{1}{\sqrt{3}}yz^2, \tag{3.16}$$

with $P_3\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}$.

c) $\mathcal{H}=\{h=x(yz+x^2)=1,\ x<0,\ y>0\}$.
With $p=\left(\begin{smallmatrix}-1\\2\\-2\end{smallmatrix}\right)\in\mathcal{H}$ *and*

$$A=\begin{pmatrix}-1 & 0 & \frac{2\sqrt{2}}{\sqrt{15}}\\ 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{30}}\\ -2 & \sqrt{2} & \frac{\sqrt{2}}{\sqrt{15}}\end{pmatrix}$$

we obtain

$$h\left(A\cdot\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2)+\frac{2\sqrt{2}}{\sqrt{15}}y^2z+\frac{14\sqrt{2}}{15\sqrt{15}}z^3. \tag{3.17}$$

d) $\mathcal{H}=\{h=z(x^2+y^2-z^2)=1,\ z<0\}$.
By re-ordering of the coordinates and switching one sign one quickly finds that $\mathcal{H}$ *is equivalent to* $\widetilde{\mathcal{H}}=\{\widetilde{h}=x^3-x(y^2+z^2)=1,\ x>0\}$*, which is precisely the connected component of* $\{\widetilde{h}=1\}$ *that contains the point* $\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$*. The corresponding point in* $\mathcal{H}$ *and transformation* $A\in\mathrm{GL}(3)$ *are given by* $p=\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\in\mathcal{H}$ *and*

$$A=\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},$$

so that indeed

$$h\left(A\cdot\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2). \tag{3.18}$$

The transformation A is not of the form (3.7) since we needed to switch the x- and z-coordinate so that $\partial_x(h\circ A)|_{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)}\neq 0$*. Note that in the sense of equation (3.12) this means that* $P_3\equiv 0$*, so one might call* $\widetilde{h}$ *the simplest possible cubic polynomial of the form (3.12) defining a CCPSR manifold.*

e) $\mathcal{H} = \{h = x(y^2 - z^2) + y^3 = 1,\ y < 0,\ x > 0\}$**.**
Consider the point $p = \left(\begin{smallmatrix} 2 \\ -1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ *and the corresponding linear transformation as in (3.7)*

$$A = \begin{pmatrix} 2 & \frac{1}{\sqrt{3}} & 0 \\ -1 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{30}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We find

$$h\left(A \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) = x^3 - x(y^2 + z^2) + \frac{2}{3\sqrt{3}} y^3 - \frac{1}{2\sqrt{3}} yz^2. \tag{3.19}$$

f) $\mathcal{H}_b = \{h = y^2 z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0,\ 2x > z\}$**,** $b \in (-1,1)$**.**
Observe that the point $p_b = \frac{1}{\sqrt[3]{1-b}} \left(\begin{smallmatrix} \frac{1}{2} \\ 0 \\ -1 \end{smallmatrix}\right)$ *is contained in* $\mathcal{H}_b$ *for all* $b \in (-1,1)$*. After switching the* x*- and* z*-coordinate via the transformation* $\left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right)$*, we can apply the construction in equation (3.7) in order to find* $\widetilde{A}_b \in \mathrm{GL}(3)$*, such that* $h \circ \left(\left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right) \cdot \widetilde{A}_b\right)$ *is of the form (3.12). We calculate that*

$$\widetilde{A}_b = \begin{pmatrix} -\frac{1}{\sqrt[3]{1-b}} & 0 & 0 \\ 0 & \sqrt[6]{1-b} & 0 \\ \frac{1}{2\sqrt[3]{1-b}} & 0 & -\frac{\sqrt[6]{1-b}}{\sqrt{6}} \end{pmatrix}.$$

With

$$A_b := \left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right) \cdot \widetilde{A}_b = \begin{pmatrix} \frac{1}{2\sqrt[3]{1-b}} & 0 & -\frac{\sqrt[6]{1-b}}{\sqrt{6}} \\ 0 & \sqrt[6]{1-b} & 0 \\ -\frac{1}{\sqrt[3]{1-b}} & 0 & 0 \end{pmatrix}$$

we obtain

$$h\left(A_b \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) = x^3 - x(y^2 + z^2) + \frac{\sqrt{2}\sqrt{1-b}}{3\sqrt{3}} z^3 \tag{3.20}$$

and have thus shown that $h \circ A_b$ *is of the form (3.12) and that* $A_b \cdot \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) = p_b$ *for all* $b \in (-1,1)$ *as required. Note that equation (3.20) allows us to interpret the one-parameter family of CCPSR surface* $\mathcal{H}_b$ *as interpolating between the CCPSR curves Theorem 2.45 c) (for* $b \to 1$*, see equation (3.18)) and e) (for* $b \to 0$*) of Theorem 2.45. To see the latter, observe that with*

$$\widetilde{A} = \begin{pmatrix} 2^{-\frac{1}{3}} \frac{4}{3} & 2^{-\frac{1}{3}} \frac{2}{3\sqrt{3}} & 0 \\ 2^{-\frac{1}{3}} \frac{1}{\sqrt{3}} & 2^{-\frac{1}{3}} \frac{5}{3} & 0 \\ 0 & 0 & 2^{-\frac{5}{6}} \sqrt{3} \end{pmatrix},$$

the polynomial

$$\widetilde{h} := x^3 - x(y^2 + z^2) + \frac{2}{3\sqrt{3}} y^3$$

transforms to

$$\widetilde{h}\left(\widetilde{A} \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) = x^3 - x(y^2 + z^2) + \frac{2}{3\sqrt{3}} y^3 - \frac{1}{2\sqrt{3}} yz^2$$

which coincides with equation (3.19). Furthermore one can check that the point $\widetilde{A} \cdot \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ *is contained in the connected component of* $\left\{x^3 - x(y^2 + z^2) + \frac{2}{3\sqrt{3}} y^3 - \frac{1}{2\sqrt{3}} yz^2 = 1\right\}$ *that contains the point* $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$*, for which we have shown that this is equivalent to the CCPSR* *surface e) in Theorem 2.45. Hence, the connected component of*

$$\left\{x^3 - x(y^2+z^2) + \frac{2}{3\sqrt{3}}y^3 = 1\right\}$$

that contains the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *is in particular also a CCPSR surface which is equivalent to the surface e).*[3]

We will use the parametrisation (3.14) of $\mathcal{H} \subset \{h=1\}$ to study infinitesimal changes of the P_k's in the standard form (3.12) of h when we vary the point $p \in \mathcal{H}$ in Prop. 3.18 (i) near $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$. The results are important tools in the following sections. Whenever we use z-variables in the general considerations in this section, we will be working with $\operatorname{dom}(\mathcal{H})$. The y-variables will be used in when working with the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H}$. Note however that in the examples in Section 6 we will in general not stick to this convention.

For the following calculations we will define the (globally smooth) functions

$$\alpha : \mathbb{R}^n \to \mathbb{R}, \quad \alpha(z) = \partial_x h|_{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)}, \tag{3.21}$$

$$\beta : \mathbb{R}^n \to \mathbb{R}, \quad \beta(z) = h\left(\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)\right). \tag{3.22}$$

Note that $\operatorname{dom}(\mathcal{H})$ coincides with the connected component of $\{\beta(z) > 0\}$ that contains the point $z = 0 \in \mathbb{R}^n$, and $\beta|_{\partial \operatorname{dom}(\mathcal{H})} \equiv 0$. Also, as shown in the proof of Proposition 3.18, $\alpha|_{\operatorname{dom}(\mathcal{H})} > 0$ if $\mathcal{H}$ is a CCGPSR manifold. If $\mathcal{H}$ is not closed, we can at least find a neighbourhood V of $z = 0 \in \mathbb{R}^n$, such that $\alpha|_V > 0$, which also follows from the proof of Proposition 3.18. Furthermore, it immediately follows from (3.14) that $\Phi(z) = \frac{1}{\sqrt[\tau]{\beta(z)}}\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)$ for $z \in \operatorname{dom}(\mathcal{H})$. While dh does not vanish on $\mathcal{H}$, it might vanish at a point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)$ for $\overline{z} \in \partial \operatorname{dom}(\mathcal{H})$ or, equivalently, on the ray $\mathbb{R}_{>0} \cdot \left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right) \subset \partial(\mathbb{R}_{>0} \cdot \mathcal{H})$. If $\mathcal{H}$ is also closed, we are in this case precisely in the setting of CCGPSR manifolds that are singular at infinity, cf. Definition 3.16. The following lemma characterises these cases for CCGPSR manifolds in terms of the functions α and β.

Lemma 3.25. *Let $\mathcal{H} \subset \{h=1\}$ be a CCGPSR manifold with $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ and h a homogeneous polynomial of homogeneity-degree $\tau \geq 3$ of the form (3.12), i.e. $h = x^\tau - x^{\tau-2}\langle y,y\rangle + \sum\limits_{k=3}^{\tau} x^{\tau-k}P_k(y)$. Let α, β be defined as in (3.21), respectively (3.22). Then for all $\overline{z} \in \partial\operatorname{dom}(\mathcal{H})$ the following are equivalent:*

(i) $dh_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)} = 0$,

(ii) $\alpha(\overline{z}) = 0$,

(iii) $d\beta_{\overline{z}} = 0$.

Proof. Assume that $dh_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)} = 0$ for a $\overline{z} \in \partial\operatorname{dom}(\mathcal{H})$. By affinely embedding $\operatorname{dom}(\mathcal{H})$ into $\mathbb{R}^{n+1}$ via $z \mapsto \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)$ and identifying y and z we obtain

$$dh_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)} = \alpha(\overline{z})dx + d\beta_{\overline{z}}.$$

Since $\alpha(\overline{z})dx$ and $d\beta_{\overline{z}}$ are linearly independent we conclude that $\alpha(\overline{z}) = 0$ and $d\beta_{\overline{z}} = 0$.

[3]In order to find the transformation $\widetilde{A}$, we have used a technique developed later in this thesis in Theorem 5.6. Specifically we used equations (5.22) and (5.23).

Now assume that $d\beta_{\overline{z}} = 0$. Then, using the Euler-identity for homogeneous functions, $0 = \tau\beta(\overline{z}) = dh_{\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right)}\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right) = \alpha(\overline{z})$ showing that $\alpha(\overline{z}) = 0$. Hence, $dh_{\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right)} = 0$.

Lastly, assume that $\alpha(\overline{z}) = 0$. Similar to above, $0 = \tau\beta(\overline{z}) = dh_{\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right)}\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right) = d\beta_{\overline{z}}(\overline{z})$. We need to show that this implies $d\beta_{\overline{z}} = 0$. Assume the latter does not hold. Then $dh_{\left(\begin{smallmatrix}1\\ \overline{z}\end{smallmatrix}\right)} \neq 0$ and, hence, we can use the implicit function theorem and conclude that $\mathrm{dom}(\mathcal{H})$ has smooth boundary near $\overline{z}$, and $d\beta_{\overline{z}}(\overline{z}) = 0$ is equivalent to the statement that $\overline{z} \in T_{\overline{z}}\partial\mathrm{dom}(\mathcal{H})$. This, however, contradicts the assumption that $\mathcal{H}$ is a CCGPSR manifold which implies that $\mathrm{dom}(\mathcal{H})$ is a convex set containing the point $0 \in \mathbb{R}^n$ (cf. Lemma 3.8). To see the contradiction, observe that for each non-singular point $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, i.e. a point satisfying $d\beta_{\overline{z}} \neq 0$, the affinely embedded tangent space $\overline{z} + T_{\overline{z}}\partial\mathrm{dom}(\mathcal{H})$ in $\mathbb{R}^n$ intersects the convex compact set $\overline{\mathrm{dom}(\mathcal{H})}$ (cf. Corollary 3.23) only at its boundary, that is $\partial\mathrm{dom}(\mathcal{H})$. But if $\overline{z} \in T_{\overline{z}}\partial\mathrm{dom}(\mathcal{H})$, the intersection of $\overline{z} + T_{\overline{z}}\partial\mathrm{dom}(\mathcal{H})$ and $\overline{\mathrm{dom}(\mathcal{H})}$ will always contain $0 \in \mathbb{R}^n$ which is, independently of any coordinate choice of the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H}$, always contained in $\mathrm{dom}(\mathcal{H})$ and in particular never contained in $\partial\mathrm{dom}(\mathcal{H})$. This follows directly from the definition of $\mathrm{dom}(\mathcal{H})$, see Definition 3.22. This is a contradiction to the convexity of $\mathrm{dom}(\mathcal{H})$, see Corollary 3.23. □

Returning to Proposition 3.18, we will now study the infinitesimal changes in the transformations $A(p)$ for $p \in \mathcal{H}$ near $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right) \in \mathcal{H}$, and in the corresponding polynomials P_i in the considered polynomial h as in equation (3.12). To do so we use the parametrisation $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ given in equation (3.14). The next result might seem a bit artificial or overcomplicated at first, but it has useful applications, see e.g. Proposition 3.34.

Proposition 3.26 (Infinitesimal standard form)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a connected GPSR manifold, $\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right) \in \mathcal{H}$, and let h be of the form $h = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum_{k=3}^{\tau} x^{\tau-k}P_k(y)$ as in equation (3.12). Furthermore, let $V \subset \mathcal{H}$ be an open neighbourhood of $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$ and $A : V \to \mathrm{GL}(n+1)$,*

$$A(p) = \left(\begin{array}{c|c} p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p \circ E(p) \\ \hline p_y & E(p) \end{array}\right),$$

as in equation (3.7) so that $A(p)$ fulfils (i) and (ii) in Proposition 3.18. Further assume that $A\left(\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)\right) = \mathbb{1}$, cf. Lemma 3.21. Let $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ be the diffeomorphism given in equation (3.14) and define

$$\mathcal{A} : \Phi^{-1}(V) \to \mathrm{GL}(n+1), \quad \mathcal{A}(z) := A(\Phi(z)). \tag{3.23}$$

Then there exists an $\mathfrak{so}(n)$-valued linear map $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$ of the form

$$dB_0 = \sum_{k=1}^{n(n-1)/2} a_k \langle \ell_k, dz\rangle,$$

where $\{a_k,\ 1 \leq k \leq n(n-1)/2\}$ is a basis of $\mathfrak{so}(n)$ and $\ell_k \in \mathbb{R}^n$ for all $1 \leq k \leq n(n-1)/2$, such that for $\tau \geq 4$

$$\begin{aligned}
&\partial_z \left(h\left(\mathcal{A}(z)\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\right)\right)\Big|_{z=0} \\
&= dh_{\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\right) \\
&= x^{\tau-3}\left(\frac{-2(\tau-2)}{\tau}\langle y, y\rangle\langle y, dz\rangle + 3P_3\left(y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dz)^T\right) + 4P_4(y, y, y, dz)\right) \\
&+ \left(\sum_{i=4}^{\tau-1} x^{\tau-i}\left(\frac{2(\tau-i+1)}{\tau}P_{i-1}(y)\langle y, dz\rangle\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+ iP_i\left(y,\ldots,y,dB_0y+\frac{3}{2}P_3(y,\cdot,dz)^T\right)\\
&+(i+1)P_{i+1}(y,\ldots,y,dz)\Big)\Big)\\
+\frac{2}{\tau}P_{\tau-1}(y)\langle y,dz\rangle+\tau P_\tau\left(y,\ldots,y,dB_0y+\frac{3}{2}P_3(y,\cdot,dz)^T\right)
\end{aligned}
\tag{3.24}
$$

and for $\tau=3$, that is for connected PSR manifolds,

$$
\begin{aligned}
\partial_z\left(h\left(\mathcal{A}(z)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\right)\Big|_{z=0} &= dh_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\\
&= -\frac{2}{3}\langle y,y\rangle\langle y,dz\rangle+3P_3\left(y,y,dB_0y+\frac{3}{2}P_3(y,\cdot,dz)^T\right).
\end{aligned}
$$

In the above equations, $P_3(y,\cdot,dz)^T$ is to be understood as the column-vector

$$
\frac{1}{6}\partial^2P_3|_y\left(\begin{pmatrix}dz_1\\ \vdots\\ dz_n\end{pmatrix},\cdot\right)^T
$$

and dB_0 is to be understood as

$$
dB_0y=\sum_{k=1}^{n(n-1)/2}a_ky\langle\ell_k,dz\rangle. \tag{3.25}
$$

Proof. Note that $z=0\in\Phi^{-1}(V)$ for all possible choices for V since $\Phi^{-1}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=0$. Observe that for all $v\in\mathbb{R}^n$, the function $-\frac{\partial_yh(v)}{\partial_xh}$ defined on $\mathbb{R}_{>0}\cdot\mathcal{H}$ is constant along rays of the form $\mathbb{R}_{>0}\cdot p$, $p\in\mathcal{H}$. With the notation $\mathcal{E}(z)=E(\Phi(z))$ and α, β defined in (3.21), respectively (3.22),

$$
\mathcal{A}(z)=\left(\begin{array}{c|c}\frac{1}{\sqrt[\tau]{\beta(z)}} & -\frac{d\beta_z(\mathcal{E}(z)\cdot)}{\alpha(z)}\\ \hline \frac{z}{\sqrt[\tau]{\beta(z)}} & \mathcal{E}(z)\end{array}\right)
$$

and

$$
\mathcal{A}(0)=\mathbb{1},\quad \mathcal{E}(0)=\mathbb{1},\quad \alpha(0)=\tau,\quad \partial^2\beta_0=-2\langle dz,dz\rangle.
$$

We obtain

$$
d\mathcal{A}_0=\left(\begin{array}{c|c}0 & \frac{2}{\tau}dz^T\\ \hline dz & d\mathcal{E}_0\end{array}\right), \tag{3.26}
$$

where we understand dz as the identity-map on $\mathbb{R}^n$ and $d\mathcal{E}_0$ as a $\mathfrak{gl}(n)$-valued 1-form, $d\mathcal{E}_0\in\Omega^1(\mathbb{R}^n,\mathfrak{gl}(n))$, both using the identification $T_0\mathrm{dom}(\mathcal{H})\cong\mathbb{R}^n$ obtained with the affine embedding $d\Phi_0$ as in equation (3.14). With

$$
\begin{aligned}
dh_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)} &= \left(\tau x^{\tau-1}-(\tau-2)x^{\tau-3}\langle y,y\rangle+\sum_{i=3}^{\tau-1}(\tau-i)x^{\tau-i-1}P_i(y)\right)dx\\
&\quad -2x^{\tau-2}\langle y,dy\rangle+\sum_{i=3}^{\tau}x^{\tau-i}iP_i(y,\ldots,y,dy)
\end{aligned}
$$

we get

$$
\begin{aligned}
dh_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) &= x^{\tau-2}\left(-2\langle y,d\mathcal{E}_0y\rangle+3P_3(y,y,dz)\right)\\
&\quad+(\text{terms of lower order in }x).
\end{aligned}
\tag{3.27}
$$

The assumption that A fulfils (i) and (ii) in Proposition (3.18) and $A(0) = \mathbb{1}$ implies that the $x^{\tau-2}$-term in the above equation (3.27) must vanish, i.e.

$$-2\langle y, d\mathcal{E}_0 y\rangle + 3P_3(y, y, dz) = 0.$$

This is true if and only if

$$d\mathcal{E}_0(y) = \frac{3}{2}P_3(y, \cdot, dz)^T + dB_0 y \tag{3.28}$$

for all $y \in \mathbb{R}^n$, with $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$ a linear map from $\mathbb{R}^n$ to $\mathfrak{so}(n)$. Here we have identified $\mathbb{R}^n$ with $T_0\mathrm{dom}(\mathcal{H})$. We will now justify our notation of the endomorphism dB_0. Consider for any smooth map $B : \mathbb{R}^n \to \mathrm{O}(n)$ with $B(0) = \mathbb{1}$ the map

$$B \cdot \mathcal{E} : \Phi^{-1}(V) \to \mathrm{GL}(n+1), \quad z \mapsto B(z) \cdot \mathcal{E}(z).$$

It is clear that if we replace E with $(B \circ \Phi^{-1}) \cdot E$ in the map A (and correspondingly $B \cdot \mathcal{E}$ in $\mathcal{A}$), it will still fulfil (i) and (ii) in Proposition 3.18 and $A(0) = \mathbb{1}$. We can thus choose for any $dB_0 \in \Omega^1(\mathbb{R}^n, \mathfrak{so}(n))$ a fitting map $B : \mathbb{R}^n \to \mathrm{O}(n)$ and a smooth map $\widetilde{\mathcal{E}} : \Phi^{-1}(V) \to \mathrm{GL}(n)$ with $d\widetilde{\mathcal{E}}_0 = \frac{3}{2}P_3(y, \cdot, dz)^T$, $\widetilde{\mathcal{E}}(0) = \mathbb{1}$, so that $\mathcal{E} := B \cdot \widetilde{\mathcal{E}}$ will fulfil equation (3.28). Also note that the requirement $B(0) = \mathbb{1}$ implies that the image of B lies in $\mathrm{SO}(n)$.

To complete the proof, we only need to replace $d\mathcal{E}_0$ in $dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(d\mathcal{A}_0 \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$ according to equation (3.28) and obtain the claimed result. □

Equation (3.24) in Proposition 3.26 determines precisely the infinitesimal changes of the P_k's in the polynomial h as in equation (3.12) when changing coordinates for $p \in \mathcal{H} \subset \{h = 1\}$ parametrised by $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ (3.14) in the way described by Proposition 3.18. Rotations in $y \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$ always preserve (3.12), which is seen in the freedom of choosing $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$. We will now assign symbols to the respective infinitesimal changes of the P_k's in order to simplify the considerations to follow.

Definition 3.27 (First variation of the P_k's)**.** *With the assumptions of Proposition 3.26 and the definition of $\mathcal{A}$ as in (3.23), we define for $\tau \geq 3$ and $3 \leq k \leq \tau$*

$$\begin{aligned} \delta P_k(y) &:= \left.\frac{1}{(\tau - k)!}\frac{\partial}{\partial x^{\tau-k}}\frac{\partial}{\partial z}h\left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)\right|_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right),\ z=0} \\ &= \left.\left(\frac{1}{(\tau - k)!}\partial_x^{\tau-k} dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(d\mathcal{A}_0 \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)\right)\right|_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)}, \end{aligned} \tag{3.29}$$

where we denote by $\frac{\partial}{\partial z} = \sum\limits_{i=1}^{n} dz_i \otimes \partial_{z_i}$ the de-Rham differential with respect to the $z = (z_1, \ldots, z_n)^T$-coordinates. In particular, we have for $\tau = 3$, that is cubic polynomials h,

$$\delta P_3(y) = -\frac{2}{3}\langle y, y\rangle\langle y, dz\rangle + 3P_3\left(y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dz)^T\right), \tag{3.30}$$

and for $\tau = 4$, that is quartic polynomials h,

$$\delta P_3(y) = -\langle y, y\rangle\langle y, dz\rangle + 3P_3\left(y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dz)^T\right) + 4P_4(y, y, y, dz), \tag{3.31}$$

$$\delta P_4(y) = \frac{1}{2}P_3(y)\langle y, dz\rangle + 4P_4\left(y, y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dz)^T\right). \tag{3.32}$$

This means that the $\delta P_k(y)$'s are precisely the factors depending on y in the summands of

$$dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = \sum_{i=3}^{\tau} x^{\tau-i}\delta P_i(y)$$

that are of order $x^{\tau-k}$, respectively. For each $3 \leq k \leq \tau$ we call δP_k the first variation of P_k along $\mathcal{H}$ with respect to the chosen dB_0 (3.28), respectively $d\mathcal{A}_0$ (3.26), and understand $\delta P_k(y)$ as a linear map $\delta P_k(y) : \mathbb{R}^n \to \mathrm{Sym}^k\left((\mathbb{R}^n)^*\right)$, so that we insert vectors $v \in \mathbb{R}^n$ into the dz in each $\delta P_k(y)$ and obtain a homogeneous polynomial in $(y_1, \ldots, y_n)$ of degree k.

The first application for Proposition 3.26 that we will consider is calculating the first derivative of the scalar curvature of a connected GPSR manifold $\mathcal{H}$ equipped with its centro-affine fundamental form at one certain point. To do so we need a closed form of the scalar curvature (at at least one point). Its calculation uses the following result.

Lemma 3.28 (Pullback metric on $\mathrm{dom}(\mathcal{H})$)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a connected GPSR manifold with centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ (cf. Lemma 3.4), h of homogeneity-degree $\tau \geq 3$ and of the form (3.12), and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$. Let $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ be the diffeomorphism given in equation (3.14) and β as in equation (3.22). Then*

$$\left(\Phi^* g_{\mathcal{H}}\right)_z = -\frac{\partial^2\beta_z}{\tau\beta(z)} + \frac{(\tau-1)d\beta_z^2}{\tau^2\beta^2(z)}. \tag{3.33}$$

Proof. This is a special case of [CNS, Cor. 1.13]. To check the claim, one uses the homogeneity of degree $\tau-2 \geq 1$ of $\partial^2 h_p$ in p and the first derivative of the diffeomorphism $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ (3.14), that is

$$d\Phi_z = \frac{1}{\sqrt[\tau]{\beta(z)}}\begin{pmatrix} 0 \\ dz \end{pmatrix} - \frac{1}{\tau\beta^{(\tau+1)/\tau}(z)}\begin{pmatrix} 1 \\ z \end{pmatrix} \otimes d\beta_z.$$

□

We will use equation (3.33) to calculate the scalar curvature of $(\mathrm{dom}(\mathcal{H}), \Phi^* g)$ at $z = 0 \in \mathrm{dom}(\mathcal{H})$.

Proposition 3.29 (Scalar curvature of GPSR manifolds)**.** *Let $\mathcal{H}$ be an n-dimensional connected GPSR manifold $\mathcal{H} \subset \{h = 1\}$, h of homogeneity-degree $\tau \geq 3$ and of the form (3.12), $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$, and let $g_{\mathcal{H}} = -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ be the centro-affine fundamental form of $\mathcal{H}$. Denote by $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ the linear coordinates of the ambient space $\mathbb{R}^{n+1}$ in accordance with equation (3.12). Then the scalar curvature $S_{\mathcal{H}} : \mathcal{H} \to \mathbb{R}$ at the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ is given by*

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = n(1-n) + \frac{9\tau}{8}\sum_{\ell}\sum_{a\neq i}\left(-P_3(\partial_a, \partial_a, \partial_\ell)P_3(\partial_i, \partial_i, \partial_\ell) + P_3(\partial_a, \partial_i, \partial_\ell)^2\right), \tag{3.34}$$

where $\partial_k = \partial_{y_k}$ for $1 \leq k \leq n$.

Proof. Note that we identify $\partial_{z_i} = \partial_{y_i} = \partial_i$ when inserting vectors in the P_k-polynomials. This is justified by the fact that $d\Phi_0$ maps $T_0\mathrm{dom}(\mathcal{H})$ to $T_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\mathcal{H}$ via $d\Phi_0 : \partial_{z_i} \mapsto \partial_{y_i}$ for all $1 \leq i \leq n$. For the following calculations we will first calculate the scalar curvature $S : \mathrm{dom}(\mathcal{H}) \to \mathbb{R}$ of $(\mathrm{dom}(\mathcal{H}), g := \tau\Phi^* g_{\mathcal{H}})$ at $z = 0$. We work with g instead of $g_{\mathcal{H}}$ because the necessary calculations will then require less symbols. Furthermore, we will for the general calculations assume that $\tau \geq 4$. The calculations for $\tau = 3$ are analogous and the formulas coincide when we set $P_4 \equiv 0$. The metric g has the form

$$g = -\frac{\partial^2\beta_z}{\beta(z)} + \frac{\tau-1}{\tau}\frac{d\beta_z^2}{\beta^2(z)}.$$

We abbreviate $\partial_{z_\mu} = \partial_\mu$ and obtain for the first differential of g in z_μ-direction

$$\begin{aligned}\partial_\mu g = {} & \beta^{-3}\left(\frac{-2\tau+2}{\tau} d\beta(\partial_\mu) d\beta^2\right) \\ & + \beta^{-2}\left(\frac{2\tau-2}{\tau} d\beta\partial^2\beta(\partial_\mu,\cdot) + d\beta(\partial_\mu)\partial^2\beta\right) \\ & + \beta^{-1}\left(-\partial^3\beta(\partial_\mu,\cdot,\cdot)\right).\end{aligned}$$

The second partial derivatives of g read

$$\begin{aligned}\partial_\nu\partial_\mu g = {} & \beta^{-4}\left(\frac{6\tau-6}{\tau} d\beta(\partial_\nu) d\beta(\partial_\mu) d\beta^2\right) \\ & + \beta^{-3}\left(\frac{-4\tau+4}{\tau}\left(d\beta(\partial_\nu) d\beta\partial^2\beta(\partial_\mu,\cdot) + d\beta(\partial_\mu) d\beta\partial^2\beta(\partial_\nu,\cdot)\right)\right. \\ & \quad \left. - 2 d\beta(\partial_\nu) d\beta(\partial_\mu)\partial^2\beta + \frac{-2\tau+2}{\tau}\partial^2(\partial_\nu,\partial_\mu) d\beta^2\right) \\ & + \beta^{-2}\left(d\beta(\partial_\nu)\partial^3\beta(\partial_\mu,\cdot,\cdot) + d\beta(\partial_\mu)\partial^3\beta(\partial_\nu,\cdot,\cdot) + \frac{2\tau-2}{\tau} d\beta\partial^3\beta(\partial_\nu,\partial_\mu,\cdot)\right. \\ & \quad \left. + \partial^2\beta(\partial_\nu,\partial_m u)\partial^2\beta + \frac{2\tau-2}{\tau}\partial^2\beta(\partial_\nu,\cdot)\partial^2\beta(\partial_\mu,\cdot)\right) \\ & + \beta^{-1}\left(-\partial^4\beta(\partial_\nu,\partial_\mu,\cdot,\cdot)\right).\end{aligned}$$

Applying the above formulas at $z = 0$, we obtain

$$g|_0 = 2\langle dz, dz\rangle, \tag{3.35}$$

$$g^{-1}|_0 = \frac{1}{2}\langle\partial_z,\partial_z\rangle, \tag{3.36}$$

$$\begin{aligned}\partial_\mu g|_0 &= -\partial^3\beta_0(\partial_\mu,\cdot,\cdot) = -6P_3(\partial_\mu,\cdot,\cdot), \\ \partial_\mu(g^{-1})|_0 &= -g^{-1}|_0\partial_\mu g|_0 g^{-1}|_0 = \frac{3}{2}P_3(\partial_\mu,\cdot,\cdot), \\ \partial_\nu\partial_\mu g|_0 &= \partial^2\beta_0(\partial_\mu,\partial_\nu)\partial^2\beta_0 + \frac{2\tau-2}{\tau}\partial^2\beta_0(\partial_\nu,\cdot)\partial^2\beta_0(\partial_\mu,\cdot) - \partial^4\beta_0(\partial_\nu,\partial_\mu,\cdot,\cdot) \\ &= 4\delta^\nu_\mu\langle\cdot,\cdot\rangle + \frac{8\tau-8}{\tau}\langle\partial_\nu,\cdot\rangle\langle\partial_\mu,\cdot\rangle - 24P_4(\partial_\nu,\partial_\mu,\cdot,\cdot), \\ \partial_\nu\partial_\mu g_{ij}|_0 &= 4\delta^\nu_\mu\delta^j_i + \frac{4\tau-4}{\tau}\left(\delta^i_\nu\delta^j_\mu + \delta^j_\nu\delta^i_\mu\right) - 24P_4(\partial_\nu,\partial_\mu,\partial_i,\partial_j),\end{aligned}$$

In order to calculate the scalar curvature of $(\mathrm{dom}(\mathcal{H}), g) \cong (\mathcal{H}, \tau g_{\mathcal{H}})$,

$$S = \sum_{a,i,j}\left(\partial_a\Gamma^a_{ji} - \partial_j\Gamma^a_{ia} + \sum_k\left(\Gamma^k_{ij}\Gamma^a_{ak} - \Gamma^k_{ia}\Gamma^a_{jk}\right)\right) g^{ij},$$

at $z = 0$, we need to calculate the Christoffel symbols and their first derivatives. We have

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2}\sum_\ell\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right) g^{\ell k}, \\ \partial_a\Gamma^k_{ij} &= \frac{1}{2}\sum_\ell\left(\left(\partial_a\partial_i g_{j\ell} + \partial_a\partial_j g_{i\ell} - \partial_a\partial_\ell g_{ij}\right) g^{\ell k} + \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right)\partial_a g^{\ell k}\right),\end{aligned}$$

and

$$\Gamma^k_{ij}\Big|_0 = -\frac{3}{2}P_3(\partial_i,\partial_j,\partial_k), \tag{3.37}$$

$$\partial_a \Gamma^k_{ij}\Big|_0 = \delta^i_a \delta^k_j + \delta^j_a \delta^k_i + \frac{\tau-2}{\tau}\delta^k_a \delta^j_i - 6P_4(\partial_a,\partial_i,\partial_j,\partial_k) - \frac{9}{2}\sum_\ell P_3(\partial_a,\partial_k,\partial_\ell)P_3(\partial_i,\partial_j,\partial_\ell). \tag{3.38}$$

Since $g^{ij}|_0 = \frac{1}{2}\delta^j_i$,

$$S(0) = \frac{1}{2}\sum_{a,i}\left(\partial_a \Gamma^a_{ii} - \partial_i \Gamma^a_{ia} + \sum_k \left(\Gamma^k_{ii}\Gamma^a_{ak} - \Gamma^k_{ia}\Gamma^a_{ik}\right)\right)\Bigg|_0 .$$

We obtain

$$\begin{aligned}
\partial_a \Gamma^a_{ii}|_0 &= \frac{\tau-2}{\tau} + 2\delta^i_a - 6P_4(\partial_a,\partial_a,\partial_i,\partial_i) - \frac{9}{2}\sum_\ell P_3(\partial_a,\partial_a,\partial_\ell)P_3(\partial_i,\partial_i,\partial_\ell),\\
\partial_i \Gamma^a_{ia}|_0 &= 1 + \frac{2\tau-2}{\tau}\delta^a_i - 6P_4(\partial_a,\partial_a,\partial_i,\partial_i) - \frac{9}{2}\sum_\ell P_3(\partial_a,\partial_i,\partial_\ell),\\
\Gamma^k_{ii}\Gamma^a_{ak}\Big|_0 &= \frac{9}{4}P_3(\partial_a,\partial_a,\partial_k)P_3(\partial_i,\partial_i,\partial_k),\\
\Gamma^k_{ia}\Gamma^a_{ik}\Big|_0 &= \frac{9}{4}\left(P_3(\partial_a,\partial_i,\partial_k)\right)^2 .
\end{aligned}$$

Hence,

$$\begin{aligned}
S(0) &= \frac{n(1-n)}{\tau} + \frac{9}{8}\sum_{a,i,\ell}\left(-P_3(\partial_a,\partial_a,\partial_\ell)P_3(\partial_i,\partial_i,\partial_\ell) + \left(P_3(\partial_a,\partial_i,\partial_\ell)\right)^2\right)\\
&= \frac{n(1-n)}{\tau} + \frac{9}{8}\sum_\ell\sum_{a\neq i}\left(-P_3(\partial_a,\partial_a,\partial_\ell)P_3(\partial_i,\partial_i,\partial_\ell) + \left(P_3(\partial_a,\partial_i,\partial_\ell)\right)^2\right).
\end{aligned}$$

Recall that $d\Phi_0(\partial_{z_i}) = \partial_{y_i}$ for all $1 \le i \le n$, which one can easily verify. Thus $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = \tau S(0)$ and the above equation prove our claim. Observe that $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$ only depends on the dimension n of $\mathcal{H}$, the degree of homogeneity τ, and the cubic polynomial P_3. Also note that $S_{\mathcal{H}} \equiv 0$ for $\dim(\mathcal{H}) = n = 1$ is consistent with the formula (3.34). □

Note that Proposition 3.29 gives us, at least in theory, a simple way of calculating the scalar curvature of a connected GPSR manifold $\mathcal{H}$ equipped with its centro-affine fundamental form (and thus of all GPSR manifolds by considering each connected component) at every point $p \in \mathcal{H}$. This, however, requires calculating $A(p)$ as in Proposition 3.18 for each $p \in \mathcal{H}$. This amounts basically to determining an orthonormal basis for some positive definite bilinear form. This is certainly easier than calculating Christoffel-symbols and their derivatives at each point, but nevertheless complicated enough to require a (both p- and $\mathcal{H}$-dependent) case-by-case study and not giving us a closed form of $S_{\mathcal{H}}(p)$ for all $p \in \mathcal{H}$. An application of these studies is Proposition 6.9.

Calculating the first derivative of the scalar curvature $S_{\mathcal{H}}$ at the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$ can of course also be done in a direct way, but the calculations require the (local) calculation of the third partial derivatives of the metric $g_{\mathcal{H}}$ and, hence, are very long and have a huge potential for human error. One can however also make use of Proposition 3.26 to obtain a formula for $dS_{\mathcal{H}}|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$.

Proposition 3.30 (First derivative of $S_{\mathcal{H}}$)**.** *With the assumptions and notations of Proposition 3.29 and Definition 3.27 and identifying $T_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}\mathcal{H}$ with the affinely embedded hyperplane $\left\{\left(\begin{smallmatrix}0\\y\end{smallmatrix}\right) \;\middle|\; y \in \mathbb{R}^n\right\} \subset \mathbb{R}^{n+1}$, we have for $\tau \geq 4$*

$$dS_{\mathcal{H}}|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \left(\sum_a \frac{3(n-1)(\tau-2)}{2}P_3(\partial_a,\partial_a,dy)\right)$$

$$
\begin{aligned}
&+ \left(\sum_{a,i,j} 9\tau(-P_3(\partial_a,\partial_a,\partial_j)P_4(\partial_i,\partial_i,\partial_j,dy) + P_3(\partial_a,\partial_i,\partial_j)P_4(\partial_a,\partial_i,\partial_j,dy)) \right) \\
&+ \sum_{a,i,j,\ell} \frac{27\tau}{8} P_3(\partial_j,\partial_\ell,dy)(-P_3(\partial_a,\partial_a,\partial_j)P_3(\partial_i,\partial_i,\partial_\ell) \\
&- 2P_3(\partial_a,\partial_a,\partial_i)P_3(\partial_i,\partial_j,\partial_\ell) + 3P_3(\partial_a,\partial_i,\partial_j)P_3(\partial_a,\partial_i,\partial_\ell)) \qquad (3.39)
\end{aligned}
$$

and for $\tau = 3$

$$
\begin{aligned}
dS_{\mathcal{H}}|_{\binom{1}{0}} &= \left(\sum_a \frac{3(n-1)}{2} P_3(\partial_a,\partial_a,dy) \right) \\
&+ \sum_{a,i,j,\ell} \frac{81}{8} P_3(\partial_j,\partial_\ell,dy)(-P_3(\partial_a,\partial_a,\partial_j)P_3(\partial_i,\partial_i,\partial_\ell) \\
&- 2P_3(\partial_a,\partial_a,\partial_i)P_3(\partial_i,\partial_j,\partial_\ell) + 3P_3(\partial_a,\partial_i,\partial_j)P_3(\partial_a,\partial_i,\partial_\ell)). \qquad (3.40)
\end{aligned}
$$

Proof. In the following calculations we will identify dz and dy, respectively each ∂_{z_i} and ∂_{y_i} (and write ∂_i instead) via $d\Phi_0$, cf. equation (3.14), which has the property that $d\Phi_0(\partial_{z_i}) = \partial_{y_i}$ for all $1 \leq i \leq n$. We start with the case $\tau \geq 4$. With the notations of Definition 3.27, Propositions 3.29 and Proposition 3.26 equation (3.24) imply

$$
dS_{\mathcal{H}}|_{\binom{1}{0}} = \frac{9\tau}{4} \sum_{a,i,\ell} \left(-P_3(\partial_a,\partial_a,\partial_\ell)\delta P_3(\partial_i,\partial_i,\partial_\ell) + P_3(\partial_a,\partial_i,\partial_\ell)\delta P_3(\partial_a,\partial_i,\partial_\ell) \right), \qquad (3.41)
$$

where

$$
\delta P_3(y) = \frac{-2(\tau-2)}{\tau}\langle y,y\rangle\langle y,dz\rangle + 3P_3\left(y,y,dB_0y + \frac{3}{2}P_3(y,\cdot,dz)^T\right) + 4P_4(y,y,y,dz). \qquad (3.42)
$$

Recall that $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$. We thus need to determine a formula for $\delta P_3(\partial_i,\partial_j,\partial_k)$ for all $1 \leq i,j,k \leq n$. The safest way in the sense that possible errors in the pre-factors do not occur is to determine $\partial^2(\delta P_3(y))$, where we regard dz in equation (3.42) as a constant vector. We obtain

$$
\begin{aligned}
d(\delta P_3)_y(v) &= \frac{-4(\tau-2)}{\tau}\langle y,v\rangle\langle y,dz\rangle + \frac{-2(\tau-2)}{\tau}\langle y,y\rangle\langle v,dz\rangle \\
&+ 6P_3\left(y,v,dB_0y + \frac{3}{2}P_3(y,\cdot,dz)^T\right) + 3P_3\left(y,y,dB_0v + \frac{3}{2}P_3(v,\cdot,dz)^T\right) \\
&+ 12P_4(y,y,v,dz) \qquad (3.43)
\end{aligned}
$$

for all $y, v \in \mathbb{R}^n$ and, hence,

$$
\begin{aligned}
\partial^2(\delta P_3)_y(v,w) &= \frac{-4(\tau-2)}{\tau}(\langle v,w\rangle\langle y,dz\rangle + \langle y,v\rangle\langle w,dz\rangle + \langle y,w\rangle\langle v,dz\rangle) \\
&+ 6P_3\left(v,w,dB_0y + \frac{3}{2}P_3(y,\cdot,dz)^T\right) \\
&+ 6P_3\left(y,v,dB_0w + \frac{3}{2}P_3(w,\cdot,dz)^T\right) \\
&+ 6P_3\left(y,w,dB_0v + \frac{3}{2}P_3(v,\cdot,dz)^T\right) \\
&+ 24P_4(y,v,w,dz)
\end{aligned}
$$

for all $y, v, w \in \mathbb{R}^n$. Since $\delta P_3(y)$ is homogeneous of degree 3 in y, we have the identities

$$
d(\delta P_3)_y(v) = 3\delta P_3(y,y,v),
$$

$$\partial^2(\delta P_3)_y(v,w) = 6\delta P_3(y,v,w),$$

when we regard dz as a constant vector and interpret δP_3 as a cubic tensor. We use the above identities and obtain

$$\begin{aligned}
&\sum_{a,i,\ell} P_3(\partial_a,\partial_a,\partial_\ell)\delta P_3(\partial_i,\partial_i,\partial_\ell)\\
&= \left(\sum_{a,\ell} P_3(\partial_a,\partial_a,\partial_\ell)\frac{(\tau-2)(-2n-4)}{3\tau}\langle\partial_\ell, dz\rangle\right)\\
&+\left(\sum_{a,i,\ell} P_3(\partial_a,\partial_a,\partial_\ell)\left(2P_3(\partial_i,\partial_\ell,dB_0\partial_i)+P_3(\partial_i,\partial_i,dB_0\partial_\ell)+4P_4(\partial_i,\partial_i,\partial_\ell,dz)\right)\right)\\
&+\left(\sum_{a,i,\ell,j} P_3(\partial_a,\partial_a,\partial_\ell)\left(3P_3(\partial_i,\partial_\ell,\partial_j)P_3(\partial_i,\partial_j,dz)+\frac{3}{2}P_3(\partial_i,\partial_i,\partial_j)P_3(\partial_\ell,\partial_j,dz)\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{a,i,\ell} P_3(\partial_a,\partial_i,\partial_\ell)\delta P_3(\partial_a,\partial_i,\partial_\ell)\\
&= \left(\sum_{a,\ell} P_3(\partial_a,\partial_\ell,\partial_\ell)\frac{-2(\tau-2)}{\tau}\langle\partial_a, dz\rangle\right)\\
&+\left(\sum_{a,i,\ell} P_3(\partial_a,\partial_i,\partial_\ell)\left(3P_3(\partial_a,\partial_i,dB_0\partial_\ell)+4P_4(\partial_a,\partial_i,\partial_\ell,dz)\right)\right)\\
&+\left(\sum_{a,i,\ell,j} P_3(\partial_a,\partial_i,\partial_\ell)\left(\frac{9}{2}P_3(\partial_a,\partial_i,\partial_j)P_3(\partial_\ell,\partial_j,dz)\right)\right).
\end{aligned}\tag{3.44}$$

To see that all terms containing $dB_0 : \mathbb{R}^n \to \mathfrak{so}(n)$ (understood as in equation (3.25)) vanish, observe that for all $1 \leq a,i,\ell \leq n$ the tensors

$$P_3(\partial_a,\partial_a,\partial_\ell)P_3(\cdot,\partial_\ell,\cdot),\quad P_3(\partial_a,\partial_a,\cdot)P_3(\partial_i,\partial_i,\cdot),\quad P_3(\partial_a,\partial_i,\cdot)P_3(\partial_a,\partial_i,\cdot)$$

are symmetric in their two arguments. Their trace with respect to the standard Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ when inserting any matrix $M \in \mathfrak{so}(n)$ in one of the arguments thus vanishes. We can now use the above formulas for $\sum\limits_{a,i,\ell} P_3(\partial_a,\partial_a,\partial_\ell)\delta P_3(\partial_i,\partial_i,\partial_\ell)$ and $\sum\limits_{a,i,\ell} P_3(\partial_a,\partial_i,\partial_\ell)\delta P_3(\partial_a,\partial_i,\partial_\ell)$ in equation (3.41) and, with the identification of dz and dy via $d\Phi_0$ (3.14), obtain our claimed result for $\tau \geq 4$. For $\tau = 3$, observe that the formulas for δP_3 in equations (3.30) and (3.42) coincide when setting $P_4 \equiv 0$. The calculations for the case $\tau = 3$ thus coincide with the cases $\tau \geq 4$ and we obtain the claimed result. □

The calculations used in Proposition 3.29 can also be used to calculate the Riemannian curvature tensor, the Ricci curvature, and the sectional curvatures of a connected GPSR manifold $(\mathcal{H}, g_{\mathcal{H}})$.

Lemma 3.31 (Riemannian, Ricci, and sectional curvature of GPSR manifolds)**.** *With the assumptions of Proposition 3.29, let R denote the Riemannian curvature tensor,* Ric *denote the Ricci curvature, and K denote the sectional curvature of an n-dimensional connected GPSR $(\mathcal{H}, g_{\mathcal{H}})$, respectively. We again identify dz and dy at $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$ via $d\Phi_0$ (3.14). Then*

$$R_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j)\partial_k = \frac{2}{\tau}\left(\delta_i^k - \delta_j^k\right)$$

$$+\frac{9}{4}\sum_{a,\ell}\left(-P_3(\partial_i,\partial_\ell,\partial_a)P_3(\partial_j,\partial_k,\partial_a)+P_3(\partial_i,\partial_k,\partial_a)P_3(\partial_j,\partial_\ell,\partial_a)\right), \tag{3.45}$$

$$\begin{aligned}\mathrm{Ric}_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_j,\partial_k) &= 2(1-n)\delta_j^k\\ &\quad+\frac{9\tau}{4}\sum_{a,i}\left(-P_3(\partial_i,\partial_i,\partial_a)P_3(\partial_j,\partial_k,\partial_a)+P_3(\partial_i,\partial_j,\partial_a)P_3(\partial_i,\partial_k,\partial_a)\right),\end{aligned} \tag{3.46}$$

and for $\dim\left(\mathrm{span}\{v,w\}\right)=2$

$$K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(v,w) = -1+\frac{9\tau}{8}\sum_{\ell}\left(-P_3(F\partial_i,F\partial_i,F\partial_\ell)P_3(F\partial_j,F\partial_j,F\partial_\ell)+P_3(F\partial_i,F\partial_j,F\partial_\ell)^2\right), \tag{3.47}$$

where $F\in \mathrm{O}(n)$ *is any orthogonal transformation with the property that* $\mathrm{span}\{v,w\}=\mathrm{span}\{F\partial_i,F\partial_j\}$*. Note that such a transformation* F *always exists for any choices of* $i\neq j$*, and that* $K(v,w)$ *does in particular not depend on that choice of* i*,* j*, and the corresponding* F *(cf. Definition 2.14).*

Proof. The formulas (3.45) and (3.46) for the Riemannian curvature tensor R and the Ricci tensor Ric, respectively, follow directly from the formulas for the Christoffel symbols (3.37), their first derivatives (3.38), and the inverse of $g_{\mathcal{H}}$ at the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (3.36) (up to the factor τ) given in the proof of Proposition 3.29. Recall that in said proof we work with $g=\tau\Phi^*g_{\mathcal{H}}$, Φ as in (3.14), hence we also need to rescale the formula for g at 0 (3.36) at the point where we take the trace with respect $g_{\mathcal{H}}$. For the sectional curvature K, the formula for $K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j)$ for $i\neq j$ follows easily from (3.45) and (3.35) (and by rescaling with the overall factor τ). To find the general formula $K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(v,w)$ (3.47) for any two linearly independent vectors $v,w\in T_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}\mathcal{H}\cong\mathbb{R}^n$, choose $i\neq j$ and $F\in\mathrm{O}(n)$ as described such that $\mathrm{span}\{v,w\}=\mathrm{span}\{F\partial_i,F\partial_j\}$. Changing the coordinates of the ambient $\mathbb{R}^{n+1}$ via

$$\begin{pmatrix}\widetilde{x}\\ \widetilde{y}\end{pmatrix}=\begin{pmatrix}x\\ F^{-1}y\end{pmatrix}$$

corresponds to rotating $\mathcal{H}$ in the y-coordinates and correspondingly changing the defining cubic h to

$$\widetilde{h}=\widetilde{x}^3-\widetilde{x}\langle\widetilde{y},\widetilde{y}\rangle+\widetilde{P}_3(\widetilde{y}),$$

with $\widetilde{P}_3(\widetilde{y})=P_3(F\widetilde{y})$. In the $\left(\begin{smallmatrix}\widetilde{x}\\ \widetilde{y}\end{smallmatrix}\right)$-coordinates, let $\widetilde{K}$ denote the sectional curvature. By identifying $\partial_{\widetilde{y}_k}=\partial_{y_k}=\partial_k$ for all $1\leq k\leq n$ (as the kth unit vector in $\mathbb{R}^n$, not via the map F) we have

$$\begin{aligned}K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(v,w) &= \widetilde{K}_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j)\\ &= -1+\frac{9\tau}{8}\sum_{\ell}\left(-\widetilde{P}_3(\partial_i,\partial_i,\partial_\ell)\widetilde{P}_3(\partial_j,\partial_j,\partial_\ell)+\widetilde{P}_3(\partial_i,\partial_j,\partial_\ell)^2\right)\\ &= -1+\frac{9\tau}{8}\sum_{\ell}\left(-P_3(F\partial_i,F\partial_i,F\partial_\ell)P_3(F\partial_j,F\partial_j,F\partial_\ell)+P_3(F\partial_i,F\partial_j,F\partial_\ell)^2\right).\end{aligned}$$

□

In the next part of this section, we will determine the second variation of the P_k-polynomials in (3.12) which we will define analogously to their first variation determined in Proposition 3.26 and defined in Definition 3.27.

Definition 3.32 (Second variation of the P_k's)**.** *With the assumptions of Proposition 3.26 we define for $\tau \geq 3$ and $3 \leq k \leq \tau$*

$$\delta^2 P_k(y) := \left(\frac{1}{(\tau-k)!} \partial_x^{\tau-k} \left(\partial_z^2 \left(h \left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right) \right) \Big|_{z=0} \right) \right) \Bigg|_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right)}, \tag{3.48}$$

where $\partial_z^2 \left(h \left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right) \right)$ denotes the second derivative with respect to the $z = (z_1, \ldots, z_n)^T$-coordinates, $\left(\partial_z^2 \left(h \left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right) \right)\right)_{ij} = \frac{\partial^2}{\partial z_i \partial z_j} \left(h \left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right) \right)$. For each $3 \leq k \leq \tau$ we call $\delta^2 P_k$ the second variation of P_k along $\mathcal{H}$ with respect to the chosen dB_0 (3.28), respectively $d\mathcal{A}_0$ (3.26), and understand $\delta^2 P_k(y)$ as a bilinear symmetric map $\delta P_k(y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Sym}^k \left((\mathbb{R}^n)^* \right)$.

Note that with the conventions introduced in Definition 3.48

$$\partial_z^2 \left(h \left(\mathcal{A}(z) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right) \right) \Big|_{z=0} = \sum_{i=3}^{\tau} x^{\tau-i} \delta^2 P_i(y).$$

Proposition 3.33 ($\delta^2 P_k$'s for $\tau = 3$ and $\tau = 4$)**.** *With the assumptions and notations from Proposition 3.26, $\delta^2 P_3(y)$ for connected PSR manifolds is of the form (3.55). For quartic connected GPSR manifolds, that is connected GPSR manifolds $\mathcal{H} \subset \{h = 1\}$ with corresponding h of homogeneity-degree $\tau = 4$, $\delta^2 P_3(y)$ is of the form (3.56) and $\delta^2 P_4(y)$ is of the form (3.57).*

Proof. The proof is, up to one step[4], just a big calculation. We will however include the main steps of this calculation, since the general result can most likely not be obtained with a computer algebra system for arbitrary $\tau \geq 3$ and arbitrary $n \in \mathbb{N}$. Furthermore, as the calculations become longer, the potential for calculation errors rises. So if one wants to check the result, intermediate steps will provide cornerstones for checking ones own calculations. We start with the formulas for the first and second derivatives of h and $\mathcal{A}$. It is easy to confuse some symmetric tensors with non-symmetric tensors, e.g. consider for any smooth $F : \mathrm{dom}(\mathcal{H}) \to \mathrm{GL}(n)$ and $v, w \in T_0 \mathrm{dom}(\mathcal{H}) \cong \mathbb{R}^n$

$$\left(\langle dz, dF_0 \cdot \rangle \right)(v, w) = \frac{1}{2} \left(\langle v, dF_0 w \rangle + \langle w, dF_0 v \rangle \right),$$

and using wrong pre-factors when forgetting about the factor $\frac{1}{2}$. For this reason we will use the identification $\langle dz, dF_0 \cdot \rangle = \frac{1}{2} \left(\langle d_1 z, d_2 F_0 \cdot \rangle + \langle d_2 z, d_1 F_0 \cdot \rangle \right)$, where $d_1 z$ and $d_2 z$ stand for the first, respectively second, direction in which we are taking the derivative. We again identify dz and dy via $d\Phi_0$ (3.14) and obtain

$$\begin{aligned} dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} = {} & \left(\tau x^{\tau-1} - (\tau-2) x^{\tau-3} \langle y, y \rangle + \sum_{i=3}^{\tau-1} (\tau-i) x^{\tau-i-1} P_i(y) \right) dx \\ & - 2 x^{\tau-2} \langle y, dy \rangle + \sum_{i=3}^{\tau} x^{\tau-i} i P_i(y, \ldots, y, dy), \end{aligned}$$

$$\begin{aligned} \partial^2 h_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} = {} & \left(\tau(\tau-1) x^{\tau-2} - (\tau-2)(\tau-3) x^{\tau-4} \langle y, y \rangle + \sum_{i=3}^{\tau-2} (\tau-i)(\tau-i-1) x^{\tau-i-2} P_i(y) \right) dx^2 \\ & + 2 \left(-2(\tau-2) x^{\tau-3} \langle y, dy \rangle + \sum_{i=3}^{\tau-1} i(\tau-i) x^{\tau-i-1} P_i(y, \ldots, y, dy) \right) dx \end{aligned}$$

[4]See implication of equation (3.49).

$$- 2x^{\tau-2}\langle dy, dy\rangle + \sum_{i=3}^{\tau} i(i-1)x^{\tau-i}P_i(y,\ldots,y,dy,dy),$$

and

$$dA_0 = \left(\begin{array}{c|c} 0 & \frac{2}{\tau}dz^T \\ \hline dz & d\mathcal{E}_0 \end{array}\right),$$

$$\partial^2 A_0 = \left(\begin{array}{c|c} \frac{2}{\tau}\langle dz, dz\rangle & \frac{2}{\tau}\left(\langle d_1 z, d_2\mathcal{E}_0\cdot\rangle + \langle d_2 z, d_1\mathcal{E}_0\cdot\rangle - 3P_3(d_1z,d_2z,\cdot)\right) \\ \hline 0 & \partial^2\mathcal{E}_0 \end{array}\right),$$

where we, similarly to Proposition 3.26 and using the identification $T_0\mathrm{dom}(\mathcal{H}) \cong \mathbb{R}^n$, understand $d\mathcal{E}_0$ as an element of $\mathrm{Lin}(\mathbb{R}^n, \mathfrak{gl}(n))$ and $\partial^2\mathcal{E}_0$ as a $\mathfrak{gl}(n)$-valued symmetric $(0,2)$-tensor, i.e. $\partial^2\mathcal{E}_0 \in \mathrm{Sym}^2\left((\mathbb{R}^n)^*\right) \otimes \mathfrak{gl}(n)$.

The following calculations are slightly different for $\tau = 3$, $\tau = 4$, and $\tau \geq 5$, respectively. The difference is that certain terms, for example terms of order $x^{\tau-5}$, do not appear for $\tau = 3$ and $\tau = 4$. We will present the calculations for $\tau \geq 5$ and present the results for the cases $\tau = 3$ and $\tau = 4$, which are easily obtain by slightly modifying the $\tau \geq 5$-case. We obtain with (3.26) and (3.28)

$$\begin{aligned}
&\partial^2 h_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(d_1A_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), d_2A_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \\
&= x^{\tau}(-2\langle d_1z, d_2z\rangle) \\
&+ x^{\tau-1}\left(-2\langle d_1z, d_2\mathcal{E}_0y\rangle - 2\langle d_2z, d_1\mathcal{E}_0y\rangle\right) \\
&+ x^{\tau-2}\left(\frac{-4\tau+12}{\tau}\langle y, d_1z\rangle\langle y, d_2z\rangle - 2\langle d_1\mathcal{E}_0y, d_2\mathcal{E}_0y\rangle\right) \\
&+ x^{\tau-3}\left(\frac{-4(\tau-2)}{\tau}\left(\langle y, d_1\mathcal{E}_0y\rangle\langle y, d_2z\rangle + \langle y, d_2\mathcal{E}_0y\rangle\langle y, d_1z\rangle\right)\right) \\
&+ x^{\tau-4}\left(\frac{-4(\tau-2)(\tau-3)}{\tau^2}\langle y, y\rangle\langle y, d_1z\rangle\langle y, d_2z\rangle\right) \\
&+ \left(\sum_{i=5}^{\tau} x^{\tau-i}\frac{4(\tau-i+2)(\tau-i+1)}{\tau^2}P_{i-2}(y)\langle y, d_1z\rangle\langle y, d_2z\rangle\right) \\
&+ \left(\sum_{i=4}^{\tau} x^{\tau-i}\frac{2(i-1)(\tau-i+1)}{\tau}\left(P_{i-1}(y,\ldots,y,d_2\mathcal{E}_0y)\langle y, d_1z\rangle\right.\right. \\
&\qquad \left.\left.+P_{i-1}(y,\ldots,y,d_1\mathcal{E}_0y)\langle y, d_2z\rangle\right)\right) \\
&+ \left(\sum_{i=3}^{\tau-1} x^{\tau-i}\frac{2i(\tau-i)}{\tau}\left(P_i(y,\ldots,y,d_2z)\langle y, d_1z\rangle + P_i(y,\ldots,y,d_1z)\langle y, d_2z\rangle\right)\right) \\
&+ \left(\sum_{i=3}^{\tau} x^{\tau-i}i(i-1)P_i(y,\ldots,y,d_1\mathcal{E}_0y,d_2\mathcal{E}_0y)\right) \\
&+ \left(\sum_{i=2}^{\tau-1} x^{\tau-i}(i+1)i\left(P_{i+1}(y,\ldots,y,d_2\mathcal{E}_0y,d_1z) + P_{i+1}(y,\ldots,y,d_1\mathcal{E}_0y,d_2z)\right)\right) \\
&+ \left(\sum_{i=1}^{\tau-2} x^{\tau-i}(i+2)(i+1)P_{i+2}(y,\ldots,y,d_1z,d_2z)\right)
\end{aligned}$$

and

$$\begin{aligned}
&dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(\partial^2A_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \\
&= x^{\tau}\left(2\langle d_1z, d_2z\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&+ x^{\tau-1}\left(2(\langle d_1 z, d_2\mathcal{E}_0 y\rangle + \langle d_2 z, d_1\mathcal{E}_0 y\rangle) - 6P_3(d_1 z, d_2 z, y)\right)\\
&+ x^{\tau-2}\left(\frac{-2(\tau-2)}{\tau}\langle y,y\rangle\langle d_1 z, d_2 z\rangle - 2\langle y, \partial^2\mathcal{E}_0 y\rangle\right)\\
&+ x^{\tau-3}\left(\frac{-2(\tau-2)}{\tau}\langle y,y\rangle(\langle d_1 z, d_2\mathcal{E}_0 y\rangle + \langle d_2 z, d_1\mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y))\right)\\
&+ \left(\sum_{i=4}^{\tau} x^{\tau-i}\frac{2(\tau-i+1)}{\tau}P_{i-1}(y)(\langle d_1 z, d_2\mathcal{E}_0 y\rangle + \langle d_2 z, d_1\mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y))\right)\\
&+ \left(\sum_{i=3}^{\tau-1} x^{\tau-i}\frac{2(\tau-i)}{\tau}P_i(y)\langle d_1 z, d_2 z\rangle\right)\\
&+ \left(\sum_{i=3}^{\tau} x^{\tau-i} i P_i(y,\ldots,y,\partial^2\mathcal{E}_0 y)\right).
\end{aligned}
$$

We immediately see that the terms of order x^τ and $x^{\tau-1}$ in the sum

$$\partial^2 h_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right), d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) + dh_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(\partial^2\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)$$

vanish as expected. Furthermore, the term of order $x^{\tau-2}$ is also required to vanish, which yields the following equation for $\partial^2\mathcal{E}_0$:

$$
\begin{aligned}
\langle y, \partial^2\mathcal{E}_0 y\rangle = {}& \frac{-2(\tau-2)}{\tau}\langle y,y\rangle\langle d_1 z, d_2 z\rangle + \frac{-4(\tau-3)}{\tau}\langle y, d_1 z\rangle\langle y, d_2 z\rangle - 2\langle d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y\rangle\\
&+ 6(P_3(y, d_2\mathcal{E}_0 y, d_1 z) + P_3(y, d_1\mathcal{E}_0 y, d_2 z)) + 12P_4(y,y,d_1 z, d_2 z).
\end{aligned}
\tag{3.49}
$$

Assuming that $\partial^2\mathcal{E}_0$ fulfils equation (3.49), we obtain

$$
\begin{aligned}
&\partial^2 h_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right), d\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) + dh_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(\partial^2\mathcal{A}_0\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\\
&= x^{\tau-3}\left(\frac{-4(\tau-2)}{\tau}(\langle y, d_1\mathcal{E}_0 y\rangle\langle y, d_2 z\rangle + \langle y, d_2\mathcal{E}_0 y\rangle\langle y, d_1 z\rangle)\right.\\
&\quad + \frac{6(\tau-3)}{\tau}(P_3(y,y,d_2 z)\langle y, d_1 z\rangle + P_3(y,y,d_1 z)\langle y, d_2 z\rangle)\\
&\quad + 6P_3(y, d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y)\\
&\quad + 12(P_4(y,y,d_2\mathcal{E}_0 y, d_1 z) + P_4(y,y,d_1\mathcal{E}_0 y, d_2 z))\\
&\quad + 20P_5(y,y,y,d_1 z, d_2 z)\\
&\quad + \frac{-2(\tau-2)}{\tau}\langle y,y\rangle(\langle d_1 z, d_2\mathcal{E}_0 y\rangle + \langle d_2 z, d_1\mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y))\\
&\quad \left. + \frac{2(\tau-3)}{\tau}P_3(y)\langle d_1 z, d_2 z\rangle + 3P_3(y,y,\partial^2\mathcal{E}_0 y)\right)\\
&+ x^{\tau-4}\left(\frac{-4(\tau-2)(\tau-3)}{\tau^2}\langle y,y\rangle\langle y, d_1 z\rangle\langle y, d_2 z\rangle\right.\\
&\quad + \frac{8(\tau-4)}{\tau}(P_4(y,y,y,d_2 z)\langle y, d_1 z\rangle + P_4(y,y,y,d_1 z)\langle y, d_2 z\rangle)\\
&\quad + \frac{6(\tau-3)}{\tau}(P_3(y,y,d_2\mathcal{E}_0 y)\langle y, d_1 z\rangle + P_3(y,y,d_1\mathcal{E}_0 y)\langle y, d_2 z\rangle)\\
&\quad + 12P_4(y,y,d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y)\\
&\quad + 20(P_5(y,y,y,d_2\mathcal{E}_0 y, d_1 z) + P_5(y,y,y,d_1\mathcal{E}_0 y, d_2 z))\\
&\quad + 30P_6(y,y,y,y,d_1 z, d_2 z)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2(\tau-4)}{\tau} P_4(y)\langle d_1 z, d_2 z\rangle \\
&+ \frac{2(\tau-3)}{\tau} P_3(y)(\langle d_1 z, d_2 \mathcal{E}_0 y\rangle + \langle d_2 z, d_1 \mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y)) + 4P_4(y,y,y,\partial^2 \mathcal{E}_0 y)\Big) \\
+ x&\Big(\frac{24}{\tau^2} P_{\tau-3}(y)\langle y, d_1 z\rangle\langle y, d_2 z\rangle \\
&+ \frac{4(\tau-2)}{\tau}(P_{\tau-2}(y,\ldots,y,d_2\mathcal{E}_0 y)\langle y, d_1 z\rangle + P_{\tau-2}(y,\ldots,y,d_1\mathcal{E}_0 y)\langle y, d_2 z\rangle) \\
&+ \frac{4}{\tau} P_{\tau-2}(y)(\langle d_1 z, d_2 \mathcal{E}_0 y\rangle + \langle d_2 z, d_1 \mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y)) \\
&+ \frac{2(\tau-1)}{\tau}(P_{\tau-1}(y,\ldots,y,d_2 z)\langle y, d_1 z\rangle + P_{\tau-1}(y,\ldots,y,d_1 z)\langle y, d_2 z\rangle) \\
&+ (\tau-1)(\tau-2)P_{\tau-1}(y,\ldots,y,d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y) \\
&+ \frac{2}{\tau} P_{\tau-1}(y)\langle d_1 z, d_2 z\rangle + (\tau-1)P_{\tau-1}(y,\ldots,y,\partial^2\mathcal{E}_y) \\
&+ \tau(\tau-1)(P_\tau(y,\ldots,y,d_2\mathcal{E}_0 y, d_1 z) + P_\tau(y,\ldots,y,d_1\mathcal{E}_0 y, d_2 z))\Big) \\
+ &\Big(\frac{8}{\tau^2} P_{\tau-2}(y)\langle y, d_1 z\rangle\langle y, d_2 z\rangle \\
&+ \frac{2(\tau-1)}{\tau}(P_{\tau-1}(y,\ldots,y,d_2\mathcal{E}_0 y)\langle y, d_1 z\rangle + P_{\tau-1}(y,\ldots,y,d_1\mathcal{E}_0 y)\langle y, d_2 z\rangle) \\
&+ \frac{2}{\tau} P_{\tau-1}(y)(\langle d_1 z, d_2 \mathcal{E}_0 y\rangle + \langle d_2 z, d_1 \mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y)) \\
&+ \tau(\tau-1)P_\tau(y,\ldots,y,d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y) + \tau P_\tau(y,\ldots,y,\partial^2\mathcal{E}_0 y)\Big) \\
+ &\sum_{i=5}^{\tau-2} x^{\tau-i}\Bigg(\frac{4(\tau-i+2)(\tau-i+1)}{\tau^2} P_{i-2}(y)\langle y, d_1 z\rangle\langle y, d_2 z\rangle \\
&+ \frac{2(i-1)(\tau-i+1)}{\tau}(P_{i-1}(y,\ldots,y,d_2\mathcal{E}_0 y)\langle y, d_1 z\rangle + P_{i-1}(y,\ldots,y,d_1\mathcal{E}_0 y)\langle y, d_2 z\rangle) \\
&+ \frac{2(\tau-i+1)}{\tau} P_{i-1}(y)(\langle d_1 z, d_2 \mathcal{E}_0 y\rangle + \langle d_2 z, d_1 \mathcal{E}_0 y\rangle - 3P_3(d_1 z, d_2 z, y)) \\
&+ \frac{2i(\tau-i)}{\tau}(P_i(y,\ldots,y,d_2 z)\langle y, d_1 z\rangle + P_i(y,\ldots,y,d_1 z)\langle y, d_2 z\rangle) \\
&+ i(i-1)P_i(y,\ldots,y,d_1\mathcal{E}_0 y, d_2\mathcal{E}_0 y) \\
&+ \frac{2(\tau-i)}{\tau} P_i(y)\langle d_1 z, d_2 z\rangle + iP_i(y,\ldots,y,\partial^2\mathcal{E}_0 y) \\
&+ (i+1)i(P_{i+1}(y,\ldots,y,d_2\mathcal{E}_0 y, d_1 z) + P_{i+1}(y,\ldots,y,d_1\mathcal{E}_0 y, d_2 z)) \\
&+ (i+2)(i+1)P_{i+2}(y,\ldots,y,d_1 z, d_2 z)\Bigg).
\end{aligned}
$$

In this proposition we want to determine $\delta^2 P_3(y)$ and $\delta^2 P_4(y)$, and are thus only interested in the terms of order $x^{\tau-3}$ and $x^{\tau-4}$. The terms of order $x^{\tau-i}$ for $5 \leq i \leq \tau$ and related analogous definitions of $\delta^2 P_i(y)$ can be determined in a similar way.Note that the summation ranges in the above formula are precisely the reason why we have to be careful in the cases $\tau = 3$ and $\tau = 4$. At this point we have shown that

$$
\delta^2 P_3(y) = \frac{-4(\tau-2)}{\tau}(\langle y, d_1\mathcal{E}_0 y\rangle\langle y, d_2 z\rangle + \langle y, d_2\mathcal{E}_0 y\rangle\langle y, d_1 z\rangle)
$$

$$
\begin{aligned}
&+\frac{6(\tau-3)}{\tau}(P_3(y,y,d_2z)\langle y,d_1z\rangle+P_3(y,y,d_1z)\langle y,d_2z\rangle)\\
&+6P_3(y,d_1\mathcal{E}_0y,d_2\mathcal{E}_0y)\\
&+12(P_4(y,y,d_2\mathcal{E}_0y,d_1z)+P_4(y,y,d_1\mathcal{E}_0y,d_2z))\\
&+20P_5(y,y,y,d_1z,d_2z)\\
&+\frac{-2(\tau-2)}{\tau}\langle y,y\rangle(\langle d_1z,d_2\mathcal{E}_0y\rangle+\langle d_2z,d_1\mathcal{E}_0y\rangle-3P_3(d_1z,d_2z,y))\\
&+\frac{2(\tau-3)}{\tau}P_3(y)\langle d_1z,d_2z\rangle+3P_3(y,y,\partial^2\mathcal{E}_0y).
\end{aligned}
\tag{3.50}
$$

Next we will replace the first and second derivatives of $\mathcal{E}$ at $z=0$. In Proposition 3.26 we have already determined that $d\mathcal{E}_0$ fulfils $d\mathcal{E}_0(y)=\frac{3}{2}P_3(y,\cdot,dz)^T+dB_0y$, $dB_0\in\mathrm{Lin}(\mathbb{R}^n,\mathfrak{so}(n))$, for details see equation (3.28) and the discussion following it. We denote by $\mathrm{End}_{\mathrm{sym.}}(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ the symmetric endomorphisms of $\mathbb{R}^n$ with respect to the standard Euclidean scalar product and uniquely decompose $\mathfrak{gl}(n)=\mathrm{End}_{\mathrm{sym.}}(\mathbb{R}^n,\langle\cdot,\cdot\rangle)\oplus\mathfrak{so}(n)$, so that we can write

$$
\partial^2\mathcal{E}_0=\mathfrak{s}+\mathfrak{a},\quad \mathfrak{s}\in\mathrm{Sym}^2\left(\mathbb{R}^n\right)^*\otimes\mathrm{End}_{\mathrm{sym.}}(\mathbb{R}^n,\langle\cdot,\cdot\rangle),\ \mathfrak{a}\in\mathrm{Sym}^2\left(\mathbb{R}^n\right)^*\otimes\mathfrak{so}(n).
\tag{3.51}
$$

Hence, we obtain for the symmetric part $\mathfrak{s}$ of $\partial^2\mathcal{E}_0$

$$
\begin{aligned}
\langle v,\mathfrak{s}w\rangle=&\frac{-(\tau-2)}{\tau}\langle v,w\rangle\langle d_1z,d_2z\rangle\\
&+\frac{-(\tau-3)}{\tau}(v,d_1z\rangle\langle w,d_2z\rangle+\langle w,d_1z\rangle\langle v,d_2z\rangle)\\
&-\frac{1}{2}(\langle d_1\mathcal{E}_0v,d_2\mathcal{E}_0w\rangle+\langle d_1\mathcal{E}_0w,d_2\mathcal{E}_0v\rangle)\\
&+\frac{3}{2}(P_3(v,d_2\mathcal{E}_0w,d_1z)+P_3(w,d_2\mathcal{E}_0v,d_1z)\\
&\quad+P_3(v,d_1\mathcal{E}_0w,d_2z)+P_3(w,d_1\mathcal{E}_0v,d_2z))\\
&+6P_4(v,w,d_1z,d_2z).
\end{aligned}
\tag{3.52}
$$

The only part in equation (3.50) containing the term $\partial^2\mathcal{E}_0$ is $3P_3(y,y,\partial^2\mathcal{E}_0y)$. By setting $v=P_3(y,y,\cdot)^T$ and $w=y$, we can insert equations (3.51) and (3.52) into $3P_3(y,y,\partial^2\mathcal{E}_0y)$:

$$
\begin{aligned}
3P_3(y,y,\partial^2\mathcal{E}_0y)=&\,3P_3(y,y,\mathfrak{a}y)\\
&+\frac{-3(\tau-2)}{\tau}P_3(y)\langle d_1z,d_2z\rangle\\
&+\frac{-3(\tau-3)}{\tau}(P_3(y,y,d_1z)\langle y,d_2z\rangle+P_3(y,y,d_2z)\langle y,d_1z\rangle)\\
&-\frac{3}{2}\left(P_3\left(y,y,\langle d_2\mathcal{E}_0y,d_1\mathcal{E}_0\cdot\rangle^T\right)+P_3\left(y,y,\langle d_1\mathcal{E}_0y,d_2\mathcal{E}_0\cdot\rangle^T\right)\right)\\
&+\frac{9}{2}\left(P_3\left(y,d_2\mathcal{E}_0P_3(y,y,\cdot)^T,d_1z\right)+P_3\left(P_3(y,y,\cdot)^T,d_2\mathcal{E}_0y,d_1z\right)\right.\\
&\quad\left.+P_3\left(y,d_1\mathcal{E}_0P_3(y,y,\cdot)^T,d_2z\right)+P_3\left(P_3(y,y,\cdot)^T,d_1\mathcal{E}_0y,d_2z\right)\right)\\
&+18P_3\left(y,y,P_4(y,d_1z,d_2z,\cdot)^T\right).
\end{aligned}
$$

Hence,

$$
\delta^2P_3(y)=\frac{-4(\tau-2)}{\tau}(\langle y,d_1\mathcal{E}_0y\rangle\langle y,d_2z\rangle+\langle y,d_2\mathcal{E}_0y\rangle\langle y,d_1z\rangle)
$$

$$
\begin{aligned}
&+\frac{3(\tau-3)}{\tau}(P_3(y,y,d_2z)\langle y,d_1z\rangle+P_3(y,y,d_1z)\langle y,d_2z\rangle)\\
&+20P_5(y,y,y,d_1z,d_2z)\\
&+12(P_4(y,y,d_2\mathcal{E}_0y,d_1z)+P_4(y,y,d_1\mathcal{E}_0y,d_2z))\\
&+6P_3(y,d_1\mathcal{E}_0y,d_2\mathcal{E}_0y)\\
&+\frac{-2(\tau-2)}{\tau}\langle y,y\rangle(\langle d_1z,d_2\mathcal{E}_0y\rangle+\langle d_2z,d_1\mathcal{E}_0y\rangle-3P_3(y,d_1z,d_2z))\\
&+\frac{-(\tau-2)}{\tau}P_3(y)\langle d_1z,d_2z\rangle\\
&+P_3(y,y,\mathfrak{a}y)\\
&-\frac{3}{2}P_3\left(y,y,\langle d_2\mathcal{E}_0y,d_1\mathcal{E}_0\cdot\rangle^T+\langle d_1\mathcal{E}_0y,d_2\mathcal{E}_0\cdot\rangle^T\right)\\
&+18P_3\left(y,y,P_4(y,d_1z,d_2z,\cdot)^T\right)\\
&+\frac{9}{2}\left(P_3\left(y,d_2\mathcal{E}_0P_3(y,y,\cdot)^T,d_1z\right)+P_3\left(P_3(y,y,\cdot)^T,d_2\mathcal{E}_0y,d_1z\right)\right.\\
&\quad\left.+P_3\left(y,d_1\mathcal{E}_0P_3(y,y,\cdot)^T,d_2z\right)+P_3\left(P_3(y,y,\cdot)^T,d_1\mathcal{E}_0y,d_2z\right)\right).
\end{aligned}
$$

The next step is replacing $d\mathcal{E}_0(y)=\frac{3}{2}P_3(y,\cdot,dz)^T+dB_0y$, cf. equation (3.28). This yields

$$
\begin{aligned}
\delta^2P_3(y)=&\frac{-3(\tau-1)}{\tau}(P_3(y,y,d_2z)\langle y,d_1z\rangle+P_3(y,y,d_1z)\langle y,d_2z\rangle)\\
&+20P_5(y,y,y,d_1z,d_2z)\\
&+18\left(P_4\left(y,y,P_3\left(y,\cdot,d_2z\right)^T,d_1z\right)\right.\\
&\quad+P_4\left(y,y,P_3\left(y,\cdot,d_1z\right)^T,d_2z\right)\\
&\quad\left.+P_4\left(y,P_3\left(y,y,\cdot\right)^T,d_1z,d_2z\right)\right)\\
&+\frac{27}{2}P_3\left(y,P_3(y,\cdot,d_1z)^T,P_3(y,\cdot,d_2z)^T\right)\\
&-\frac{27}{8}P_3\left(y,y,P_3\left(P_3(y,\cdot,d_2z)^T,\cdot,d_1z\right)^T+P_3\left(P_3(y,\cdot,d_1z)^T,\cdot,d_2z\right)^T\right)\\
&+\frac{-(\tau-2)}{\tau}P_3(y)\langle d_1z,d_2z\rangle\\
&+\frac{27}{4}\left(P_3\left(y,d_1z,P_3\left(P_3(y,y,\cdot)^T,\cdot,d_2z)^T\right)\right)+P_3\left(P_3(y,y,\cdot)^T,P_3(y,\cdot,d_2z)^T,d_1z\right)\right.\\
&\quad\left.+P_3\left(y,d_2z,P_3\left(P_3(y,y,\cdot)^T,\cdot,d_1z)^T\right)\right)+P_3\left(P_3(y,y,\cdot)^T,P_3(y,\cdot,d_1z)^T,d_2z\right)\right)\\
&+12(P_4(y,y,d_2B_0y,d_1z)+P_4(y,y,d_1B_0y,d_2z))\\
&+6P_3(y,d_1B_0y,d_2B_0y)\\
&+9\left(P_3\left(y,d_1B_0y,P_3(y,\cdot,d_2z)^T\right)+P_3\left(y,d_2B_0y,P_3(y,\cdot,d_1z)^T\right)\right)\\
&+\frac{-2(\tau-2)}{\tau}\langle y,y\rangle(\langle d_1z,d_2B_0y\rangle+\langle d_2z,d_1B_0y\rangle)\\
&+3P_3(y,y,\mathfrak{a}y)\\
&-\frac{3}{2}P_3\left(y,y,\langle d_2B_0y,d_1B_0\cdot\rangle^T+\langle d_1B_0y,d_2B_0\cdot\rangle^T\right)\\
&-\frac{9}{4}\left(P_3\left(y,y,P_3(d_2B_0y,\cdot,d_1z)^T\right)+P_3\left(y,y,P_3(d_1B_0y,\cdot,d_2z)^T\right)\right.\\
&\quad\left.+P_3\left(y,y,P_3(y,d_2B_0\cdot,d_1z)^T\right)+P_3\left(y,y,P_3(y,d_1B_0\cdot,d_2z)^T\right)\right)
\end{aligned}
$$

$$+\frac{9}{2}\left(-P_3\left(y,P_3(y,y,d_2B_0\cdot)^T,d_1z\right)+P_3\left(P_3(y,y,\cdot)^T,d_2B_0y,d_1z\right)\right.$$
$$\left.-P_3\left(y,P_3(y,y,d_1B_0\cdot)^T,d_2z\right)+P_3\left(P_3(y,y,\cdot)^T,d_1B_0y,d_2z\right)\right). \tag{3.53}$$

Note that we have used that dB_0 has image in $\mathfrak{so}(n)$, that is $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$ or, equivalently, $dB_0^T = -dB_0$ and, hence,

$$dB_0P_3(y,y,\cdot)^T = \left(P_3(y,y,\cdot)dB_0^T\cdot\right)^T = (-P_3(y,y,dB_0\cdot))^T = -P_3(y,y,dB_0\cdot)^T.$$

The formula for $\delta^2P_4(y)$ is obtained with methods analogous to the ones used for the calculation of $\delta^2P_3(y)$. We obtain

$$\begin{aligned}
\delta^2P_4(y) = &\frac{-4(\tau-2)(\tau-3)}{\tau^2}\langle y,y\rangle\langle y,d_1z\rangle\langle y,d_2z\rangle\\
&+\frac{4(\tau-5)}{\tau}(P_4(y,y,y,d_2z)\langle y,d_1z\rangle+P_4(y,y,y,d_1z)\langle y,d_2z\rangle)\\
&+\frac{9(\tau-3)}{\tau}\left(P_3\left(y,y,P_3(y,\cdot,d_2z)^T\right)\langle y,d_1z\rangle+P_3\left(y,y,P_3(y,\cdot,d_1z)^T\right)\langle y,d_2z\rangle\right)\\
&\quad+18P_4\left(y,y,P_3(y,\cdot,d_1z)^T,P_3(y,\cdot,d_2z)^T\right)\\
&+30\left(P_5\left(y,y,y,P_3(y,\cdot,d_2z)^T,d_1z\right)+P_5\left(y,y,y,P_3(y,\cdot,d_1z)^T,d_2z\right)\right)\\
&+30P_6(y,y,y,y,d_1z,d_2z)\\
&-2P_4(y)\langle d_1z,d_2z\rangle\\
&+\frac{27}{2}P_4\left(y,y,y,P_3\left(P_3(y,\cdot,d_2z)^T,\cdot,d_1z\right)+P_3\left(P_3(y,\cdot,d_1z)^T,\cdot,d_2z\right)\right)\\
&+24P_4\left(y,y,y,P_4(y,\cdot,d_1z,d_2z)^T\right)\\
&+\frac{6(\tau-3)}{\tau}(P_3(y,y,d_2B_0y)\langle y,d_1z\rangle+P_3(y,y,d_1B_0y)\langle y,d_2z\rangle)\\
&+18\left(P_4\left(y,y,d_1B_0y,P_3(y,\cdot,d_2z)^T\right)+P_4\left(y,y,d_2B_0y,P_3(y,\cdot,d_1z)^T\right)\right)\\
&+12P_4(y,y,d_1B_0y,d_2B_0y)\\
&+20(P_5(y,y,y,d_2B_0y,d_1z)+P_5(y,y,y,d_1B_0y,d_2z))\\
&+\frac{2(\tau-3)}{\tau}P_3(y)(\langle d_1z,d_2B_0y\rangle+\langle d_2z,d_1B_0y\rangle)\\
&+4P_4(y,y,y,\mathfrak{a}y)\\
&+3P_4\left(y,y,y,P_3(d_2B_0y,\cdot,d_1z)^T+P_3(y,d_2B_0\cdot,d_1z)^T\right.\\
&\quad\left.+P_3(d_1B_0y,\cdot,d_2z)^T+P_3(y,d_1B_0\cdot,d_2z)^T\right)\\
&-2P_4\left(y,y,y,\langle d_2B_0y,d_1B_0\cdot\rangle^T+\langle d_1B_0y,d_2B_0\cdot\rangle^T\right).
\end{aligned} \tag{3.54}$$

To obtain $\partial^2P_3(y)$ for $\tau=3$, respectively $\partial^2P_3(y)$ and $\partial^2P_4(y)$ for $\tau=4$, it turns out that we can simply set the P_i's for $i>3$, respectively $i>4$, to zero. Also not that we do not run into any difficulty with P_i's of the form $P_{\tau-k}$, which might not be defined, if we are only interested in $\partial^2P_3(y)$ and $\partial^2P_4(y)$. We obtain for $\tau=3$, that is for cubic polynomials h, (see equation (3.53))

$$\begin{aligned}
\delta^2P_3(y) = &-2(P_3(y,y,d_2z)\langle y,d_1z\rangle+P_3(y,y,d_1z)\langle y,d_2z\rangle)\\
&+\frac{27}{2}P_3\left(y,P_3(y,\cdot,d_1z)^T,P_3(y,\cdot,d_2z)^T\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{27}{8} P_3\left(y, y, P_3\left(P_3(y,\cdot,d_2z)^T,\cdot,d_1z\right)^T + P_3\left(P_3(y,\cdot,d_1z)^T,\cdot,d_2z\right)^T\right)\\
&- \frac{1}{3} P_3(y)\langle d_1z, d_2z\rangle\\
&+ \frac{27}{4}\left(P_3\left(y, d_1z, P_3\left(P_3(y,y,\cdot)^T,\cdot,d_2z)^T\right)\right) + P_3\left(P_3(y,y,\cdot)^T, P_3(y,\cdot,d_2z)^T, d_1z\right)\right.\\
&\quad \left. + P_3\left(y, d_2z, P_3\left(P_3(y,y,\cdot)^T,\cdot,d_1z)^T\right)\right) + P_3\left(P_3(y,y,\cdot)^T, P_3(y,\cdot,d_1z)^T, d_2z\right)\right)\\
&+ 6P_3(y, d_1B_0y, d_2B_0y)\\
&+ 9\left(P_3\left(y, d_1B_0y, P_3(y,\cdot,d_2z)^T\right) + P_3\left(y, d_2B_0y, P_3(y,\cdot,d_1z)^T\right)\right)\\
&- \frac{2}{3}\langle y,y\rangle(\langle d_1z, d_2B_0y\rangle + \langle d_2z, d_1B_0y\rangle)\\
&+ 3P_3(y,y,\mathfrak{a}y)\\
&- \frac{3}{2} P_3\left(y, y, \langle d_2B_0y, d_1B_0\cdot\rangle^T + \langle d_1B_0y, d_2B_0\cdot\rangle^T\right)\\
&- \frac{9}{4}\left(P_3\left(y,y,P_3(d_2B_0y,\cdot,d_1z)^T\right) + P_3\left(y,y,P_3(d_1B_0y,\cdot,d_2z)^T\right)\right.\\
&\quad \left. + P_3\left(y,y,P_3(y,d_2B_0\cdot,d_1z)^T\right) + P_3\left(y,y,P_3(y,d_1B_0\cdot,d_2z)^T\right)\right)\\
&+ \frac{9}{2}\left(-P_3\left(y, P_3(y,y,d_2B_0\cdot)^T, d_1z\right) + P_3\left(P_3(y,y,\cdot)^T, d_2B_0y, d_1z\right)\right.\\
&\quad \left. - P_3\left(y, P_3(y,y,d_1B_0\cdot)^T, d_2z\right) + P_3\left(P_3(y,y,\cdot)^T, d_1B_0y, d_2z\right)\right),
\end{aligned}
\tag{3.55}
$$

and for $\tau = 4$, that is for quartic polynomials h (see equations (3.53) and (3.54)),

$$
\begin{aligned}
\delta^2 P_3(y) = &- \frac{9}{4}(P_3(y,y,d_2z)\langle y, d_1z\rangle + P_3(y,y,d_1z)\langle y, d_2z\rangle)\\
&+ 18\left(P_4\left(y,y,P_3\left(y,\cdot,d_2z\right)^T, d_1z\right)\right.\\
&\quad + P_4\left(y,y,P_3\left(y,\cdot,d_1z\right)^T, d_2z\right)\\
&\quad \left. + P_4\left(y, P_3\left(y,y,\cdot\right)^T, d_1z, d_2z\right)\right)\\
&+ \frac{27}{2} P_3\left(y, P_3(y,\cdot,d_1z)^T, P_3(y,\cdot,d_2z)^T\right)\\
&- \frac{27}{8} P_3\left(y, y, P_3\left(P_3(y,\cdot,d_2z)^T,\cdot,d_1z\right)^T + P_3\left(P_3(y,\cdot,d_1z)^T,\cdot,d_2z\right)^T\right)\\
&- \frac{1}{2} P_3(y)\langle d_1z, d_2z\rangle\\
&+ \frac{27}{4}\left(P_3\left(y, d_1z, P_3\left(P_3(y,y,\cdot)^T,\cdot,d_2z)^T\right)\right) + P_3\left(P_3(y,y,\cdot)^T, P_3(y,\cdot,d_2z)^T, d_1z\right)\right.\\
&\quad \left. + P_3\left(y, d_2z, P_3\left(P_3(y,y,\cdot)^T,\cdot,d_1z)^T\right)\right) + P_3\left(P_3(y,y,\cdot)^T, P_3(y,\cdot,d_1z)^T, d_2z\right)\right)\\
&+ 12(P_4(y,y,d_2B_0y,d_1z) + P_4(y,y,d_1B_0y,d_2z))\\
&+ 6P_3(y, d_1B_0y, d_2B_0y)\\
&+ 9\left(P_3\left(y, d_1B_0y, P_3(y,\cdot,d_2z)^T\right) + P_3\left(y, d_2B_0y, P_3(y,\cdot,d_1z)^T\right)\right)\\
&- \langle y,y\rangle(\langle d_1z, d_2B_0y\rangle + \langle d_2z, d_1B_0y\rangle)\\
&+ 3P_3(y,y,\mathfrak{a}y)\\
&- \frac{3}{2} P_3\left(y, y, \langle d_2B_0y, d_1B_0\cdot\rangle^T + \langle d_1B_0y, d_2B_0\cdot\rangle^T\right)\\
&- \frac{9}{4}\left(P_3\left(y,y,P_3(d_2B_0y,\cdot,d_1z)^T\right) + P_3\left(y,y,P_3(d_1B_0y,\cdot,d_2z)^T\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +P_3\left(y,y,P_3(y,d_2B_0\cdot,d_1z)^T\right)+P_3\left(y,y,P_3(y,d_1B_0\cdot,d_2z)^T\right)\Big)\\
&+\frac{9}{2}\Big(-P_3\left(y,P_3(y,y,d_2B_0\cdot)^T,d_1z\right)+P_3\left(P_3(y,y,\cdot)^T,d_2B_0y,d_1z\right)\\
&\quad -P_3\left(y,P_3(y,y,d_1B_0\cdot)^T,d_2z\right)+P_3\left(P_3(y,y,\cdot)^T,d_1B_0y,d_2z\right)\Big), \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
\delta^2P_4(y)=&-\frac{1}{2}\langle y,y\rangle\langle y,d_1z\rangle\langle y,d_2z\rangle\\
&-\left(P_4(y,y,y,d_2z)\langle y,d_1z\rangle+P_4(y,y,y,d_1z)\langle y,d_2z\rangle\right)\\
&+\frac{9}{4}\Big(P_3\left(y,y,P_3(y,\cdot,d_2z)^T\right)\langle y,d_1z\rangle+P_3\left(y,y,P_3(y,\cdot,d_1z)^T\right)\langle y,d_2z\rangle\Big)\\
&\quad +12P_4\left(y,y,P_3(y,\cdot,d_1z)^T,P_3(y,\cdot,d_2z)^T\right)\\
&-2P_4(y)\langle d_1z,d_2z\rangle\\
&+\frac{27}{2}P_4\Big(y,y,y,P_3\left(P_3(y,\cdot,d_2z)^T,\cdot,d_1z\right)+P_3\left(P_3(y,\cdot,d_1z)^T,\cdot,d_2z\right)\Big)\\
&+24P_4\left(y,y,y,P_4(y,\cdot,d_1z,d_2z)^T\right)\\
&+\frac{3}{2}\left(P_3(y,y,d_2B_0y)\langle y,d_1z\rangle+P_3(y,y,d_1B_0y)\langle y,d_2z\rangle\right)\\
&+18\Big(P_4\left(y,y,d_1B_0y,P_3(y,\cdot,d_2z)^T\right)+P_4\left(y,y,d_2B_0y,P_3(y,\cdot,d_1z)^T\right)\Big)\\
&+12P_4(y,y,d_1B_0y,d_2B_0y)\\
&+\frac{1}{2}P_3(y)(\langle d_1z,d_2B_0y\rangle+\langle d_2z,d_1B_0y\rangle)\\
&+4P_4(y,y,y,\mathfrak{a}y)\\
&+3P_4\Big(y,y,y,P_3(d_2B_0y,\cdot,d_1z)^T+P_3(y,d_2B_0\cdot,d_1z)^T\\
&\quad +P_3(d_1B_0y,\cdot,d_2z)^T+P_3(y,d_1B_0\cdot,d_2z)^T\Big)\\
&-2P_4\left(y,y,y,\langle d_2B_0y,d_1B_0\cdot\rangle^T+\langle d_1B_0y,d_2B_0\cdot\rangle^T\right). \qquad (3.57)
\end{aligned}
$$

□

This would enable us to calculate the second derivative of the scalar curvature $S_{\mathcal{H}}$ of a connected GPSR manifold $(\mathcal{H},g_{\mathcal{H}})$ with relatively low effort in comparison with a direct calculation which would require calculating the 4-jet of the metric $g_{\mathcal{H}}$ at $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$. As a motivation why one really does not want to do this, the interested reader is encouraged to try calculating the 4th derivative of $g_{\mathcal{H}}$ at $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$ of her or his favourite connected GPSR manifold $\mathcal{H}$ without the help of a computer algebra system.

One application of the first variation of the P_k's as defined in Definition 3.27 is the study of homogeneous spaces that are CCGPSR manifolds. We will derive a sufficient condition for a connected GPSR manifold $\mathcal{H}\subset\{h=1\}$ to be a homogeneous space with respect to the action of G_0^h, that is the identity-component of the automorphism group G^h of h, cf. Definition 3.3.

Proposition 3.34 (Sufficient condition for homogeneity of CCGPSR manifolds)**.** *Let $\mathcal{H}\subset\{h=1\}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$, h of the form (3.12), that is*

$$h=x^\tau-x^{\tau-2}\langle y,y\rangle+\sum_{k=3}^{\tau}x^{\tau-k}P_k(y),$$

be a maximal connected GPSR manifold of dimension $n\geq 1$. Let $\delta P_k(y):\mathbb{R}^n\to\mathrm{Sym}^k\left(\mathbb{R}^n\right)^*$ as in equation (3.29) depending on $dB_0\in\mathrm{Lin}\left(\mathbb{R}^n,\mathfrak{so}(n)\right)$ (3.28), cf. Proposition 3.26. Then *the connected component containing the neutral element of the automorphism group of* h*, that is* G_0^h*, acts transitively on* $\mathcal{H}$ *if and only if there exists a choice for* $dB_0 \in \mathrm{Lin}\,(\mathbb{R}^n, \mathfrak{so}(n))$*, such that* $\delta P_k(y) \equiv 0$ *for all* $3 \leq k \leq \tau$*. Furthermore, each of the latter two equivalent statements imply that* $\mathcal{H}$ *is a CCGPSR manifold.*

Proof. By Lemma 3.14, the action $G_0^h \times \mathcal{H} \to \mathcal{H}$ is well defined. Assume that G_0^h acts transitively on the considered maximally extended GPSR manifold $\mathcal{H} \subset \{h = 1\}$. Then $\mathcal{H}$ is, in particular, a CCGPSR manifold. For $p = \left(\begin{smallmatrix} p_x \\ p_y \end{smallmatrix}\right) \in \mathcal{H}$ let $M(p) \in G_0^h$, such that $M(p) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = p$. We will show that $M(p)$ is necessarily of the form (3.7). It is clear that $M(p)$ is of the form

$$M(p) = \left(\begin{array}{c|c} p_x & v_p^T \\ \hline p_y & W(p) \end{array}\right)$$

for some $v_p \in \mathbb{R}^n$ and $W(p) \in \mathrm{Mat}(n \times n, \mathbb{R})$. We calculate

$$\begin{aligned} h\left(M(p) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = {} & x^\tau + x^{\tau-1} dh_p\left(\left(\begin{smallmatrix} \langle v_p, y\rangle \\ W(p)y \end{smallmatrix}\right)\right) + 2x^{\tau-2}\partial^2 h_p\left(\left(\begin{smallmatrix} \langle v_p, y\rangle \\ W(p)y \end{smallmatrix}\right), \left(\begin{smallmatrix} \langle v_p, y\rangle \\ W(p)y \end{smallmatrix}\right)\right) \\ & + (\text{terms of lower order in } x). \end{aligned}$$

Since by assumption $h \equiv h \circ M(p)$ it follows that

$$\partial_x h_p \langle v_p, y\rangle + \partial_y h_p(W(p)y) = 0 \tag{3.58}$$

and

$$2\partial^2 h_p\left(\left(\begin{smallmatrix} \langle v_p, y\rangle \\ W(p)y \end{smallmatrix}\right), \left(\begin{smallmatrix} \langle v_p, y\rangle \\ W(p)y \end{smallmatrix}\right)\right) = -\langle y, y\rangle. \tag{3.59}$$

Suppose that $W(p) \notin \mathrm{GL}(n)$. Then there exists $\overline{y} \in \mathbb{R}^n \setminus \{0\}$, such that $W(p)\overline{y} = 0$. Then by (3.59)

$$2\partial_x^2 h_p \left\langle v_p, \overline{y}\right\rangle^2 = -\left\langle \overline{y}, \overline{y}\right\rangle < 0.$$

This in particular shows that $\langle v_p, \overline{y}\rangle \neq 0$. But then equation (3.58) cannot be fulfilled since $\partial_y h_p(W(p)\overline{y}) = \partial_y h_p(0) = 0$ and $\partial_x h|_{\mathcal{H}} > 0$ is true since $\mathcal{H}$ is a CCGPSR manifold, cf. proof of Proposition 3.18 equation (3.8). We deduce that $W(p) \in \mathrm{GL}(n)$. Hence, setting $\widetilde{v}_p = W(p)^T v$ in equation (3.58) implies that $\widetilde{v}_p = -\left.\frac{\partial_y h}{\partial_x h}\right|_p$. This shows that

$$M(p) = \left(\begin{array}{c|c} p_x & -\left.\frac{\partial_y h}{\partial_x h}\right|_p \circ W(p) \\ \hline p_y & W(p) \end{array}\right)$$

is of the form (3.7) as claimed. The action $G_0^h \times \mathcal{H} \to \mathcal{H}$ might not be simply transitive, but near $p = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$, that is on some open neighbourhood $U \subset \mathcal{H}$ of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, we can choose a smooth branch of the possible maps $W : U \to \mathrm{GL}(n)$ by the implicit function theorem. Then, using the diffeomorphism $\Phi : \mathrm{dom}(\mathcal{H}) \to \mathcal{H}$ (3.14), $W \circ \Phi$ is locally on $\Phi^{-1}(U)$ a valid choice for $\mathcal{A}$ as in equation (3.23) and $d(W \circ \Phi)_0$ must fulfil the same equation as $\mathcal{E}$ in (3.28) in the proof of Proposition 3.26. We now use the equality

$$h\left(W(\Phi(z)) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = h\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$$

for all $z \in \Phi^{-1}(U)$ to conclude with the definition of the δP_k's (3.29) that there exists a linear map $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$, such that the corresponding functions $\delta P_k(y) : \mathbb{R}^n \to \mathrm{Sym}^k\left((\mathbb{R}^n)^*\right)$ identically vanish for all $y \in \mathbb{R}^n$ and all $3 \leq k \leq \tau$.

Now assume that there exists $dB_0 \in \mathrm{Lin}(\mathbb{R}^n, \mathfrak{so}(n))$, such that $\delta P_k(y) \equiv 0$ for all $3 \leq k \leq \tau$. Consider the corresponding map $\mathcal{A} : \Phi^{-1}(V) \to \mathrm{GL}(n+1)$ (3.23) for any open neighbourhood $V \subset \mathcal{H}$ of the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ so that $\mathcal{A}$ is defined, with

$$d\mathcal{A}_0 = \left(\begin{array}{c|c} 0 & \frac{2}{\tau} dz^T \\ \hline dz & \frac{3}{2} P_3(\cdot, \cdot, dz)^T + dB_0 \end{array}\right), \tag{3.60}$$

cf. equations (3.28) and (3.26). Then $\delta P_k(y) \equiv 0$ for all $3 \leq k \leq \tau$ implies that for all $v \in T_0\mathrm{dom}(\mathcal{H}) \cong \mathbb{R}^n$

$$dh_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(d\mathcal{A}_0(v) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \equiv 0,$$

where $d\mathcal{A}_0(v)$ denotes the $\mathfrak{gl}(n+1)$-valued 1-form $d\mathcal{A}$ at $z = 0$ applied to $v \in T_0\mathrm{dom}(\mathcal{H})$. With

$$a_i := d\mathcal{A}_0\left(\partial_{z_i}\right)$$

for $1 \leq i \leq n$, the set of matrices $\{a_1, \ldots, a_n\}$ is linearly independent. Furthermore $\{a_1, \ldots, a_n\} \subset T_{\mathbb{1}}G^h = T_{\mathbb{1}}G_0^h$, see (3.2). Let $\mu : G_0^h \to \mathcal{H}$, $\mu(a) = a \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, denote the action of G_0^h on the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$. Then

$$d\mu_{\mathbb{1}}(a_i) = \partial_{y_i}$$

for all $1 \leq i \leq n$. Hence, $d\mu_{\mathbb{1}} : T_{\mathbb{1}}G_0^h \to T_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\mathcal{H}$ is surjective (recall that with h of the form (3.12), we view $T_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\mathcal{H}$ as the vector subspace $\{\left(\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$). This shows that there exists an open subset $U \subset \mathcal{H}$, such that $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in U$ and $U \subset G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. We can without loss of generality assume that U is diffeomorphic to the open ball $B_1(0) \subset \mathbb{R}^n$ with radius 1 in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ and that ∂U is diffeomorphic to $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$. Suppose that the orbit $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subset \mathcal{H}$ is not open in $\mathcal{H}$. Then the set $\mathcal{H} \cap \partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \cap \left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ is non-empty. Let $q \in \mathcal{H} \cap \partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \cap \left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ and let $a(q) \in G_0^h$, such that $q = a(q) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Since q is by assumption an element of $\partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ and $a(q)$ acts via linear transformations on $\mathbb{R}^{n+1}$ restricted to $\mathcal{H}$, there must exist $p \in \partial U$, such that $a(q)p = q$, because otherwise $q \notin a(q) \cdot \partial U$ and $q \in a(q) \cdot U$ would imply that $q \notin \partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$. But we have by definition of G_0^h that $G_0^h \subset \mathrm{GL}(n+1)$ and, hence, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = a(q)^{-1}q = p$, this is a contradiction to $p \in \partial U$. We conclude that the orbit $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subset \mathcal{H}$ is open in $\mathcal{H}$. Since $\mathcal{H} \subset \mathbb{R}^{n+1}$ is maximally extended and being a hyperbolic point of h is an open condition in $\mathbb{R}^{n+1}$ it follows that $\mathcal{H} \cap \partial\mathcal{H} = \emptyset$. This shows that the same also holds for the relative to $\mathcal{H}$ open orbit $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, i.e. that $\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \cap \partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = \emptyset$ where the boundary of $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is relative to $\mathbb{R}^{n+1}$. This implies that $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Furthermore, $\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_{\mathcal{H}}|_{G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\right)$ is also by construction a homogeneous Riemannian manifold and, hence, in particular geodesically complete (see Remark 3.10). This implies that $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subset \mathbb{R}^{n+1}$ is closed, which can be seen the following way. Suppose that $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is not closed in $\mathbb{R}^{n+1}$ but geodesically complete with respect to the restriction of $g_{\mathcal{H}}$ and let p_0 be a point in the boundary $\partial\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$. For any other point $p \in G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ consider a curve $\gamma : [0, 1) \to G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with $\gamma(0) = p$ and $\lim\limits_{t \to 1,\, t<1} \gamma(t) = p_0$. Since $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subset \mathcal{H} \subset \{h = 1\}$ and $h : \mathbb{R}^{n+1} \to \mathbb{R}$, we conclude that $1 = \lim\limits_{t \to 1,\, t<1} h(\gamma(t)) = h\left(\lim\limits_{t \to 1,\, t<1} \gamma(t)\right) = h(p_0)$. Since $g_{\mathcal{H}} = -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H} \times T\mathcal{H}}$ it in particular follows from the fact that $h(p_0) = 1$ and that h is a homogeneous polynomial of homogeneity-degree τ that $g_{\mathcal{H}}$ can be smoothly extended to $p_0 \in G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. This implies

$$\lim_{t \to 1,\, t<1} \int_0^t \sqrt{g_{\mathcal{H}}(\dot{\gamma}(t), \dot{\gamma}(t))}\,dt < \infty.$$

This is a contradiction to the geodesic completeness of $\left(G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_{\mathcal{H}}|_{G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\right)$ by Lemma 2.20. By assumption, $\mathcal{H} \subset \mathbb{R}^{n+1}$ is maximally extended, and we have shown that $G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subset \mathbb{R}^{n+1}$ is closed. We deduce that $\mathcal{H} = G_0^h \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and that the action of G_0^h on $\mathcal{H}$ is, in fact, transitive. In particular, $\mathcal{H}$ is a CCGPSR manifold. $\square$

It is however not true in general that the Lie group G_0^h corresponding to a homogeneous CCGPSR manifold $\mathcal{H} \subset \{h = 1\}$ acts transitively on $\mathcal{H}$. We will prove that statement in the next lemma.

Lemma 3.35 (Homogeneity of CCGPSR curves)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a CCGPSR curve, h of homogeneity-degree $\tau \geq 3$. Then $\mathcal{H}$ is a homogeneous space.*

Proof. Every one-dimensional CCGPSR manifold $\mathcal{H}$ is complete, independent of the homogeneity-degree τ of h, cf. [CNS, Thm. 2.9]. Furthermore, $\mathcal{H}$ is diffeomorphic to $\mathbb{R}$ since $\mathcal{H} \subset \mathbb{R}^2$ is closed by assumption. We now choose an arbitrary non-constant maximal unit-speed geodesic $\gamma : \mathbb{R} \to \mathcal{H}$ of $(\mathcal{H}, g_{\mathcal{H}})$, which is by the connectedness of $\mathcal{H}$ automatically a diffeomorphism with the property

$$\gamma^* g_{\mathcal{H}} = dt^2.$$

This shows that $(\mathcal{H}, g_{\mathcal{H}})$ is isometric to $(\mathbb{R}, dt^2)$, which is in particular a Lie group with bi-invariant metric. Hence, $\mathcal{H}$ is a homogeneous space when viewed as a Riemannian manifold. □

Remark 3.36 (Examples of CCGPSR curves that do not fulfil $\delta P_k \equiv 0$-criterion, $\tau = 3$ and $\tau = 4$)**.** One can check that for all $\tau \in \{3, 4\}$, the homogeneous polynomial $h : \mathbb{R}^2 \to \mathbb{R}$, $h = x^\tau - x^{\tau-2}y^2$, defines a CCPSR curve for $\tau = 3$ and a quartic CCGPSR curve for $\tau = 4$, in each case given by the connected component of $\{h = 1\} \subset \mathbb{R}^2$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. In both cases one can now verify that $\delta P_3(y) \neq 0$.

Open problem 3.37 ("Non-linear" homogeneous CCGPSR manifolds)**.** *Are there any homogeneous CCGPSR manifolds $\mathcal{H} \subset \{h = 1\}$ of dimension $\dim(\mathcal{H}) \geq 2$, such that the corresponding connected component that contains the neutral element of the automorphism group of h, that is G_0^h, does not act transitively on $\mathcal{H}$? Note that Proposition 3.34 immediately implies that then the orbits of the action of G_0^h on $\mathcal{H}$ must everywhere locally be of dimension smaller than $\dim(\mathcal{H})$.*

The above open problem 3.37 is in particular interesting for homogeneous connected PSR manifolds $\mathcal{H} \subset \{h = 1\}$ which have been classified under the assumption that G_0^h acts transitively on $\mathcal{H}$ in [DV]. An open problem related to the above open problem 3.37 is the following question.

Open problem 3.38 (CCGPSR manifolds of constant scalar curvature)**.** *Is it possible to find a classification of CCGPSR manifolds $\mathcal{H} \subset \{h = 1\}$ of arbitrary dimension $n \in \mathbb{N}$ and arbitrary homogeneity-degree $\tau \geq 3$ of h that have constant scalar curvature $S_{\mathcal{H}}$? Is such a classification possible for some fixed $\tau \geq 3$? Are there such CCGPSR manifolds that are <u>not</u> homogeneous spaces?*

If one manages to classify CCGPSR manifolds with with constant scalar curvature, at least corresponding to some specific degree of homogeneity $\tau \geq 3$ of the respective polynomials h, one would have all possible candidates that might solve the open problem 3.37 for that specific degree τ.

4 Curvature bounds of complete projective special real manifolds

In this section we will study curvature properties of CCPSR manifolds. We will use the formulas for the curvature tensors obtained in Section 3. Recall that we can assume without loss of generality that the defining cubic polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ of an n-dimensional CCPSR manifold $\mathcal{H} \subset \{h = 1\}$ is of the form

$$h = x^3 - x\langle y, y\rangle + P_3(y)$$

and that $\mathcal{H}$ is precisely the connected component of $\{h = 1\}$ containing the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{R}^{n+1}$, cf. Proposition 3.18 and equation (3.12). In the case of connected PSR manifolds, the cubic polynomial $P_3 : \mathbb{R}^n \to \mathbb{R}$ is never unique:

Lemma 4.1 (Non-uniqueness of P_3 for connected PSR manifolds)**.** *The cubic polynomial P_3 in equation (3.12) is never unique for any n-dimensional connected PSR manifold, $n \geq 1$.*

Proof. Assume $P_3(y) \not\equiv 0$. then the linear transformation $\mathbb{R}^{n+1} \ni \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}x\\-y\end{smallmatrix}\right) \in \mathbb{R}^{n+1}$ preserves the form (3.12) of h and maps $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. The induced map for the cubic polynomial h maps P_3 to $-P_3$, which are not equal since P_3 does not identically vanish.

Next, assume that $P_3(y) \equiv 0$. It suffices to show that $\delta P_3(y) \neq 0$, cf. Definition 3.29. We obtain $\delta P_3(y) = -\frac{2}{3}\langle y, y\rangle\langle y, dz\rangle \not\equiv 0$. Thus, if we move the reference point in $\mathcal{H}$ away from $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$ for which the procedure of Proposition 3.18 was applied and calculate the corresponding form (3.12) of h, we will always change $P_3(y)$ in a non-trivial way if the initial $P_3(y)$ does vanish identically. $\square$

We will now construct bounds for the different curvature tensors for CCPSR manifolds. To do so, we will investigate for every such manifold and corresponding cubic polynomial h the properties of the associated cubic tensors $P_3(\cdot,\cdot,\cdot)$. It will turn out that we can find bounds for that tensor independent of the CCPSR manifold or any of the other choices involved in determining equation (3.12) (i.e. the choice of the point $p \in \mathcal{H}$ for which to calculate the standard form (3.12) of h and the freedom of transforming the y-coordinates in the latter equation via transformations in $\mathrm{O}(n)$, see proof of Proposition 3.18).

Lemma 4.2. *Let $\mathcal{H} \subset \{h = 1\}$ be a CCPSR manifold, $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$, and $h = x^3 - x\langle y, y\rangle + P_3(y)$ as in equation (3.12). Then*

$$\forall \widehat{z} \in \{z \in \mathbb{R}^n \mid \langle z, z\rangle = 1\}: \quad |P_3(\widehat{z})| \leq \frac{2}{3\sqrt{3}}. \tag{4.1}$$

Proof. Consider $f(t) := \beta(t\widehat{z}) = 1 - t^2 + t^3 P_3(\widehat{z})$, where $\beta : \mathbb{R}^n \to \mathbb{R}$ as in equation (3.22). Since $\mathrm{dom}(\mathcal{H})$ is precompact (Lemma 3.8), f must have at least one positive and one negative real root. We will determine the range for $P_3(\widehat{z})$ such that this holds. The first and second derivative of f are

$$\dot{f}(t) = -2t + 3t^2 P_3(\widehat{z}), \quad \ddot{f}(t) = -2 + 6tP_3(\widehat{z}).$$

Hence, $\dot{f}(t) = 0$ if and only if $t = 0$ or $t = \frac{2}{3P_3(\widehat{z})}$. We obtain $\ddot{f}(0) = -2$ and $\ddot{f}\left(\frac{2}{3P_3(\widehat{z})}\right) = 2$. This implies that $f(t)$ has a local maximum at $t = 0$ and a local minimum at $t = \frac{2}{3P_3(\widehat{z})}$. If $P_3(\widehat{z}) = 0$, $f(t) = 0$ if and only if $t = \pm 1$, so in this case $f(t)$ has exactly one positive and one negative real root. Now assume $P_3(\widehat{z}) > 0$. In that case, $\frac{2}{3P_3(\widehat{z})} > 0$ and $\lim\limits_{t\to-\infty} f(t) = -\infty$.

Since $f(0)=1$, this implies that $f(t)$ has at least one negative real root (one can show that it is the only negative real root by showing that $\dot{f}(t)>0$ for all $t<0$ if $P_3(\widehat{z})>0$). We have seen that $f(t)$ attains its unique local minimum at $t=\frac{2}{3P_3(\widehat{z})}$. Furthermore $f(0)=0$, and $\lim\limits_{t\to\infty} f(t)=\infty$. Hence, $f(t)$ has a positive real root if and only if

$$f\left(\frac{2}{3P_3(\widehat{z})}\right)\leq 0 \quad\Leftrightarrow\quad 1-\frac{4}{27P_3(\widehat{z})^2}\leq 0 \quad\Leftrightarrow\quad P_3(\widehat{z})\leq\frac{2}{3\sqrt{3}}.$$

For $P_3(\widehat{z})<0$ we define $\widetilde{f}(t):=1-t^2+t^3(-P_3(\widehat{z}))$. Similarly as for $P_3(\widehat{z})$ we then obtain

$$-P_3(\widehat{z})\leq\frac{2}{3\sqrt{3}}.$$

Summarising, we have shown that $|P_3(\widehat{z})|\leq\frac{2}{3\sqrt{3}}$. □

Note that the bounds (4.1) for $P_3(\widehat{z})$, $\widehat{z}\in\{z\in\mathbb{R}^n \mid \langle z,z\rangle=1\}$, are independent of the CCPSR manifold and of its dimension. We will later in this thesis show that these bounds are in fact sharp in all dimensions, see Theorem 5.6. An immediate consequence of the calculations in Lemma 4.2 is the following

Corollary 4.3. *Let $h:\mathbb{R}^{n+1}\to\mathbb{R}$, $h=x^3-x\langle y,y\rangle+P_3(y)$, be a cubic homogeneous polynomial and let $\mathcal{H}$ denote the connected component of $\{h=1\}\subset\mathbb{R}^{n+1}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Then the connected component of the set*

$$\{h>0\}\cap\left\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\ \middle|\ z\in\mathbb{R}^n\right\}\subset\mathbb{R}^{n+1}$$

which contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ coincides[5] *with the set*

$$(\mathbb{R}_{>0}\cdot\mathcal{H})\cap\left\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\ \middle|\ z\in\mathbb{R}^n\right\}\subset\mathbb{R}^{n+1}$$

and is precompact if and only if $\max\limits_{\|z\|=1}|P_3(z)|\leq\frac{2}{3\sqrt{3}}$.

Note that it follows from Lemma 3.8 that the connected component of the set $\{h>0\}\cap\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\mid z\in\mathbb{R}^n\}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ being pre-compact is a necessary condition for the connected component of $\{h=1\}$ that contains $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to be a CCPSR manifold. Also note that if the connected component $\mathcal{H}\subset\{h=1\}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is a CCPSR manifold, then the connected component of the set $\{h>0\}\cap\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\mid z\in\mathbb{R}^n\}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, the set $(\mathbb{R}_{>0}\cdot\mathcal{H})\cap\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\mid z\in\mathbb{R}^n\}$, and $\{1\}\times\mathrm{dom}(\mathcal{H})$ coincide. One could ask if we can find similar bounds for CCGPSR manifolds of homogeneity-degree $\tau\geq 4$, but unfortunately this is in general not true as we will see in Lemma 7.9. Lemma 4.2 also means that we have determined positive and negative bounds for $P_3(\widehat{z})$, $\widehat{z}\in\{z\in\mathbb{R}^n\mid\langle z,z\rangle=1\}$, that guaranty that the corresponding hypersurface which is the connected component of $\{h=1\}$ containing the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathbb{R}^{n+1}$ is closed. However, it does a priori not give us information about hyperbolicity when we are studying some specific connected PSR manifold and want to know whether it is a CCPSR manifold or not. It will later turn out that this condition also shows hyperbolicity of all points contained in the connected component of $\{h=1\}$ that contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathbb{R}^{n+1}$, see Theorem 5.6.

Next, we will use Lemma 4.2 to determine upper and lower positive bounds for the norm of points in the boundary of $\mathrm{dom}(\mathcal{H})\subset\mathbb{R}^n$, that is $\partial\mathrm{dom}(\mathcal{H})$, corresponding to a CCPSR manifold $\mathcal{H}$.

[5] This holds true by definition.

Lemma 4.4. *In the setting of Lemma 4.2, assume without loss of generality that $P_3(\widehat{z}) \geq 0$. Let $\mathcal{N}_{P_3(\widehat{z})}$ be the biggest negative real root of $f(t)$ and $\mathcal{P}_{P_3(\widehat{z})}$ be the smallest positive real root of $f(t)$, where $f(t)$ is associated to a CCPSR manifold $\mathcal{H}$ as in the previous lemma and $|P_3(\widehat{z})| \leq \frac{2}{3\sqrt{3}}$. Then*

$$-1 \leq \mathcal{N}_{P_3(\widehat{z})} \leq -\frac{\sqrt{3}}{2},$$
$$1 \leq \mathcal{P}_{P_3(\widehat{z})} \leq \sqrt{3}.$$

Proof. Let $0 \leq A < B \leq \frac{2}{3\sqrt{3}}$, and define

$$f_A(t) := 1 - t^2 + t^3 A, \quad f_B(t) := 1 - t^2 + t^3 B.$$

$f_A(t)$ and $f_B(t)$ have a unique negative real root $\mathcal{N}_A$ and $\mathcal{N}_B$, respectively. Furthermore, $\mathcal{N}_A < \mathcal{N}_B$. To see this, consider

$$\dot{f}_A(t) = -2t + 3t^2 A, \quad \dot{f}_B(t) = -2t + 3t^2 B.$$

This implies that

$$\forall t < 0 : \quad \dot{f}_A(t) > 0,\ \dot{f}_B(t) > 0.$$

Since $\lim\limits_{t\to-\infty} f_A(t) = -\infty$, $\lim\limits_{t\to-\infty} f_B(t) = -\infty$, and $f_A(0) = f_B(0) = 1$ this implies that $\mathcal{N}_A$ and $\mathcal{N}_B$ exist and are the unique negative real roots of $f_A(t)$, respectively $f_B(t)$. We further obtain

$$\begin{aligned} f_B(\mathcal{N}_A) &= 1 - \mathcal{N}_A^2 + \mathcal{N}_A^3 B \\ &= f_A(\mathcal{N}_A) + (B - A)\mathcal{N}_A^3 \\ &= (B - A)\mathcal{N}_A^3 < 0. \end{aligned}$$

Using $\dot{f}_B|_{t<0} > 0$ this shows that

$$\mathcal{N}_B > \mathcal{N}_A. \tag{4.2}$$

We apply this result to $\mathcal{N}_{P_3(\widehat{z})}$ and obtain

$$-1 = \mathcal{N}_0 \leq \mathcal{N}_{P_3(\widehat{z})} \leq \mathcal{N}_{\frac{2}{3\sqrt{3}}} = -\frac{\sqrt{3}}{2}.$$

$\mathcal{N}_{\frac{2}{3\sqrt{3}}}$ can easily be found by guessing or using a computer algebra system.

Now let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the smallest positive root of $f_A(t)$ and $f_B(t)$, respectively. Then $\mathcal{P}_A < \mathcal{P}_B$. To see this, note that the existence of $\mathcal{P}_A$ and $\mathcal{P}_B$ is ensured by the estimate (4.1) in Lemma 4.2. We obtain

$$\begin{aligned} f_A(\mathcal{P}_B) &= 1 - \mathcal{P}_B^2 + \mathcal{P}_B^3 A \\ &= f_B(\mathcal{P}_B) + (A - B)\mathcal{P}_B^3 \\ &= (A - B)\mathcal{P}_B^3 < 0. \end{aligned}$$

Since $f_A(0) = 1$ this shows that $f_A(t)$ has a positive real root that is smaller than $\mathcal{P}_B$, and in particular that

$$\mathcal{P}_A < \mathcal{P}_B. \tag{4.3}$$

Again, we apply this result to $\mathcal{P}_{P_3(\widehat{z})}$ and obtain

$$1 = \mathcal{P}_0 \leq \mathcal{P}_{P_3(\widehat{z})} \leq \mathcal{P}_{\frac{2}{3\sqrt{3}}} = \sqrt{3}.$$

In order to show that $\mathcal{P}_{\frac{2}{3\sqrt{3}}} = \sqrt{3}$, consider $f_{\frac{2}{3\sqrt{3}}}(t) = 1 - t^2 + t^3 \frac{2}{3\sqrt{3}}$. We obtain $f_{\frac{2}{3\sqrt{3}}}(\sqrt{3}) = 0$, $\dot{f}_{\frac{2}{3\sqrt{3}}}(\sqrt{3}) = 0$, and $\ddot{f}_{\frac{2}{3\sqrt{3}}}(\sqrt{3}) = 2$. Hence, $f_{\frac{2}{3\sqrt{3}}}(t)$ has a local minimum at $t = \frac{2}{3\sqrt{3}}$. Furthermore, $\dot{f}_{\frac{2}{3\sqrt{3}}}(\sqrt{3}+s) = 4s + \frac{2}{\sqrt{3}}s^2$ and, hence, $\dot{f}_{\frac{2}{3\sqrt{3}}}(\sqrt{3}+s) > 0$ for all $s > 0$. Summarising this shows that $\mathcal{P}_{\frac{2}{3\sqrt{3}}} = \sqrt{3}$ is the only and in particular smallest positive real root of $f_{\frac{2}{3\sqrt{3}}}(t)$. $\square$

Lemma 4.4 implies the following result for the Euclidean norm of points in $\partial\mathrm{dom}(\mathcal{H})$.

Corollary 4.5. *For a CCPSR manifold $\mathcal{H}$ with the assumptions from Lemma 4.2, and corresponding* $\mathrm{dom}(\mathcal{H})$ *as in Definition 3.22, the following holds true:*

$$\forall \overline{z} \in \partial\mathrm{dom}(\mathcal{H}) : \quad \frac{\sqrt{3}}{2} \leq \|\overline{z}\| \leq \sqrt{3}, \tag{4.4}$$

where $\|\cdot\|$ denotes the norm with respect to the standard Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ in the y-coordinates from equation (3.12).

Hence, with the notation

$$B_r(0) = \left\{ z \in \mathbb{R}^n \mid \langle z, z\rangle < r^2 \right\}$$

for $r > 0$, we have the inclusion $B_{\frac{\sqrt{3}}{2}}(0) \subset \mathrm{dom}(\mathcal{H}) \subset B_{\sqrt{3}}(0)$ for all CCPSR manifolds $\mathcal{H}$. In particular this is also independent of the point chosen in the process (see Proposition 3.18) of obtaining h in the form (3.12) for any given CCPSR manifold $\mathcal{H} \subset \{h = 1\}$. Note however that the inclusion $B_{\frac{\sqrt{3}}{2}}(0) \subset \mathrm{dom}(\mathcal{H})$ might not be compact in the sense that $\partial B_{\frac{\sqrt{3}}{2}}(0) \cap \partial\mathrm{dom}(\mathcal{H})$ might not be empty. If we choose any $0 < R < \frac{\sqrt{3}}{2}$, $B_R(0)$ will always be compactly embedded via the inclusion in $\mathrm{dom}(\mathcal{H})$ since the inclusion $B_R(0) \subset B_{\frac{\sqrt{3}}{2}}(0)$ is a compact embedding.

Another consequence of Lemma 4.4 is the following characterisation of CCPSR manifolds that are singular at infinity, cf. Definition 3.16.

Lemma 4.6. *Let $\mathcal{H} \subset \{h = 1\}$ be an $n \geq 1$-dimensional CCPSR manifold and assume without loss of generality that $h = x^3 - x\langle y, y\rangle + P_3(y)$ as in (3.12) and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$. Then $\mathcal{H}$ is singular at infinity in the sense of Definition 3.16 if and only if $\max\limits_{\|z\|=1} |P_3(z)| = \frac{2}{3\sqrt{3}}$, where $\|\cdot\|$ is the Euclidean norm induced by the choice of the y-coordinates.*

Proof. First note that with our assumptions for $\mathcal{H}$ and h, $\partial(\mathbb{R}_{>0}\cdot\mathcal{H}) \setminus \{0\} = \mathbb{R}_{>0}\cdot(\{1\} \times \partial\mathrm{dom}(\mathcal{H}))$. Since dh_p is homogeneous of degree 2 in p, it thus suffices to show that there exists a $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, such that $dh_{\left(\begin{smallmatrix}1\\\overline{z}\end{smallmatrix}\right)} = 0$ if and only if $\max\limits_{\|z\|=1} |P_3(z)| = \frac{2}{3\sqrt{3}}$. In Lemma 3.25 we have shown that for $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, $dh_{\left(\begin{smallmatrix}1\\\overline{z}\end{smallmatrix}\right)} = 0$ is equivalent to $\frac{\partial h}{\partial x}\left(\left(\begin{smallmatrix}1\\\overline{z}\end{smallmatrix}\right)\right) = \alpha(\overline{z}) = 0$, which is by the Euler identity for homogeneous functions equivalent to $d\beta_{\overline{z}}(\overline{z}) = 0$. Hence, $\mathcal{H}$ is singular at infinity if and only if there exists a point $\widehat{z} \in \{\|z\| = 1\}$, such that the 1-dimensional CCPSR manifold $\mathcal{H}^{\widehat{z}}$ defined by restricting h to the 2-dimensional linear subspace

$$E = \mathrm{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\widehat{z}\end{pmatrix} \right\} \subset \mathbb{R}^{n+1}$$

is singular at infinity. More precisely, $\mathcal{H}^{\widehat{z}}$ is the connected component of $\left\{h^{\widehat{z}} := x^3 - xt^2 + t^3 P_3(\widehat{z}) = 1\right\}$ that contains the point $\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\{1\} \times \mathrm{dom}\left(\mathcal{H}^{\widehat{z}}\right) = E \cap (\{1\} \times \mathrm{dom}(\mathcal{H}))$. The corresponding function $\beta^{\widehat{z}}$ as in (3.22) for $h^{\widehat{z}}$ is given by

$$\beta^{\widehat{z}}(t) = \beta(t\widehat{z}) = 1 - t^2 + t^3 P_3(\widehat{z}).$$

Let t_+ and t_- denote the smallest positive root and the biggest negative root of $\beta^{\widehat{z}}(t)$, respectively. Then $\partial\mathrm{dom}\left(\mathcal{H}^{\widehat{z}}\right) = \{t_+\widehat{z}, t_-\widehat{z}\}$. We have shown in Lemma 4.4 (with the notation $\beta^{\widehat{z}}(t) = f_{|P_3(\widehat{z})|}(t)$) that $\partial_t\beta^{\widehat{z}}(t_+) = 0$ or $\partial_t\beta^{\widehat{z}}(t_-) = 0$ if $|P_3(\widehat{z})| = \frac{2}{3\sqrt{3}}$. It remains to show that $|P_3(\widehat{z})| < \frac{2}{3\sqrt{3}}$ implies that $\partial_t\beta^{\widehat{z}}(t)$ does not vanish at neither t_+ nor t_-. To do that, assume without loss of generality $P_3(\widehat{z}) \geq 0$. For $P_3(\widehat{z}) < 0$ we can simply use the reflection $t \to -t$ and consider $\beta^{\widehat{z}}(-t)$. For $P_3(\widehat{z}) = 0$ it is easy to check that $\partial_t\beta^{\widehat{z}}(t_\pm) = \mp 2$. Now assume $P_3(\widehat{z}) > 0$. We have

$$\partial_t\beta^{\widehat{z}}(t) = t(-2 + 3tP_3(\widehat{z})),$$

hence $\partial_t\beta^{\widehat{z}}(t_-) > 0$ is always true and $\partial_t\beta^{\widehat{z}}(t_+) = 0$ if and only if $t_+ = \frac{2}{3P_3(\widehat{z})}$. One quickly finds that $\beta^{\widehat{z}}(t_+) = 0$ and $P_3(\widehat{z}) > 0$ if and only if $P_3(\widehat{z}) = \frac{2}{3\sqrt{3}}$. This shows that $\partial_t\beta^{\widehat{z}}$ vanishes at a point $\overline{z} \in \mathrm{dom}\left(\mathcal{H}^{\widehat{z}}\right) = \{t_+\widehat{z}, t_-\widehat{z}\}$ (which is equivalent to $\mathcal{H}^{\widehat{z}}$ being singular at infinity) if and only if $|P_3(\widehat{z})| = \frac{2}{3\sqrt{3}}$. Summarising, we have shown that there exists a point $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, such that $d\beta_{\overline{z}}(\overline{z}) = 0$ if and only if there exists a point $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, such that $\left|P_3\left(\frac{\overline{z}}{\|\overline{z}\|}\right)\right| = \frac{2}{3\sqrt{3}}$. In Lemma 4.4 we have shown that this is precisely the maximal possible value for $|P_3(z)|$ on $\{\|z\| = 1\}$ that does not exclude the property of $\mathcal{H}$ being closed in $\mathbb{R}^{n+1}$. We conclude that $\max\limits_{\|z\|=1} |P_3(z)| = \frac{2}{3\sqrt{3}}$ if and only if $\mathcal{H}$ is singular at infinity. $\square$

Remark 4.7. There exist a CCPSR manifold of dimension $n \geq 1$ which is singular at infinity for all $n \geq 1$. For examples consider A) and a) in Theorem 2.45 for $n = 1$ and $n = 2$, respectively, and for $n \geq 3$ see Proposition 6.6. For a general description of the set of all n-dimensional singular-at-infinity CCPSR manifolds see Proposition 5.8.

We will now determine an estimate for the bilinear form $P_3(z, \cdot, \cdot)$ for all $z \in \mathrm{dom}(\mathcal{H})$. It will use the hyperbolicity property of the CCPSR manifold $\mathcal{H}$, which we first need to reformulate.

Lemma 4.8. *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a cubic homogeneous polynomial of the form (3.12), that is $h = x^3 - x\langle y, y\rangle + P_3(y)$, and let $\mathcal{H} \subset \{h = 1\}$ be the connected component of the level set $\{h = 1\} \subset \mathbb{R}^{n+1}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Further assume that $\mathcal{H}$ is a hypersurface in $\mathbb{R}^{n+1}$. Then $\mathcal{H}$ is a CCPSR manifold if and only if*

$$\forall \left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in (\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathbb{R}^n\right\} : \quad 3\langle dz, dz\rangle - 9P_3(z, dz, dz) + \langle z, dz\rangle^2 > 0. \quad (4.5)$$

Proof. Assumption that $\mathcal{H}$ is a CCPSR manifold. Then $\mathcal{H}$ fulfils the assumptions of this lemma and $(\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ coincides with $\mathrm{dom}(\mathcal{H})$, cf. Definition 3.22. We will show that condition (4.5) follows from the hyperbolicity of each point in $\mathcal{H}$. For each $p \in \mathcal{H} \subset \mathbb{R}^{n+1}$, the tangent space $T_p\mathcal{H}$ viewed as a the hyperplane $\ker(dh_p) \subset \mathbb{R}^{n+1}$ and the line $\mathbb{R}p \subset \mathbb{R}^{n+1}$ are orthogonal with respect to the Lorenzian inner product $-\partial^2 h_p$. Recall that $-\partial^2 h_p$ being Lorenzian precisely means that p is a hyperbolic point, see Definition 2.33. Since $-\partial^2 h_p$ is homogeneous of degree $\tau - 2 \geq 1$ in p, it follows that the property that $\mathcal{H}$ consists only of hyperbolic points is equivalent to the statement that $-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)}$ is Lorenzian for all $z \in \mathrm{dom}(\mathcal{H})$. Since $-\partial^2 h_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$ is always Lorenzian if h is of the form (3.12), $-\partial^2 h_{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)}$ being Lorenzian on $\mathrm{dom}(\mathcal{H})$ is equivalent to $\det\left(-\partial^2 h_{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)}\right) < 0$ for all $z \in \mathrm{dom}(\mathcal{H})$. Consider

$$\begin{aligned}\det\left(-\partial^2 h_{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)}\right) &= \det\left(\begin{array}{c|c}-6 & 2z^T\\ \hline 2z & 2\mathbb{1} - 6P_3(z,\cdot,\cdot)\end{array}\right)\\ &= \det\left(\begin{array}{c|c}-6 & 2z^T\\ \hline 0 & 2\mathbb{1} - 6P_3(z,\cdot,\cdot) + \frac{2}{3}z\otimes\langle z,\cdot\rangle\end{array}\right)\\ &= -\frac{2^{n+1}}{3^{n-1}}\det\left(3\mathbb{1} - 9P_3(z,\cdot,\cdot) + z\otimes\langle z,\cdot\rangle\right). \end{aligned} \tag{4.6}$$

Since $\left(3\mathbb{1} - 9P_3(z,\cdot,\cdot) + z\otimes\langle z,\cdot\rangle\right)\big|_{z=0} = 3\mathbb{1}$, it follows that $\det\left(-\partial^2 h_{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)}\right) < 0$ for all $z \in \mathrm{dom}(\mathcal{H})$ is equivalent to $3\langle dz, dz\rangle - 9P_3(z,dz,dz) + \langle z, dz\rangle^2 > 0$ for all $z \in \mathrm{dom}(\mathcal{H})$.

For the other direction, the conditions that $\mathcal{H}$ is a connected component of $\{h = 1\}$ implies that it is closed as a subset of $\mathbb{R}^{n+1}$. Furthermore, $\mathcal{H}$ is a hypersurface by assumption. With the same argument as before for the homogeneity of $-\partial^2 h_p$ in p and the same calculations as above, it follows that $\mathcal{H}$ consists only of hyperbolic points. $\mathcal{H}$ is thus a connected and also closed PSR manifold, and the set $(\mathbb{R}_{>0}\cdot\mathcal{H}) \cap \{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ and $\mathrm{dom}(\mathcal{H})$ coincide. □

We will use the results from Corollary 4.5 and Lemma 4.8 to find upper and lower bounds of the eigenvalues of $P_3(z,dz,dz)$ (when viewed as a symmetric matrix) for $z \in \mathrm{dom}(\mathcal{H})$ that are valid for all CCPSR manifolds $\mathcal{H}$ (and thus also for non-connected closed PSR manifolds).

Proposition 4.9 (Bounds for eigenvalues of $P_3(z,dz,dz)$ for CCPSR manifolds)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an n-dimensional CCPSR manifold, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$, and $h = x^3 - x\langle y,y\rangle + P_3(y)$, cf. Proposition 3.18. Then*

$$\forall z \in \mathrm{dom}(\mathcal{H}): \quad -\frac{5}{6}\langle dz,dz\rangle < P_3(z,dz,dz) < \frac{2}{3}\langle dz,dz\rangle. \tag{4.7}$$

This is equivalent to the statement that for all $z \in \mathrm{dom}(\mathcal{H})$, the eigenvalues $\lambda \in \mathbb{R}$ of the representation matrix of the symmetric bilinear form $P_3(z,dz,dz)$ induced by the z-coordinates fulfil $-\frac{5}{6} < \lambda < \frac{2}{3}$. Furthermore, the upper bound in (4.7) is sharp in the sense that for all $n \geq 1$ there exists a CCPSR manifold $\mathcal{H}$ and a point $\check{z}$, such that the representation matrix of $P_3(\check{z},dz,dz)$ has one eigenvalue $\lambda = \frac{2}{3}$.

Proof. We start with the upper bound in (4.7). Equation (4.5) in Lemma 4.8 and equation (4.4) in Corollary 4.5 imply for all $z \in \mathrm{dom}(\mathcal{H})$

$$P_3(z,dz,dz) < \frac{3\langle dz,dz\rangle + \langle z,dz\rangle^2}{9} \leq \frac{3\langle dz,dz\rangle + \langle z,z\rangle\langle dz,dz\rangle}{9} < \frac{2}{3}\langle dz,dz\rangle. \tag{4.8}$$

Obtaining the alleged lower bound in equation (4.7) for $P_3(z,dz,dz)$ needs more work. A "naive" lower bound can be obtained the following way. For all $\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$ with $\|\check{z}\| = \frac{\sqrt{3}}{2}$ (recall that $\overline{B_{\frac{\sqrt{3}}{2}}(0)} \subset \overline{\mathrm{dom}(\mathcal{H})}$ is always true, see Corollary 4.5), the biggest positive eigenvalue of the representation matrix of $P_3(\check{z},dz,dz)$ is bound from above by $\frac{2}{3}$. Using that $P_3(z,dz,dz)$ is linear in z, we obtain that the smallest eigenvalue of the representation matrix of $P_3(-2\check{z},dz,dz)$ is bounded from below by $-\frac{4}{3}$. Since $\check{z} \in \partial B_{\frac{\sqrt{3}}{2}}(0)$ was arbitrary, we obtain for all $\tilde{z} \in \partial B_{\sqrt{3}}(0)$ the estimate $P_3(\tilde{z},dz,dz) \geq -\frac{4}{3}\langle dz,dz\rangle$. Since for all CCPSR manifolds with the assumptions of this lemma $\mathrm{dom}(\mathcal{H}) \subset B_{\sqrt{3}}(0)$, we can use the linearity of $P_3(z, dz, dz)$ in z again to conclude that for all $z \in \mathrm{dom}(\mathcal{H})$ we have the estimate $P_3(z, dz, dz) > -\frac{4}{3}\langle dz, dz\rangle$. This bound is worse than $-\frac{5}{6}\langle dz, dz\rangle$, which we will derive now.

The estimate (4.8) shows that for all $\check{z} \in \partial\mathrm{dom}(\mathcal{H})$, every positive eigenvalue λ_+ of the representation matrix of $P_3(\check{z}, dz, dz)$ fulfils

$$\lambda_+ \leq \frac{3 + \|\check{z}\|^2}{9}. \tag{4.9}$$

Fix $\check{z} \in \partial\mathrm{dom}(\mathcal{H}) \subset \mathbb{R}^2$ and let λ_- be a negative eigenvalue of the representation matrix of $P_3(\check{z}, dz, dz)$. The linearity of $P_3(z, dz, dz)$ in z implies that $-\lambda_-$ is a positive eigenvalue of the representation matrix of $P_3(-\check{z}, dz, dz)$. However, $-\check{z}$ might not be an element of $\overline{\mathrm{dom}(\mathcal{H})}$. In fact, $-\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$ if and only if $\|\check{z}\| \leq 1$, which holds if and only if $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) \in \left[-\frac{2}{3\sqrt{3}}, 0\right]$ (cf. Lemma 4.2 and see Figure 2).

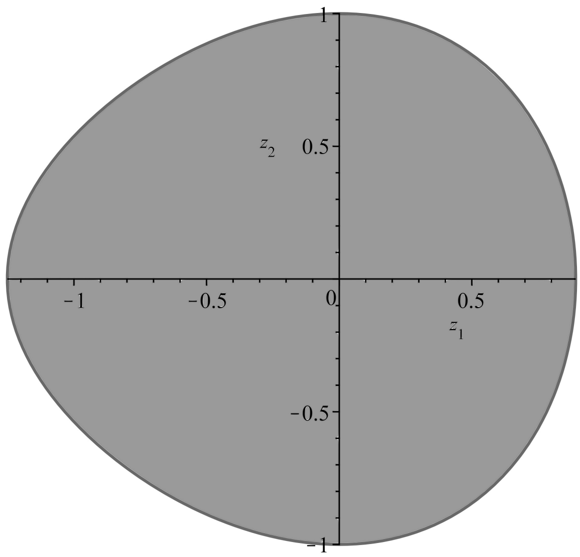


Figure 2: The set $\overline{\mathrm{dom}(\mathcal{H})} \subset \mathbb{R}^2$ corresponding to $P_3\left((z_1, z_2)^T\right) = -\frac{1}{2\sqrt{3}} z_1^3$. Observe for example that for all $\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$ of the form $\check{z} = (z_1, 0)^T$, $z_1 > 0$, we have $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) \in \left[-\frac{2}{3\sqrt{3}}, 0\right]$ and one can see that $-\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$.

For such a given $\check{z} \in \partial\mathrm{dom}(\mathcal{H})$ we want to find $\check{t} > 0$, such that $\check{t}(-\check{z}) \in \partial\mathrm{dom}(\mathcal{H})$. When we have determined said $\check{t}$, the linearity of $P_3(z, dz, dz)$ in z implies that $\check{t}(-\lambda_-)$ is a positive eigenvalue of the representation matrix of $P_3(\check{t}(-\check{z}), dz, dz)$. Using the upper bound (4.9), we can thus estimate

$$\check{t}(-\lambda_-) \leq \frac{3 + \check{t}^2\|\check{z}\|^2}{9} \quad \Leftrightarrow \quad -\lambda_- \leq \frac{3 + \check{t}^2\|\check{z}\|^2}{9\check{t}} =: F(\|\check{z}\|). \tag{4.10}$$

Our asserted lower bound $-\frac{5}{6}$ for λ_- is now obtained via showing that the function $F : \left[\frac{\sqrt{3}}{2}, \sqrt{3}\right] \to \mathbb{R}_{>0}$ defined in (4.10) is continuous and by determining its maximal value, where we recall that the elements in the closed interval $\left[\frac{\sqrt{3}}{2}, \sqrt{3}\right]$ are precisely all possible values for $\|\check{z}\|$ when considering all possible n-dimensional CCPSR manifolds $\mathcal{H}$ (cf. Corollary 4.5). To find a closed formula for $\check{t}$ depending on $\|\check{z}\|$, consider the function $f(t) = \beta\left(t\frac{\check{z}}{\|\check{z}\|}\right) = 1 - t^2 + P_3\left(\frac{\check{z}}{\|\check{z}\|}\right)t^3$ (compare with equation (3.22)) and assume that $f(\|\check{z}\|) = 0$. By assumption, $\mathcal{H}$ is a CCPSR manifold, implying that $\mathrm{dom}(\mathcal{H}) \subset \mathbb{R}^n$ is precompact and, hence, $f(t)$ must have at least one more negative real root in addition to its root $t = \|\check{z}\| > 0$. Hence, $(t - \|\check{z}\|) \mid f(t)$ and we obtain with $a, b \in \mathbb{R}$

$$\begin{aligned} f(t) &= (t - \|\check{z}\|)\left(\frac{-1}{\|\check{z}\|} + at + bt^2\right) \\ &= 1 + \left(-a\|\check{z}\| - \frac{1}{\|\check{z}\|}\right)t + (a - b\|\check{z}\|)\,t^2 + bt^3. \end{aligned}$$

This implies that $a = \frac{-1}{\|\check{z}\|^2}$ and $b = \frac{1}{\|\check{z}\|} - \frac{1}{\|\check{z}\|^3}$. Note that this determines $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right)$ depending on $\|\check{z}\|$, and as we would expect $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) = \frac{2}{3\sqrt{3}}$ if $\|\check{z}\| = \sqrt{3}$, and $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) = -\frac{2}{3\sqrt{3}}$ if $\|\check{z}\| = \frac{\sqrt{3}}{2}$ (see Lemma 4.2 and Corollary 4.5). We define

$$\widetilde{f}(t) := \frac{f(t)}{t - \|\check{z}\|} = -\frac{1}{\|\check{z}\|} - \frac{1}{\|\check{z}\|}t + \left(\frac{1}{\|\check{z}\|} - \frac{1}{\|\check{z}\|^3}\right)t^2.$$

In order to determine $\check{t}$ in dependence of $\|\check{z}\|$ we need to find the roots of $\widetilde{f}(t)$, for (at least) one of the roots coincides with $\check{t}(-\|\check{z}\|)$. We will differentiate between the three cases $\|\check{z}\| = 1$, $\|\check{z}\| \in \left(1, \sqrt{3}\right]$, and $\|\check{z}\| \in \left[\frac{\sqrt{3}}{2}, 1\right)$. We will also use these results to show that F is continuous.

Case 1: $\|\check{z}\| = 1$.
In that case $f(t) = 1 - t^2$, so the roots of $f(t)$ are $t_\pm = \pm 1$ and the root of $\widetilde{f}(t) = -1 - t$ is $t = -1$. Hence, $\check{t} = 1$ and (4.10) thus yields the estimate $-\lambda_- \leq \frac{4}{9} = F(1)$.

Case 2: $\|\check{z}\| \in \left[\frac{\sqrt{3}}{2}, \sqrt{3}\right] \setminus \{1\}$.
In this case,

$$\widetilde{f}(t_\pm) = 0 \quad \Leftrightarrow \quad t_\pm = \frac{\|\check{z}\|}{2(\|\check{z}\|^2 - 1)} \pm \sqrt{\frac{(4\|\check{z}\|^2 - 3)\|\check{z}\|^2}{4(\|\check{z}\|^2 - 1)^2}}.$$

Note that the sign of $\|\check{z}\|^2 - 1$ depends on whether $\|\check{z}\| < 1$ or $\|\check{z}\| > 1$. We will treat these cases separately.

Case 2.1: $\|\check{z}\| \in \left(1, \sqrt{3}\right]$.
In this case, the plot of $f(t)$ is of the form as in Figure 3 (except when $\|\check{z}\| = \sqrt{3}$, in which case $f(t)$ has the unique positive double root $\sqrt{3}$). Also $\|\check{z}\|^2 - 1 > 0$ and, hence, $t_- = \check{t}(-\|\check{z}\|)$. We obtain

$$\begin{aligned} t_- &= \frac{\|\check{z}\|}{2(\|\check{z}\|^2 - 1)} - \sqrt{\frac{(4\|\check{z}\|^2 - 3)\|\check{z}\|^2}{4(\|\check{z}\|^2 - 1)^2}} \\ &= \frac{\|\check{z}\|}{2(\|\check{z}\|^2 - 1)}\left(1 - \sqrt{4\|\check{z}\|^2 - 3}\right), \\ \check{t} &= \frac{1}{2(\|\check{z}\|^2 - 1)}\left(-1 + \sqrt{4\|\check{z}\|^2 - 3}\right), \end{aligned}$$

and, hence,

$$F(\|\check{z}\|) = \frac{2\|\check{z}\|^4 - 5\|\check{z}\|^2 + 3 + \sqrt{4\|\check{z}\|^2 - 3}(4\|\check{z}\|^4 - 7\|\check{z}\|^2 + 3)}{18(\|\check{z}\|^2 - 1)^2}.$$

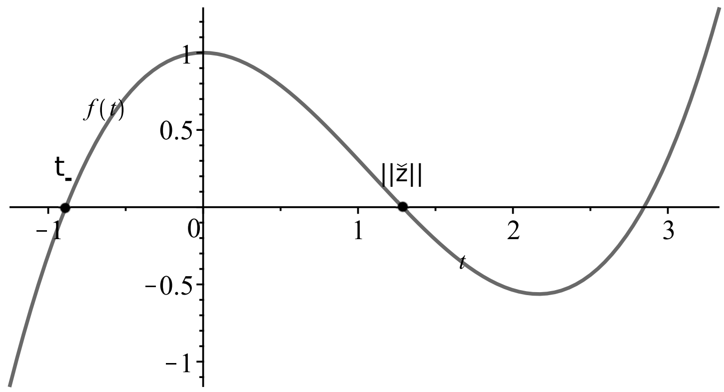


Figure 3: A typical plot of $f(t)$. Here, $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) = \frac{4}{5}\cdot\frac{2}{3\sqrt{3}}$, so that $\|\check{z}\| \in \left(1, \frac{2}{3\sqrt{3}}\right)$. The unique negative root of $f(t)$, that is $\check{t}(-\|\check{z}\|)$, and t_- coincide in Case 2.1.

It is clear that $F|_{(1,\sqrt{3}]}$ is continuous and furthermore smooth. Using L'Hôspital's rule for limits twice at $\|\check{z}\| = 1$ yields

$$\lim_{\|\check{z}\|\to 1,\ \|\check{z}\|>1} F(\|\check{z}\|) = \frac{4}{9},$$

which coincides with $F(1)$ determined in Case 1. This means that F is continuous from the right at $\|\check{z}\| = 1$. Next we will show that $F|_{(1,\sqrt{3}]}$ attains its maximum, namely at $\|\check{z}\| = \sqrt{3}$. To prove that we show that $\left.\frac{\partial F}{\partial\|\check{z}\|}\right|_{(1,\sqrt{3})} > 0$. The first derivative of F is given by

$$\frac{\partial F}{\partial\|\check{z}\|}(\|\check{z}\|) = \frac{\|\check{z}\|\left(8\|\check{z}\|^4 - 18\|\check{z}\|^2 + 9 + \sqrt{4\|\check{z}\|^2-3}\right)}{9\sqrt{4\|\check{z}\|^2-3}(\|\check{z}\|^2-1)^2},$$

and $\|\check{z}\|-1>0$ implies that in order to solve $\frac{\partial F}{\partial\|\check{z}\|}(\|\check{z}\|) = 0$ with the restriction $\|\check{z}\| \in \left(1,\sqrt{3}\right)$ we only need to solve $8\|\check{z}\|^4 - 18\|\check{z}\|^2 + 9 + \sqrt{4\|\check{z}\|^2-3} = 0$. Using MAPLE or any other computer algebra system one finds that the latter equation has no solutions in $\left(1,\sqrt{3}\right)$. It thus suffices check the sign of $\left.\frac{\partial F}{\partial\|\check{z}\|}\right|_{(1,\sqrt{3})}$ at one point in the interval, say $\frac{1+\sqrt{3}}{2}$, to determine its global sign. We calculate

$$\frac{\partial F}{\partial\|\check{z}\|}\left(\frac{1+\sqrt{3}}{2}\right) = \frac{4\sqrt{1+2\sqrt{3}}+2\sqrt{3}+2}{27} > 0.$$

We conclude that

$$\sup_{\|\check{z}\|\in\left(1,\sqrt{3}\right]} F(\|\check{z}\|) = F\left(\sqrt{3}\right) = \frac{5}{6}.$$

Case 2.2: $\|\check{z}\| \in \left[\frac{\sqrt{3}}{2}, 1\right)$**.**
This case works similarly to Case 2.1. Here, $f(t)$ has the shape as in Figure 4 (except for the case $\|\check{z}\| = \frac{\sqrt{3}}{2}$, where $f(t)$ has the unique negative double root $\frac{\sqrt{3}}{2}$). In this case, $f(t)$ has, except if $\|\check{z}\| = \frac{\sqrt{3}}{2}$, precisely two negative roots, of which we need to consider the bigger one. Since $\|\check{z}\|^2 - 1 < 0$, we see that this is

$$t_+ = \frac{\|\check{z}\|}{2(\|\check{z}\|^2-1)}\left(1-\sqrt{4\|\check{z}\|^2-3}\right),$$

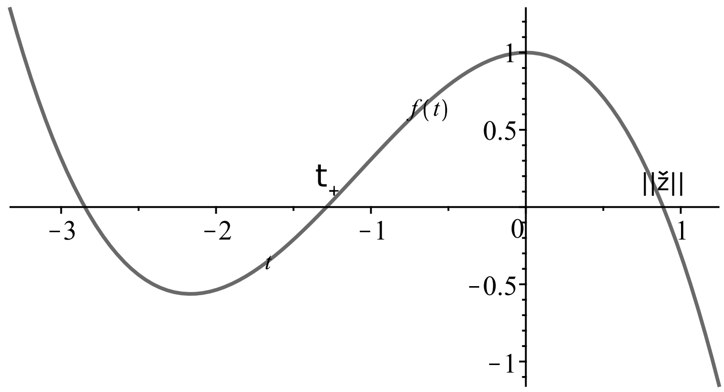


Figure 4: A plot of $f(t)$ with $P_3\left(\frac{\check{z}}{\|\check{z}\|}\right) = -\frac{4}{5}\cdot\frac{2}{3\sqrt{3}}$, so that $\|\check{z}\| \in \left(\frac{\sqrt{3}}{2},1\right)$. In Case 2.2, the biggest negative root of $f(t)$ which is $\check{t}(-\|\check{z}\|)$ by construction, and t_+ coincide.

so that $t_+ = \check{t}(-\|\check{z}\|)$ has the form

$$\check{t} = \frac{1}{2(\|\check{z}\|^2-1)}\left(-1+\sqrt{4\|\check{z}\|^2-3}\right).$$

We see that formally the function F for this case and F in Case 2.1 coincide, i.e. we have for $F|_{\left[\frac{\sqrt{3}}{2},1\right)}$

$$F(\|\check{z}\|) = \frac{2\|\check{z}\|^4 - 5\|\check{z}\|^2 + 3 + \sqrt{4\|\check{z}\|^2-3}(4\|\check{z}\|^4 - 7\|\check{z}\|^2+3)}{18(\|\check{z}\|^2-1)^2}$$

and for the derivative $\frac{\partial F}{\partial\|\check{z}\|}\Big|_{\left(\frac{\sqrt{3}}{2},1\right)}$

$$\frac{\partial F}{\partial\|\check{z}\|}(\|\check{z}\|) = \frac{\|\check{z}\|\left(8\|\check{z}\|^4 - 18\|\check{z}\|^2 + 9 + \sqrt{4\|\check{z}\|^2-3}\right)}{9\sqrt{4\|\check{z}\|^2-3}(\|\check{z}\|^2-1)^2}.$$

Proceeding analogously to Case 2.1 we will show that $\frac{\partial F}{\partial\|\check{z}\|}\Big|_{\left(\frac{\sqrt{3}}{2},1\right)} > 0$. Note that the denominator of the formula for the first derivative of F has no zeros in $\left(\frac{\sqrt{3}}{2},1\right)$, so we will not run into trouble with possibly singular values. Again, we use MAPLE to show that $8\|\check{z}\|^4 - 18\|\check{z}\|^2 + 9 + \sqrt{4\|\check{z}\|^2-3} = 0$ has no solutions in $\left(\frac{\sqrt{3}}{2},1\right)$. Hence, the global sign of $\frac{\partial F}{\partial\|\check{z}\|}\Big|_{\left(\frac{\sqrt{3}}{2},1\right)}$ coincides with the sign of

$$\frac{\partial F}{\partial\|\check{z}\|}\left(\frac{1}{2}\left(\frac{\sqrt{3}}{2}+1\right)\right) = -\frac{1220}{1089}\sqrt{3}\sqrt{-5+4\sqrt{3}} + \frac{5824}{3267}\sqrt{3} - \frac{6376}{3267}\sqrt{-5+4\sqrt{3}} + \frac{10112}{3267} > 0.$$

Hence, $F_{\left[\frac{\sqrt{3}}{2},1\right)}$ does not attain its maximum in its domain of definition, but at the limit $\|\check{z}\| \to 1$, assuming that limit exists. For the existence we need to check that F is continuous from the left at $\|\check{z}\| = 1$. This is done in the same way we have shown continuity from the right, that is by applying L'Hôspital's rule twice. As expected, we obtain

$$\lim_{\|\check{z}\|\to 1,\ \|\check{z}\|<1} F(\|\check{z}\|) = \frac{4}{9},$$

Summarising, we have shown that $F:\left[\frac{\sqrt{3}}{2},\sqrt{3}\right]\to\mathbb{R}_{>0}$ is continuous and attains its maximum at $\|\check{z}\|=\sqrt{3}$, $F\left(\sqrt{3}\right)=\frac{5}{6}$. Since the negative eigenvalue λ_- of the representation matrix of $P_3(\check{z},dz,dz)$ was arbitrary, we conclude with (4.10) that for all such negative eigenvalues λ_- we have

$$\lambda_-\geq-\max_{\|\check{z}\|\in\left[\frac{\sqrt{3}}{2},\sqrt{3}\right]}F(\|\check{z}\|)=-\frac{5}{6}.$$

The point $\check{z}\in\partial\mathrm{dom}(\mathcal{H})$ was also arbitrary and, thus, using the linearity of $P_3(z,dz,dz)$ we obtain

$$\forall z\in\mathrm{dom}(\mathcal{H}):\quad P_3(z,dz,dz)>-\frac{5}{6}\langle dz,dz\rangle.$$

Note that our calculations also show that $\lambda_-=-\frac{5}{6}$ can only possibly be a negative eigenvalue of the representation matrix of $P_3(\check{z},dz,dz)$ at a point $\check{z}\in\partial\mathrm{dom}(\mathcal{H})$ with norm $\|\check{z}\|=\sqrt{3}$.

We want to stress again at this point that the obtained lower and upper bounds for $P_3(z,dz,dz)$ hold for all CCPSR manifolds $\mathcal{H}\subset\{h=1\}$ of dimension $n\geq1$ with h of the form (3.12) and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$.

It remains to show that the upper bound in (4.7) is sharp in the stated sense, and that the lower bound in (4.7) can never be sharp. For the upper bound, we will give an example of a CCPSR manifold of dimension n for each $n\in\mathbb{N}$. For any $n\in\mathbb{N}$, let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=(x,y_1,\ldots,y_n)^T$ denote linear coordinates of $\mathbb{R}^{n+1}$ as usual and consider the cubic polynomial

$$h:\mathbb{R}^n\to\mathbb{R},\quad h=x^3-x\langle y,y\rangle+\frac{2}{3\sqrt{3}}y_n^3,\tag{4.11}$$

and the corresponding centro-affine hypersurface $\mathcal{H}\subset\{h=1\}$, which is the connected component of $\{h=1\}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Then $\mathcal{H}$ is a closed PSR manifold of dimension n. We will not prove this here, since for $n\geq3$, $\mathcal{H}$ is an element of a family of CCPSR manifolds constructed later in Theorem 6.1 (for this statement, consider also Proposition 6.6, equation (6.40) with $\mu_i=\sqrt{2}\eta_i>0$ for all $1\leq i\leq n-1$.) For $n=1$, one can check that h is equivalent to x^2y, which is one of the two 1-dimensional closed PSR manifolds classified in [CHM, Thm. 7], and for $n=2$, h is linearly equivalent to the polynomial e) in [CDL, Thm. 1] and $\mathcal{H}$ is the corresponding described CCPSR manifold. For each of these cases consider the point $\check{z}_+=\left(0,\ldots,0,\sqrt{3}\right)^T\in\partial\mathrm{dom}(\mathcal{H})$. We obtain $P_3(\check{z}_+,dz,dz)=\frac{2}{3}dz_n^2$ and the corresponding symmetric matrix has precisely the eigenvalues $\lambda_1=0$ with eigenspace-dimension $n-1$, and $\lambda_2=\frac{2}{3}$ with eigenspace-dimension 1. This proves our claim. □

We will now prove a statement similar to Proposition 4.9 but for points in $S^{n-1}=\{z\in\mathbb{R}^n\mid\langle z,z\rangle=1\}$.

Lemma 4.10 (Bounds of $P_3(\widehat{z},dz,dz)$ for $\widehat{z}\in S^{n-1}$)**.** *In the setting of Proposition 4.9 we have*

$$\forall\widehat{z}\in S^{n-1}:\quad-\frac{5}{6\sqrt{3}}\langle dz,dz\rangle\leq P_3(\widehat{z},dz,dz)\leq\frac{5}{6\sqrt{3}}\langle dz,dz\rangle.\tag{4.12}$$

Proof. The linearity in z of $P_3(z,dz,dz)$ implies that it suffices to find the maximal positive eigenvalue of the representation matrix of $P_3(\widehat{z},dz,dz)$, $\widehat{z}\in S^{n-1}$ in order to prove (4.12). Note that $P_3(y)$ being an odd function in the sense that $P_3(-y)=-P_3(y)$ implies that $S^{n-1}\in\overline{\mathrm{dom}(\mathcal{H})}$ if and only if $P_3(y)\equiv0$, since this is the only case where the solutions of $h=x^3-x\langle y,y\rangle+P_3(y)=1$ that form $\partial\mathrm{dom}(\mathcal{H})$ have Euclidean norm 1. This forbids us to simply maximise the formula for positive eigenvalues (4.9) valid for points in $\partial\mathrm{dom}(\mathcal{H})$

over S^{n-1}. Let $\check{z} \in \partial\mathrm{dom}(\mathcal{H})$ and λ_+ be a positive eigenvalue of the representation matrix $P_3(\check{z}, dz, dz)$. Then $\widehat{\lambda}_+ := \frac{1}{\|\check{z}\|}\lambda_+$ is a positive eigenvalue of the representation matrix of $P_3\left(\frac{\check{z}}{\|\check{z}\|}, dz, dz\right) = \frac{1}{\|\check{z}\|}P_3(\check{z}, dz, dz)$. The map

$$\partial\mathrm{dom}(\mathcal{H}) \ni \check{z} \mapsto \frac{\check{z}}{\|\check{z}\|} \in S^{n-1}$$

is continuous and bijective for all CCPSR manifolds $\mathcal{H}$. It is however not necessary smooth since dh might vanish at some point $\left(\begin{smallmatrix}1\\ \check{z}\end{smallmatrix}\right)$ and does thus allow for $\partial\mathrm{dom}(\mathcal{H})$ to be a continuous non-smooth submanifold of $\mathbb{R}^n$. Using (4.9) we see that

$$\widehat{\lambda}_+ = \frac{1}{\|\check{z}\|}\lambda_+ \leq \frac{3 + \|\check{z}\|^2}{9\|\check{z}\|} =: \rho(\|\check{z}\|) \tag{4.13}$$

for all $\check{z} \in \partial\mathrm{dom}(\mathcal{H})$. Hence, we can obtain an upper bound for $\widehat{\lambda}_+$ by finding

$$\max_{\|\check{z}\| \in \left[\frac{\sqrt{3}}{2}, \sqrt{3}\right]} \rho(\|\check{z}\|),$$

see Corollary 4.5. We obtain

$$\rho\left(\frac{\sqrt{3}}{2}\right) = \frac{5}{6\sqrt{3}}, \quad \rho\left(\sqrt{3}\right) = \frac{2}{3\sqrt{3}}$$

(note: $\frac{2}{3\sqrt{3}} < \frac{5}{6\sqrt{3}}$), and

$$\left.\frac{\partial\rho}{\partial\|\check{z}\|}\right|_{\left(\frac{\sqrt{3}}{2}, \sqrt{3}\right)} = \frac{-3 + \|\check{z}\|}{9\|\check{z}\|^2} < 0 \quad \forall \|\check{z}\| \in \left(\frac{\sqrt{3}}{2}, \sqrt{3}\right).$$

Hence, $\max\limits_{\|\check{z}\| \in \left[\frac{\sqrt{3}}{2}, \sqrt{3}\right]} \rho(\|\check{z}\|) = \frac{5}{6\sqrt{3}}$. This shows that the maximal positive eigenvalue possible of the representation matrix of $P_3(\widehat{z}, dz, dz)$ for $\widehat{z} \in S^{n-1}$ is $\frac{5}{6\sqrt{3}}$, and with our remark at the beginning of the proof the minimal negative eigenvalue is $-\frac{5}{6\sqrt{3}}$. □

Remark 4.11. The result for negative eigenvalues obtained in Lemma 4.10 can be used to find the exact same lower bound for $P_3(\check{z}, dz, dz)$ for $\check{z} \in \partial\mathrm{dom}(\mathcal{H})$ that is obtained via the estimate (4.7) derived in Proposition 4.9. Furthermore, we do not expect the lower bound in Proposition 4.9, equation 4.7, or equivalently the upper and lower bounds in Lemma (4.10), equation (4.12), to be sharp. This is motivated by the following Lemma.

Lemma 4.12. *For* $\dim(\mathcal{H}) = 1$*, (4.12) is never sharp. Instead we have the estimate*

$$\forall \widehat{z} \in S^0: \quad -\frac{2}{3\sqrt{3}}dz^2 \leq P_3(\widehat{z}, dz, dz) \leq \frac{2}{3\sqrt{3}}dz^2, \tag{4.14}$$

which is sharp.

Proof. We will see in Remark 7.4 that every possible standard form for a cubic polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ (3.12) describing a CCPSR curve $\mathcal{H}$ that is precisely the connected component $\mathcal{H} \subset \{h = 1\}$ containing the point $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$, with h given by

$$h = x^3 - xy^2 + Ly^3, \quad L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right],$$

see Lemma 4.2. Note that we will also show in Remark 7.4 that two such CCPSR curves $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are equivalent if and only if either the respective $L, \widetilde{L} \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$, or $L, \widetilde{L} \in \left\{-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right\}$. Alternatively, we know from Lemma 4.2 that for $h = x^3 - xy^2 + Ly^3$, the condition $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$ is a necessary requirement for the corresponding maximal connected PSR curve $\mathcal{H}$ to be a CCPSR curve. In order to show that this is also a sufficient condition, one checks using formula (3.33) and the function β as in (3.22) that

$$(\Phi^* g_{\mathcal{H}})_z = \frac{2}{9\beta^2(z)} \left(z^2 - 9Lz + 3\right).$$

for all $z \in \operatorname{dom}(\mathcal{H})$. One then finds that the equation $z^2 - 9Lz + 3 = 0$ has real solutions if and only if $|L| > \frac{2}{3\sqrt{3}}$ (and, hence, that $z^2 - 9Lz + 3 > 0$ for all $z \in \mathbb{R}$ if $|L| \leq \frac{2}{3\sqrt{3}}$), which implies that for all $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$ the pullback of the centro-affine fundamental form $(\Phi^* g_{\mathcal{H}})_z$ is positive definite for all points in the connected component of $\{h\left(\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\right) > 0\}$ that contains $z = 0$. Hence, said connected component coincides with $\operatorname{dom}(\mathcal{H})$, which shows that $\mathcal{H} \subset \mathbb{R}^2$ is closed and, hence, a CCPSR curve.

Now, in order to show that the estimate (4.12) is never sharp, we need to check this estimate for every $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$. Let $\check{z}$ be either the smallest positive root or the biggest negative root of $\beta(z) = h\left(\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\right) = 1 - z^2 + Lz^3$, which both are precisely the elements of $\partial\operatorname{dom}(\mathcal{H})$, and consider the corresponding point $\frac{\check{z}}{\|\check{z}\|} \in S^0 = \{-1, 1\} \subset \mathbb{R}$. Observe that $P_3(y) = Ly^3$ and, hence,

$$P_3\left(\frac{\check{z}}{\|\check{z}\|}, dz, dz\right) = L\frac{\check{z}}{\|\check{z}_+\|}dz^2 = L\operatorname{sgn}(\check{z})dz^2.$$

The sole eigenvalue of the representation matrix of $P_3(\frac{\check{z}}{\|\check{z}\|}, dz, dz)$, namely $L\operatorname{sgn}(\check{z})$, depends thus on L and the sign of $\check{z}$, but in particular not on the absolute value of $\check{z}$. We conclude by maximising, respectively minimising, over $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$, that for all CCPSR curves $\mathcal{H}$ with our assumptions for the standard form of the corresponding cubic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ we have the estimate

$$-\frac{2}{3\sqrt{3}}dz^2 \leq P_3(\widehat{z}, dz, dz) \leq \frac{2}{3\sqrt{3}}dz^2$$

for all $\widehat{z} \in S^0$. This estimate is sharp for $L = \pm\frac{2}{3\sqrt{3}}$. One can check that the corresponding CCPSR curve is equivalent to each of the connected components of the one described in [CHM, Thm. 8, a)], respectively A) in Theorem 2.45. □

For higher dimensions, that is CCPSR manifolds $\mathcal{H}$ of dimension $\dim(\mathcal{H}) \geq 2$, the question of finding the best possible estimate for $P_3(\widehat{z}, dz, dz)$, $\widehat{z} \in S^{n-1}$, is very difficult. We would need to classify all cubic polynomials $P_3 : \mathbb{R}^n \to \mathbb{R}$ (at least up to rotations in $\mathbb{R}^n$) to obtain an estimate for $P_3(\widehat{z}, dz, dz)$ for all $h = x^3 - x\langle y, y\rangle + P_3(y)$ that define a CCPSR manifold $\mathcal{H}$, $\dim(\mathcal{H}) \geq 2$, as in Proposition 4.9. Equivalently, we would need a classification of all CCPSR manifolds of dimension $n \geq 2$ and for each a corresponding $h = x^3 - x\langle y, y\rangle + P_3(y)$, but we would still need transformations of the form $\mathcal{A}(p)$ for all $p \in \mathcal{H}$ as in Proposition 3.18 to obtain for each CCPSR manifold $\mathcal{H}$ all $P_3(y)$'s depending on p. The explicit classification of CCPSR manifolds of arbitrary dimensions is a very difficult, and probably unsolvable, open problem, see Remark 2.44 and also the comment under Theorem 3 in [CDJL]. Even in dimension 2 where we have a classification [CDL, Thm. 1] we would still need all transformations $\mathcal{A}(p)$ defined in (3.7), which would require (after finding each standard form (3.12) for the corresponding cubic polynomial h) explicit knowledge of $\operatorname{dom}(\mathcal{H})$ in each case.

We will now use Proposition 3.18 and Lemma 4.10 to find a general global estimate for the scalar curvature of CCPSR manifolds derived in Proposition 3.29.

Theorem 4.13 (Bounds for the scalar curvature of CCPSR manifolds)**.** *The scalar curvature $S_{\mathcal{H}}$ of every n-dimensional CCPSR manifolds $\mathcal{H}$ equipped with their respective centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h\big|_{T\mathcal{H}\times T\mathcal{H}}$ is globally bounded by*

$$n(n-1)\left(-1-\frac{25}{16}n\right) \leq S_{\mathcal{H}} \leq n(n-1)\left(-1+\frac{25}{16}n\right). \tag{4.15}$$

Proof. For 0- and 1-dimensional CCPSR manifolds the estimate is always true since $S_{\mathcal{H}}$ and the bounds vanish. Assume now that $n = \dim(\mathcal{H}) \geq 2$. Proposition 3.18 implies that we can without loss of generality assume that $\mathcal{H}$ is the connected component of $\{h = x^3 - x\langle y, y\rangle + P_3(y) = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h = 1\} \subset \mathbb{R}^{n+1}$. If we can show that (4.15) holds at the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ independent of which specific form the cubic polynomial $P_3 : \mathbb{R}^n \to \mathbb{R}$ might have, we can, using linear transformations of the form $\mathcal{A}(p)$ as in (3.7) for all other points $p \in \mathcal{H}$, conclude that (4.15) holds globally on $\mathcal{H}$. The formula (3.34) for the scalar curvature of PSR manifolds (h is of homogeneity-degree $\tau = 3$) reads

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = n(1-n) + \frac{27}{8}\sum_{\ell}\sum_{a\neq i}\left(-P_3(\partial_a,\partial_a,\partial_\ell)P_3(\partial_i,\partial_i,\partial_\ell) + P_3(\partial_a,\partial_i,\partial_\ell)^2\right).$$

We rewrite

$$\begin{aligned} P_3(\partial_a,\partial_i,\partial_\ell)^2 = \frac{1}{16}\Big(&P_3(\partial_a+\partial_i,\partial_a+\partial_i,\partial_\ell)^2 + P_3(\partial_a-\partial_i,\partial_a-\partial_i,\partial_\ell)^2 \\ &- 2P_3(\partial_a+\partial_i,\partial_a+\partial_i,\partial_\ell)P_3(\partial_a-\partial_i,\partial_a-\partial_i,\partial_\ell)\Big) \end{aligned} \tag{4.16}$$

for all $1 \leq a, i \leq n$. Note that for all $1 \leq \ell \leq n$ and for $a \neq i$, the vector ∂_ℓ has Euclidean length 1 and the vectors $\partial_a \pm \partial_i$ always have Euclidean length $\sqrt{2}$. Now, using the estimate (4.12) in Lemma 4.10 yields

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \leq n(1-n) + \frac{27}{8}\sum_{\ell}\sum_{a\neq i}\frac{25}{54} = n(n-1)\left(-1+\frac{25}{16}n\right),$$

and analogously

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \geq n(n-1)\left(-1-\frac{25}{16}n\right).$$

These estimates do not depend on the specific form of P_3 as required. □

Remark 4.14. We will later in this thesis, in Proposition 5.12, find a sharp estimate for the scalar curvature of CCPSR surfaces, i.e. CCPSR manifolds $\mathcal{H}$ of dimension $\dim(\mathcal{H}) = 2$. This estimate (5.38) does not coincide with the estimate (4.15) in Theorem 4.13 for $n = 2$. This indicates that the general estimate in Theorem 4.13 is most likely not sharp for $\dim(\mathcal{H}) = n \geq 2$, to prove this for $n \geq 3$ is a task for future studies.

Note that Theorem 4.13 is also true for all closed PSR manifolds with multiple connected components by simply considering each connected component seperately. Similarly to global bounds for the scalar curvature $S_{\mathcal{H}}$ that hold for all CCPSR manifolds $\mathcal{H}$ of fixed dimension, we can also derive global bounds for their sectional curvatures (3.47), again independent of the considered CCPSR manifolds of fixed dimension.

Proposition 4.15 (Bounds for the sectional curvature of CCPSR manifolds)**.** *The sectional curvature K of every $n \geq 2$-dimensional CCPSR manifold $\mathcal{H}$ equipped with their respective centro-affine fundamental form $g_{\mathcal{H}} = -\frac{1}{3}\partial^2 h\big|_{T\mathcal{H}\times T\mathcal{H}}$ is globally bounded by*

$$-1-\frac{25}{16}n \leq K_p(V) \leq -1+\frac{25}{16}n \tag{4.17}$$

for all $p \in \mathcal{H}$ and all 2-planes $V \subset T_p\mathcal{H}$.

Proof. Proposition 3.18 and Lemma 3.31 imply that it suffices to prove the estimate (4.17) for
$$K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j)=-1+\frac{27}{8}\sum_{\ell}\left(-P_3(\partial_i,\partial_i,\partial_\ell)P_3(\partial_j,\partial_j,\partial_\ell)+P_3(\partial_i,\partial_j,\partial_\ell)^2\right)$$
(cf. equation (3.47) for $\tau=3$, $F=\mathbb{1}$) for all $i\neq j$ and all n-dimensional CCPSR manifolds $\mathcal{H}\subset\{h=x^3-x\langle y,y\rangle+P_3(y)=1\}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$. We now proceed as in the proof of Theorem 4.13, rewrite $(P_3(\partial_i,\partial_j,\partial_\ell))^2$ as in equation (4.16), and obtain
$$-1-\frac{25}{16}n\leq K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j)\leq -1+\frac{25}{16}n$$
which is independent of i and j, $i\neq j$, as required. □

Remark 4.16. The global estimate (4.17) can also be used to obtain a global estimate for $S_{\mathcal{H}}$. Since ∂_i and ∂_j are orthogonal at $p=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for all PSR manifolds $\mathcal{H}$ with $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}$ and corresponding h of the form (3.12),
$$S_{\mathcal{H}}=\sum_{i\neq j}K_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_i,\partial_j),\tag{4.18}$$
see Remark 2.15. Hence, the estimate (4.17) in Proposition 4.15 implies
$$n(n-1)\left(-1-\frac{25}{16}n\right)\leq S_{\mathcal{H}}\leq n(n-1)\left(-1+\frac{25}{16}n\right),$$
which coincides with the estimate (4.15) in Theorem 4.13.

Lastly in this section we will give a proof that all closed PSR manifolds $\mathcal{H}$ equipped with their centro-affine fundamental form $g_{\mathcal{H}}$ are geodesically complete. This was first shown in [CNS, Thm. 2.5], in the corresponding proof it was used that the moduli space of closed PSR curves under the action of GL(2), which consists precisely of two elements, is compact (cf. [CHM, Cor. 4]). Our proof will instead make use of the estimates (4.7) for $P_3(z,dz,dz)$ and (4.4) for the diameter of $\partial\mathrm{dom}(\mathcal{H})$. Note that geodesically complete PSR manifolds are necessarily closed, since otherwise we could always continuously extend $g_{\mathcal{H}}$ to each boundary point and construct a geodesic in $\mathcal{H}$ which reaches said point in finite time, cf. [CNS, Prop. 2.4].

Proposition 4.17 (Alternative closed PSR manifolds completeness proof №1)**.** *Closed PSR manifolds $(\mathcal{H},g_{\mathcal{H}})$ are geodesically complete.*

Proof. Let $\mathcal{H}$ be a closed PSR manifold and assume without loss of generality that $\mathcal{H}$ is connected, that is a CCPSR manifold. For $\dim\mathcal{H}=0$ there is nothing to show. Assume that $\dim\mathcal{H}\geq 1$. We will show that for all $p\in\mathcal{H}$ the closure of the geodesic ball $B^{g_{\mathcal{H}}}_{\frac{11}{64\sqrt{3}}}(p)\subset\mathcal{H}$ with respect to the centro-affine (Riemannian) metric $g_{\mathcal{H}}$ is always compact in $\mathcal{H}$. We can then use Lemma 2.21 to conclude that $(\mathcal{H},g_{\mathcal{H}})$ is geodesically complete.

Let $p\in\mathcal{H}$ be arbitrary. Proposition 3.18 implies that we can without loss of generality assume that $p=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and that $\mathcal{H}\subset\{h=x^3-x\langle y,y\rangle+P_3(y)=1\}$. Corollary 4.5 implies that the closure of the Euclidean ball $B_{\frac{\sqrt{3}}{8}}(0)\subset\mathrm{dom}(\mathcal{H})$ with respect to $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ is always compact in $\mathrm{dom}(\mathcal{H})$. Hence with $\Phi:\mathrm{dom}(\mathcal{H})\to\mathcal{H}$ as in (3.14), $\Phi\left(\overline{B_{\frac{\sqrt{3}}{8}}(0)}\right)\subset\mathcal{H}$ is also compact. The upper bound in estimate (4.12) Lemma 4.10 implies that
$$\forall\overline{z}\in\partial B_{\frac{\sqrt{3}}{8}}(0):\quad P_3(\overline{z},dz,dz)\leq\frac{\sqrt{3}}{8}\cdot\frac{5}{6\sqrt{3}}\langle dz,dz\rangle=\frac{5}{48}\langle dz,dz\rangle.\tag{4.19}$$

The linearity of $P_3(z,dz,dz)$ in z implies that (4.19) also holds for all $\overline{z} \in \overline{B_{\frac{\sqrt{3}}{8}}(0)}$. We use this and $0 < \beta(z) \leq 1$ on $\mathrm{dom}(\mathcal{H})$ to estimate $\Phi^* g_{\mathcal{H}}$ (see (3.33) with $\tau = 3$) on $\overline{B_{\frac{\sqrt{3}}{8}}(0)}$ and obtain for all $\overline{z} \in \overline{B_{\frac{\sqrt{3}}{8}}(0)}$

$$\begin{aligned}(\Phi^* g_{\mathcal{H}})_{\overline{z}} &= \frac{-\partial^2 \beta_{\overline{z}}}{3\beta(\overline{z})} + \frac{2d\beta_{\overline{z}}^2}{9\beta^2(\overline{z})} \\ &\geq \frac{-\partial^2 \beta_{\overline{z}}}{3\beta(\overline{z})} = \frac{1}{3\beta(\overline{z})}(2\langle dz, dz\rangle - 6P_3(\overline{z}, dz, dz)) \\ &\geq \frac{1}{3}\left(2 - \frac{5}{8}\right)\langle dz, dz\rangle = \frac{11}{24}\langle dz, dz\rangle.\end{aligned}$$

Hence,

$$\overline{B^{\Phi^* g_{\mathcal{H}}}_{\frac{\sqrt{3}}{8}\cdot\frac{11}{24}}(0)} = \overline{B^{\Phi^* g_{\mathcal{H}}}_{\frac{11}{64\sqrt{3}}}(0)} \subset \overline{B_{\frac{\sqrt{3}}{8}}(0)},$$

which implies that the closure of the geodesic ball around $z = 0$ with radius $\frac{11}{64\sqrt{3}}$ is always compactly embedded in $\mathrm{dom}(\mathcal{H})$. Since $\Phi : (\mathrm{dom}(\mathcal{H}), \Phi^* g_{\mathcal{H}}) \to (\mathcal{H}, g_{\mathcal{H}})$ is an isometry, it follows that $\overline{B^{g_{\mathcal{H}}}_{\frac{11}{64\sqrt{3}}}(0)} \subset \mathcal{H}$ is compact. This holds independently of the $p \in \mathcal{H}$ we started with and, hence, we can now use Lemma 2.21 and conclude that $(\mathcal{H}, g_{\mathcal{H}})$ is geodesically complete. □

To summarize this section, we have seen that the condition for the cubic polynomial $P_3(y)$ in $h = x^3 - x\langle y, y\rangle + P_3(y)$, cf. equation (3.12) in Proposition 3.18, so that h corresponds to a CCPSR manifold, yield various new results about the different curvature tensors, and can also be used to prove the known result about completeness of closed PSR manifolds in a different way. In particular we did not need to consider any regularity conditions of the boundary of $\mathrm{dom}(\mathcal{H})$ (cf. [CNS, Prop. 2.4] or Definition 5.1 in the following section) or properties the quotient space of closed PSR curves (cf. [CHM, Cor. 4]). This might make one hope that a similar way can be used to solve the open problem of completeness of closed GPSR manifolds (with $\tau \geq 4$, τ being the degree of homogeneity of the corresponding polynomials h). Unfortunately, we have not found a way to do that. We will however illustrate occurring problems in that endeavour for quartic GPSR manifolds ($\tau = 4$), which in even more generality will appear also for $\tau > 4$, in Section 7 and present partial results.

Next, we will be concerned with the moduli space of CCPSR manifolds. As mentioned before in Remark 2.44, one cannot expect to classify all CCPSR manifolds in a fixed dimension without further restrictions (recall that PSR manifolds that are homogeneous spaces under the action of the respective Lie group G_0^h as in Definition 3.13 have been classified in [DV]). We will however present an idea for a deformation theory for CCPSR manifolds. This might also be of particular interest in physics, i.e. in the theory of supergravity, see [GST, FS, DV, CHM] and also Section 8.

5 Deformation theory of projective special real manifolds

In this section we will study how one can deform CCPSR manifolds. This is motivated by the following problem. Let $\mathcal{H} \subset \{h = 1\}$ be an $n \geq 1$-dimensional CCPSR manifold where we assume without loss of generality that $\mathcal{H}$ is precisely the connected component of the level set $\{h = 1\}$ which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h = 1\} \subset \mathbb{R}^{n+1}$ and that h is of the form $h = x^3 - x\langle y, y\rangle + P_3(y)$ as in equation (3.12). Let $V : \mathbb{R}^n \to \mathbb{R}$, $V \in \mathrm{Sym}^3(\mathbb{R}^n)^*$, be another cubic polynomial. We now want to determine all $\varepsilon > 0$, such that

$$h_\varepsilon := x^3 - x\langle y, y\rangle + P_3(y) + \varepsilon V(y) \tag{5.1}$$

also defines a CCPSR manifold $\mathcal{H}_\varepsilon$ in the sense that, as for the initial CCPSR manifold $\mathcal{H}$, $\mathcal{H}_\varepsilon$ is the connected component of $\{h_\varepsilon = 1\}$ which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h_\varepsilon = 1\} \subset \mathbb{R}^{n+1}$. It will turn out that the answer to this question and in particular the existence of one such $\varepsilon > 0$ will only depend on the behaviour of the continuous function

$$\varepsilon \mapsto \max_{\|z\|=1} \left(P_3(z) + \varepsilon V(z)\right). \tag{5.2}$$

We start with a description of the boundary behaviour of the centro-affine fundamental form of CCGPSR manifolds.

Definition 5.1 (Regular boundary behaviour). *Let $\mathcal{H} \subset \{h = 1\}$ be a CCGPSR manifold of dimension $n \geq 1$ and let $U = \mathbb{R}_{>0} \cdot \mathcal{H}$ be the corresponding convex cone (cf. Proposition 3.7). Then $\mathcal{H}$ has regular boundary behaviour if*

(i) $dh_p \neq 0$ for all $p \in \partial U \setminus \{0\}$, i.e. $\mathcal{H}$ is not singular at infinity in the sense of Definition 3.16,

(ii) $-\partial^2 h|_{T(\partial U\setminus\{0\})\times T(\partial U\setminus\{0\})} \geq 0$ and $\dim\ker\left(-\partial^2 h|_{T(\partial U\setminus\{0\})\times T(\partial U\setminus\{0\})}\right) = 1$ for all $p \in \partial U \setminus \{0\}$.

Note that Definition 5.1 is equivalent to [CNS, Def. 1.17] restricted to CCGPSR manifolds. We also want to stress that Definition 5.1 is independent of the chosen linear coordinates of the ambient space $\mathbb{R}^{n+1}$.

Remark 5.2. With the functions α and β as in (3.21) and (3.22), Lemma 3.25 shows that the conditions (i) and (ii) in Definition 5.1 are equivalent to

(i) $\alpha(\overline{z}) \neq 0$ (or, equivalently, $d\beta_{\overline{z}}(\overline{z}) \neq 0$) for all $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$,

(ii) $-\partial^2 \beta_{\overline{z}} > 0$ for all $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$,

respectively. The second equivalence might not be immediately obvious. It follows from the equality

$$T_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)}(\partial U) = \mathbb{R} \cdot \left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right) \oplus T_{\overline{z}}(\partial\mathrm{dom}(\mathcal{H}))$$

and the property that this decomposition is orthogonal with respect to $-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)}$, since

$$-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)}\left(\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right), \cdot\right) = -(\tau - 1) dh_{\left(\begin{smallmatrix} 1 \\ \overline{z} \end{smallmatrix}\right)}(\cdot),$$

which vanishes on $T_{\overline{z}}(\partial\mathrm{dom}(\mathcal{H}))$ viewed as a linear subspace of $\mathbb{R}^{n+1}$.

We will now prove that for CCPSR manifolds, the condition Def. 5.1 (i) always implies Def. 5.1 (ii). We formulate this as follows.

Theorem 5.3 (Regularity conditions for CCPSR manifolds)**.** *CCPSR manifolds of dimension $n \geq 1$ are not singular at infinity in the sense of Definition 3.16 if and only if they have regular boundary behaviour as defined in Definition 5.1.*

Proof. A CCPSR manifold $\mathcal{H}$ that has regular boundary behaviour is by definition not singular at infinity. For the other direction, consider first $n = 1$. Then Def. 5.1 (ii) is trivially satisfied.

To prove the statement of this theorem for $n \geq 2$, it suffices to prove it for $n = 2$. To see this, consider any CCPSR manifold $\mathcal{H}$ of dimension $n > 2$ and assume that Def. 5.1 (i) holds for $\mathcal{H}$. Assume without loss of generality that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ and that h is of the form (3.12). Considering Remark 5.2, Def. 5.1 (ii) holds true if and only if Rem. 5.2 (ii) holds true. To show the latter we need to show that $-\partial^2\beta_{\overline{z}}(v,v) > 0$ for all $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$ and all $0 \neq v \in T_{\overline{z}}(\partial\mathrm{dom}(\mathcal{H})) \subset \mathbb{R}^n$. Observe that for any 2-dimensional linear subspace $E = \mathrm{span}\{w_1, w_2\} \subset \mathbb{R}^n$, where w_1 and w_2 are chosen such that they are orthonormal with respect to $\langle\cdot,\cdot\rangle$, the restricted polynomial

$$h^E\left(\left(\begin{smallmatrix} x \\ t_1 \\ t_2 \end{smallmatrix}\right)\right) := x^3 - x(t_1^2 + t_2^2) + P_3(t_1 w_1 + t_2 w_2)$$

defines a 2-dimensional CCPSR manifold $\mathcal{H}^E \subset \left\{h^E = 1\right\} \subset \mathbb{R}^3$ as the connected component containing the point $\left(\begin{smallmatrix} x \\ t_1 \\ t_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$. Furthermore,

$$\mathrm{dom}\left(\mathcal{H}^E\right) \ni \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) \mapsto t_1 w_1 + t_2 w_2 \in \mathrm{dom}(\mathcal{H})$$

is an embedding. Note that the explicit formula for h^E in general depends on the choice of basis for E. Hence, if we want to show that $-\partial^2\beta_{\overline{z}}(v,v) > 0$ for some fixed $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$ and $0 \neq v \in T_{\overline{z}}(\partial\mathrm{dom}(\mathcal{H}))$, it suffices to show Rem. 5.2 (ii) for $\mathcal{H}^E$ and h^E, respectively $\beta^E\left(\left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right)\right) = h^E\left(\left(\begin{smallmatrix} 1 \\ t_1 \\ t_2 \end{smallmatrix}\right)\right)$, with $E = \mathrm{span}\{\overline{z}, v\}$ where we view v as an element of $\mathbb{R}^n$. Hence, proving the statement of this theorem for all 2-dimensional CCPSR manifolds will also prove it for these of higher dimension. Since the conditions in Definition 5.1 are independent of the linear coordinates chosen for the ambient space $\mathbb{R}^{n+1}$, we can reduce our studies to the classification of 2-dimensional CCPSR manifolds up to equivalence given in [CDL, Thm. 1][6], see Theorem 2.45. We will do a case-by-case check for the surfaces a)–e) and the one-parameter family of surfaces f) in Theorem 2.45. For the cases a)–e) we will study the P_3-part the calculated standard form $\widetilde{h} = x^3 - x(y^2+z^2) + P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)$ (3.12) of each cubic h corresponding to a CCPSR surface $\mathcal{H} \subset \{h = 1\}$ obtained in Example 3.24 with the property that $\mathcal{H}$ is equivalent to the connected component of $\left\{\widetilde{h} = 1\right\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$. We can then use Lemma 4.6, which says that the value of $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} \left|P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)\right| \in \left[0, \frac{2}{3\sqrt{3}}\right]$ determines whether $\mathcal{H}$ is singular at infinity or not. In the cases where $\mathcal{H}$ is not singular at infinity, that is fulfils Def. 5.1 (i), we need to show that it also fulfils Def. 5.1 (ii). For the one-parameter family f) we will use another method and explain why in this case the form (3.12) is not the best choice to work with in order to prove our claim.

[6] At the time the article [CDL] was written and published, it was still an open problem to show that a PSR manifold is closed if and only if it is geodesically complete, which has first been proven in [CNS].

a) $\mathcal{H} = \{h = xyz = 1,\ x > 0,\ y > 0\}$**.**
Equation (3.15) implies that $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = -\frac{2}{3\sqrt{3}}y^3 + \frac{2}{\sqrt{3}}yz^2$. Since $\mathcal{H}$ is a CCPSR surface and $P_3\left(\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$, Lemma 4.6 implies that $\mathcal{H}$ is singular at infinity.

b) $\mathcal{H} = \{h = x(xy - z^2) = 1,\ x > 0\}$**.**
By equation (3.16), $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}y^3 + \frac{1}{\sqrt{3}}yz^2$ with $P_3\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$. Hence, $\mathcal{H}$ is singular at infinity.

c) $\mathcal{H} = \{h = x(yz + x^2) = 1,\ x < 0,\ y > 0\}$**.**
This case is a little more complicated in comparison with **a)** and **b)**. Equation (3.17) implies that $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2\sqrt{2}}{\sqrt{15}}y^2z + \frac{14\sqrt{2}}{15\sqrt{15}}z^3$. We now need to determine $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} \left|P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)\right|$. We find for $v = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{5}}{2\sqrt{2}} \end{pmatrix}$, $\|v\| = 1$, that $P_3(v) = \frac{2}{3\sqrt{3}}$. Hence, $\mathcal{H}$ being closed and connected implies that $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} \left|P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)\right| = \frac{2}{3\sqrt{3}}$. This shows that $\mathcal{H}$ is singular at infinity. Note that v can be found without the help of a computer algebra system like MAPLE by considering the equation $dP_3|_{\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)} = r\left\langle \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right), \cdot \right\rangle$, $r > 0$, which is not difficult to solve in this case since $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)$ is reducible.

d) $\mathcal{H} = \{h = z(x^2 + y^2 - z^2) = 1,\ z < 0\}$**.**
From equation (3.18) we obtain that in this case $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) \equiv 0$. Hence, $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} \left|P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)\right| = 0$ and $\mathcal{H}$ is thus not singular at infinity. It is immediate that $\mathrm{dom}\left(\widetilde{\mathcal{H}}\right) = \left\{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| < 1\right\}$ and that for the corresponding function $\beta(y, z) = 1 - y^2 - z^2$ as in (3.22) we have $d\beta = -2ydy - 2zdz$. Hence, $d\beta$ vanishes at no point in $\partial\mathrm{dom}\left(\widetilde{\mathcal{H}}\right)$, so Lemma 3.25 implies that $\widetilde{\mathcal{H}}$, and thus also $\mathcal{H}$, fulfils Def. 5.1 (i). Furthermore

$$\partial^2\beta_{\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} < 0 \quad \forall \begin{pmatrix} y \\ z \end{pmatrix} \in \partial\mathrm{dom}\left(\widetilde{\mathcal{H}}\right),$$

so $\widetilde{\mathcal{H}}$, and equivalently $\mathcal{H}$, fulfils Def. 5.1 (ii).

e) $\mathcal{H} = \{h = x(y^2 - z^2) + y^3 = 1,\ y < 0,\ x > 0\}$**.**
From equation (3.19) we know that $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}y^3 - \frac{1}{2\sqrt{3}}yz^2$. Hence, $P_3\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$, which shows that $\mathcal{H}$ is singular at infinity.

f)[7] $\mathcal{H}_b = \{h = y^2z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0,\ 2x > z\}$**,** $b \in (-1, 1)$**.**
For all $b \in (-1, 1)$, the projective curve $C := \{h = y^2z - 4x^3 + 3xz^2 + bz^3 = 0\} \subset \mathbb{R}\mathrm{P}^2$ has no singularities, cf. [CDL, Prop. 3], which means that $dh_p \neq 0$ for all $p \in \{h = 0\} \setminus \{0\} \subset \mathbb{R}^3$. Hence, each $\mathcal{H}_b$, $b \in (-1, 1)$, is not singular at infinity in the sense of Definition 3.16 and, hence, fulfils condition Def. 5.1 (i). Note that $\mathcal{H}_b$ not being singular at infinity for all

[7] For this one-parameter family of CCPSR surfaces which are each contained in the level set of the respective Weierstaß cubic with positive discriminant h, the method used for **a)**–**e)** has proven itself to be unsuitable. This is because the formulas for the corresponding function β as in (3.22) and the derivatives corresponding to h when brought to the form (3.12) might not depend on $b \in (-1, 1)$ in complicated way, but studying the system of equations $v \in \ker d\beta$, $v \in \ker \partial^2\beta$, $\beta = 0$, turned out to be quite difficult. We will thus consider Definition 5.1 and not the equivalent conditions in Remark 5.2 to prove our claim for this one-parameter family.

$b \in (-1,1)$ also follows easily from equation (3.20) in Example 3.24. We need to show that each $\mathcal{H}_b$, $b \in (-1,1)$, also fulfils Def. 5.1 (ii). In order to prove this, we need to determine $\partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b) \subset \{h = 0,\ z \leq 0,\ 2x \geq z\} \subset \mathbb{R}^3$ for each $b \in (-1,1)$. Observe that $\{h = 0,\ z \leq 0,\ 2x \geq z\} \cap \{z = 0\} = \{x = 0,\ z = 0\}$. Hence, the line $\{x = 0,\ z = 0\}$ is contained in $\{h = 0,\ z \leq 0,\ 2x \geq z\}$, but $\mathbb{R}_{>0} \cdot \mathcal{H}_b$ being a convex cone which has the property described in Lemma 3.8 shows that $\{x = 0,\ z = 0\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b) = \left\{\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)\right\}$. For $z < 0$ we will determine the section $\{z = -1\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$, which can then be used with the homogeneity of h to obtain the whole set $\partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$. We find

$$h\left(\left(\begin{smallmatrix}x\\y\\-1\end{smallmatrix}\right)\right) = 0 \quad \Leftrightarrow \quad \underbrace{-y^2 - 4x^3 + 3x - b}_{=:\rho_b} = 0, \tag{5.3}$$

where $\rho_b\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) = h\left(\left(\begin{smallmatrix}x\\y\\-1\end{smallmatrix}\right)\right)$. We consider ρ_b to be defined for all $b \in \mathbb{R}$, not just for $b \in (-1,1)$. Let

$$V = \operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)\right\}$$

and observe that $\mathcal{H}_b$ not being singular at infinity implies that the tangent space $T\partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$ fulfils

$$T_p\partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b) = \mathbb{R} \cdot p \oplus (\ker dh_p \cap V) \quad \forall p \in \{z = -1\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b).$$

Furthermore, the 1-dimensional linear subspaces $\mathbb{R} \cdot p$ and $\ker dh_p \cap V$ of $T_p\partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$ are orthogonal with respect to the positive-semidefinite bilinear form $-\partial^2 h_p$, which follows from $-\partial^2 h_p(p,\cdot) = -2dh_p(\cdot)$. Also note that $\ker dh_p \cap V$ is always 1-dimensional since the position vector $p \neq 0$ is always an element of $\ker dh_p$ for all $p \in \mathbb{R}_{>0} \cdot \mathcal{H}_b$. Thus, in order to prove that Def. 5.1 (ii) is fulfilled for each $\mathcal{H}_b$, $b \in (-1,1)$, it suffices to show that $-\partial^2 h|_{(\ker dh_p \cap V) \times (\ker dh_p \cap V)} > 0$. We obtain

$$dh = (-12x^2 + 3z^2)dx + 2yzdy + (y^2 + 6xz + 3bz^2)dz$$

and

$$\partial^2 h = \begin{pmatrix} -24x & 0 & 6z \\ 0 & 2z & 2y \\ 6z & 2y & 6x + 6bz \end{pmatrix}.$$

Since $\mathcal{H}_b$ is not singular at infinity, it follows that at each point $p = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right) \in \{z = -1\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$, $\ker dh_p \cap V$ is given by

$$\ker dh_p \cap V = \operatorname{span}\left\{\begin{pmatrix} -\partial_y h_p \\ \partial_x h_p \\ 0 \end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix} 2y \\ -12x^2 + 3 \\ 0 \end{pmatrix}\right\}.$$

Hence, $-\partial^2 h|_{(\ker dh_p \cap V) \times (\ker dh_p \cap V)} > 0$ if and only if

$$\begin{aligned} & -\partial^2 h_p\left(\left(\begin{smallmatrix}2y\\-12x^2+3\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2y\\-12x^2+3\\0\end{smallmatrix}\right)\right) > 0 \\ \Leftrightarrow \quad & 96xy^2 + 288x^4 - 144x^2 + 18 > 0 \\ \Leftrightarrow \quad & 16xy^2 + 48x^4 - 24x^2 + 3 > 0 \\ \Leftrightarrow \quad & 16xy^2 + \left(4\sqrt{3}x^2 - \sqrt{3}\right)^2 > 0 \end{aligned} \tag{5.4}$$

for all $p = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right) \in \{h = 0,\ z = -1\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$. We will first check the above inequality (5.4) for $y = 0$. In that case, (5.4) can only be false if $x = \pm\frac{1}{2}$. Then with ρ_b defined as in (5.3) we obtain

$$0 = \rho_b\left(\left(\begin{smallmatrix}\pm\frac{1}{2}\\0\end{smallmatrix}\right)\right) = \mp\frac{1}{2} \pm \frac{3}{2} - b = \pm 1 - b.$$

This is however a contradiction to $b \in (-1,1)$ and, hence, (5.4) holds at all points in $\{z = -1,\ y = 0\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$. Now let $y \neq 0$. We see that then (5.4) is true for all $x \geq 0$, independent of $b \in (-1,1)$. It thus remains to check the inequality (5.4) for points in $\{z = -1,\ x < 0\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$. Note that the latter set might be empty, in fact one can show that it is empty if and only if $0 < b < 1$, but we will not need this information for our proof. Observe that for all $b_1, b_2 \in \mathbb{R}$ with $b_1 < b_2$,

$$\rho_{b_1}\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) > \rho_{b_2}\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$$

for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$. Hence, $\rho_{b_1}|_{\{\rho_{b_2}=0\}} > 0$, which in particular implies that

$$\rho_{-1}|_{\{\rho_b=0\}} > 0 \tag{5.5}$$

for all $b \in (-1,1)$. With the fact that $\begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \end{pmatrix} \in \{z = -1\} \cap (\mathbb{R}_{>0} \cdot \mathcal{H}_b)$, cf. Example 3.24, and $\rho_{-1}\left(\left(\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right)\right) = 2 > 0$ it follows that $\{z = -1\} \cap \overline{(\mathbb{R}_{>0} \cdot \mathcal{H}_b)}$ is a subset of the connected component of $\{\rho_{-1} > 0\} \times \{-1\} \subset \mathbb{R}^3$ that contains the point $\begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \end{pmatrix}$, see Figure 5. Further observe that $\rho_b\left(\left(\begin{smallmatrix} -\frac{1}{2} \\ y \end{smallmatrix}\right)\right) = -y^2 - 1 - b < 0$ for all $y \in \mathbb{R}$ and $b \in (-1,1)$, and that

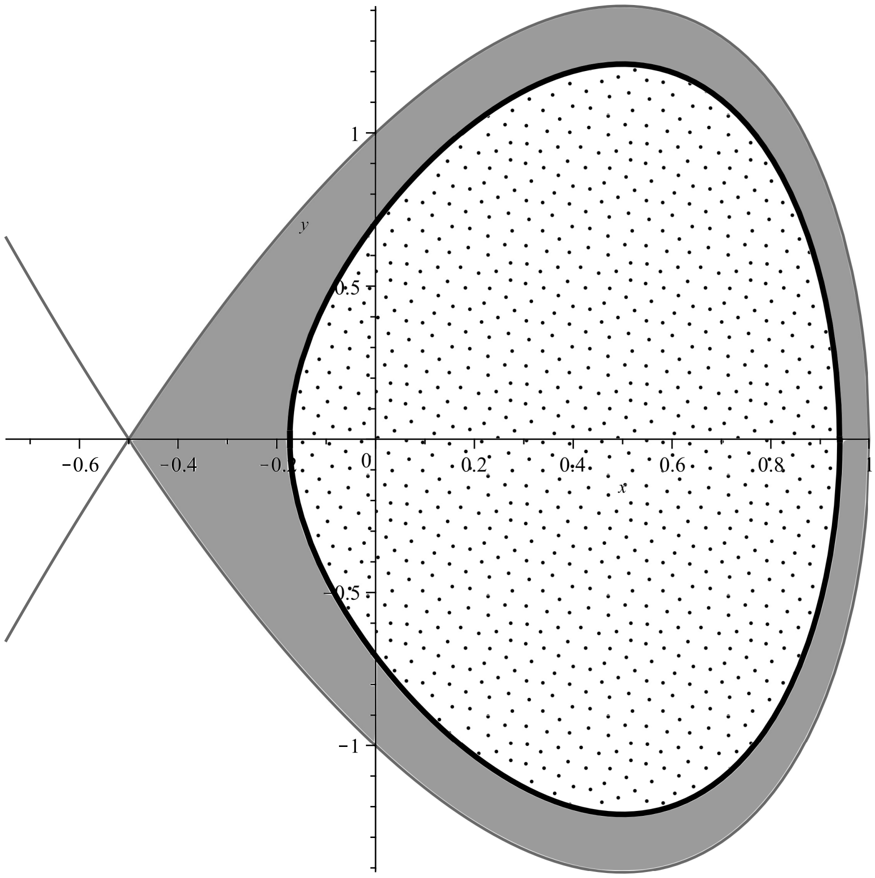


Figure 5: The connected component of $\{\rho_{-1} > 0\}$ that contains the point $\left(\frac{1}{2}, 0, -1\right)^T$ is (partly) marked in grey, its boundary is a part of the set $\{\rho_{-1} = 0\}$ which is also shown in $\{-1 < x < 1\} \subset \mathbb{R}^2$. The dotted area in the plot is the connected component of $\left\{\rho_{-\frac{1}{2}} > 0\right\}$ that contains the point $\left(\frac{1}{2}, 0, -1\right)^T$.

$\mathcal{H}_b \subset \{z < 0,\ 2x > z\}$ implies that $\{z = -1\} \cap \partial(\mathbb{R}_{>0} \cdot \mathcal{H}_b)$ is contained in $\left\{z = -1,\ x > -\frac{1}{2}\right\}$ for all $b \in (-1,1)$. In particular there exists no $b \in (-1,1)$, such that the x-coordinate of an element in $\{z=-1\}\cap\partial(\mathbb{R}_{>0}\cdot\mathcal{H}_b)$ has the value $-\frac{1}{2}$. Hence, (5.4) and (5.5) imply that in order to prove that $\mathcal{H}_b$ fulfils Def. 5.1 (ii) it suffices to show

$$\left\{16xy^2+\left(4\sqrt{3}x^2-\sqrt{3}\right)^2=0\right\}\cap\left\{\rho_{-1}=0,\ x\geq-\frac{1}{2}\right\}\cap\{x<0\}=\left\{\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\0\end{pmatrix}\right\},\quad(5.6)$$

since $\left(16xy^2+(4\sqrt{3}x^2-\sqrt{3})^2\right)\Big|_{\{x>-\frac{1}{2},\ y=0\}}>0$, see also Figure 6. We insert $\rho_{-1}=0$, which

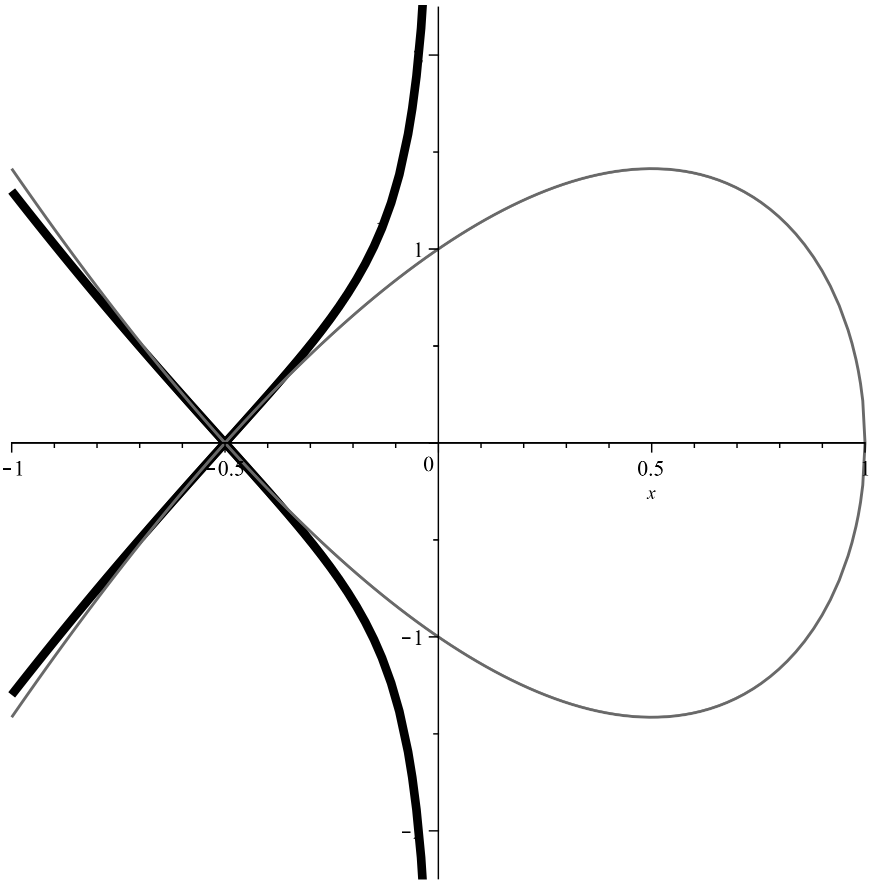


Figure 6: The thick black curves represent the set $\left\{16xy^2+\left(4\sqrt{3}x^2-\sqrt{3}\right)^2=0\right\}\cap\{-1<x<0\}$, the thinner grey curve is the set $\{\rho_{-1}=0\}\cap\{-1<x<1\}$.

is equivalent to $y^2=-4x^3+3x+1$, into $16xy^2+\left(4\sqrt{3}x^2-\sqrt{3}\right)^2=0$ and obtain

$$F(x):=-16x^4+24x^2+16x+3=0.$$

One can now use a computer algebra system like MAPLE and find that $F(x)=0$ and $-\frac{1}{2}\leq x<0$ if and only if $x=-\frac{1}{2}$. This proves (5.6) and, hence, shows that each $\mathcal{H}_b$, $b\in(-1,1)$, fulfils Def. 5.1 (ii).

This finishes the proof of Theorem 5.3. □

Lemma 4.6 and Theorem 5.3 show the following.

Corollary 5.4 (Critical values of $P_3|_{\{\|z\|=1\}}$ and regularity of CCPSR manifolds)**.** *An $n\geq 1$-dimensional CCPSR manifold $\mathcal{H}\subset\{h=x^3-x\langle y,y\rangle+P_3(y)=1\}$, $\binom{x}{y}=\binom{1}{0}\in\mathcal{H}$, has regular boundary behaviour if and only if* $\max\limits_{\|z\|=1}P_3(z)<\frac{2}{3\sqrt{3}}$.

Proposition 5.5 (Starshape and path-connectedness of the moduli space of CCPSR manifolds)**.** *Let $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ be an $n \geq 1$-dimensional CCPSR manifold and assume without loss of generality that $h = x^3 - x\langle y, y\rangle + P_3(y)$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ (cf. Proposition 3.18). Let $\|\cdot\|$ denote the norm on $\mathbb{R}^n$ induced by the Euclidean standard scalar product $\langle \cdot, \cdot \rangle$ determined by the choice of the coordinates $y = (y_1, \ldots, y_n)^T$. Then for all $s \in [0, 1]$, the connected component $\mathcal{H}_s \subset \{h_s := x^3 - x\langle y, y\rangle + sP_3(y) = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is a CCPSR manifold.*

Proof. For all $s \in [0, 1]$,

$$\max_{\|z\|=1} |sP_3(z)| \leq \max_{\|z\|=1} |P_3(z)| \leq \frac{2}{3\sqrt{3}}.$$

Hence Corollary 4.3 shows that for each corresponding $\mathcal{H}_s$, which is by definition closed as a subset of $\mathbb{R}^{n+1}$, the necessary condition for $\mathcal{H}_s$ to be a CCPSR manifold, namely that the set $(\mathbb{R}_{>0} \cdot \mathcal{H}_s) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is precompact, is satisfied. For $s = 1$, $\mathcal{H}_1$ and $\mathcal{H}$ coincide. For $s = 0$, (4.5) in Lemma 4.8 immediately shows that $\mathcal{H}_0$ is a CCPSR manifold. Now consider $s \in (0, 1)$ and let $\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in (\mathbb{R}_{>0} \cdot \mathcal{H}_s) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ be arbitrary. For $z = 0$, (4.5) in Lemma 4.8 is always true. For $z \neq 0$, we will differentiate between the cases $P_3(z) \geq 0$ and $P_3(z) < 0$. In the first case, that is $P_3(z) \geq 0$, the estimate (4.3) in Lemma 4.2 for $f_{sP_3\left(\frac{z}{\|z\|}\right)}(t) = h_s\left(\left(\begin{smallmatrix} 1 \\ t\frac{z}{\|z\|} \end{smallmatrix}\right)\right)$ (note: $B = P_3\left(\frac{z}{\|z\|}\right)$ and $A = sP_3\left(\frac{z}{\|z\|}\right)$) show that $z \in \mathrm{dom}(\mathcal{H})$ for all $s \in (0, 1)$. Hence, using the hyperbolicity of $\mathcal{H}$ we estimate

$$3\langle dz, dz\rangle - 9sP_3(z, dz, dz) + \langle z, dz\rangle^2 > s\left(3\langle dz, dz\rangle - 9P_3(z, dz, dz) + \langle z, dz\rangle^2\right) > 0. \quad (5.7)$$

This shows that all points in $(\mathbb{R}_{>0} \cdot \mathcal{H}_s) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ with $P_3(z) \geq 0$ satisfy (4.5) in Lemma 4.8 for all $s \in (0, 1)$.

Next, consider the case $P_3(z) < 0$. This case is a bit more complicated, since the estimate (4.2)[8] in Lemma 4.4 for $f_{-sP_3\left(\frac{z}{\|z\|}\right)}(t) = h_s\left(\left(\begin{smallmatrix} 1 \\ -t\frac{z}{\|z\|} \end{smallmatrix}\right)\right)$ shows that for all $s \in (0, 1)$ there exist points in $(\mathbb{R}_{>0} \cdot \mathcal{H}_s) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ that are not contained in the set

$$\left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathrm{dom}(\mathcal{H})\right\} = (\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathbb{R}^n\right\}$$

(see Figure 7 for an example). Consider for $\overline{z} \in \mathbb{R}_{>0} \cdot z$, such that $\overline{z} \in \partial\mathrm{dom}(\mathcal{H})$, and for $t \in [0, 1]$ the function $r : [0, 1] \to [1, \infty)$ implicitly defined by

$$F(r, t) = 1 - r^2\langle \overline{z}, \overline{z}\rangle + (1 - t)r^3 P_3(\overline{z}) = 0.$$

The condition that $r(t)$ is a positive function and the uniqueness of the positive real root of $r \mapsto F(r, t)$ for all $t \in [0, 1]$ show that $F(r, t) = 0$ indeed defines $r(t)$ in a unique way, and furthermore that $r(t)$ is smooth for $t \in (0, 1)$ and continuous for $t \in [0, 1]$ (note: $P_3(\overline{z}) < 0$). The map

$$\begin{aligned} \Psi : \{1\} \times ((\mathbb{R}_{>0} \cdot z) \cap \mathrm{dom}(\mathcal{H})) \to{}& (\{1\} \times \mathbb{R}_{>0} \cdot z) \\ &\cap \left((\mathbb{R}_{>0} \cdot \mathcal{H}_s) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathbb{R}^n\right\}\right), \\ \begin{pmatrix} 1 \\ \widetilde{z} \end{pmatrix} \mapsto{}& \begin{pmatrix} 1 \\ r(1-s)\widetilde{z} \end{pmatrix}, \end{aligned}$$

[8] With corresponding values $B = -P_3\left(\frac{z}{\|z\|}\right)$ and $A = -sP_3\left(\frac{z}{\|z\|}\right)$.

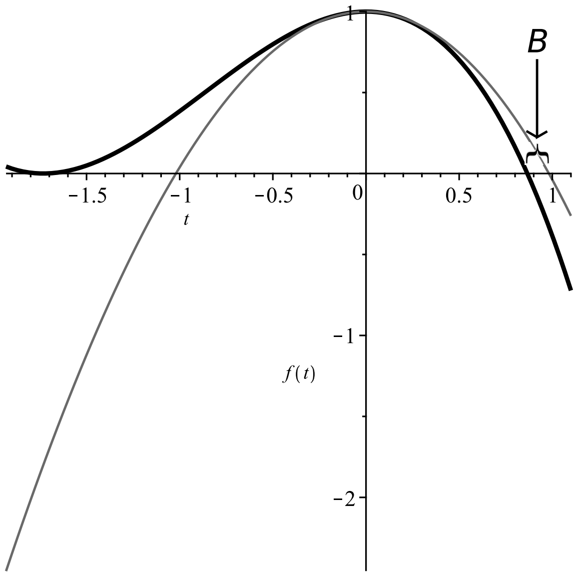


Figure 7: The black curve is a plot of $f_{-\frac{2}{3\sqrt{3}}}(t)$ corresponding to $s=1$, that is $\mathcal{H}_1$. The grey curve is a plot of $f_{-\frac{1}{10}\cdot\frac{2}{3\sqrt{3}}}(t)$ corresponding to $s=\frac{1}{10}\in(0,1)$, that is $\mathcal{H}_{-\frac{1}{10}}$. The set B is to be understood as points in that are not contained in $\mathrm{dom}(\mathcal{H}_1)$, but are contained in (a fitting projection of the set) $(\mathbb{R}_{>0}\cdot\mathcal{H}_s)\cap\left\{(1,z)^T\in\mathbb{R}^{n+1}\ \middle|\ z\in\mathbb{R}^n\right\}$.

is thus a diffeomorphism for all $s\in[0,1]$. Furthermore, Ψ can be continuously extended to be defined on $\{1\}\times\left((\mathbb{R}_{>0}\cdot z)\cap\overline{\mathrm{dom}(\mathcal{H})}\right)$ for all $s\in[0,1]$, with the property that

$$\Psi(\overline{z})\in\partial\left((\mathbb{R}_{>0}\cdot\mathcal{H}_s)\cap\left\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\ \middle|\ z\in\mathbb{R}^n\right\}\right).$$

We obtain for the first t-derivative of $r=r(t)$ for all $t\in(0,1)$

$$\begin{aligned}&-2r(t)\dot r(t)\langle\overline{z},\overline{z}\rangle-r^3(t)P_3(\overline{z})+3(1-t)r^2(t)\dot r(t)P_3(\overline{z})=0\\ \Leftrightarrow\quad &\dot r(t)=\frac{-r^2(t)P_3(\overline{z})}{2\langle\overline{z},\overline{z}\rangle-3(1-t)r(t)P_3(\overline{z})}.\end{aligned}\tag{5.8}$$

Since $P_3(\overline{z})<0$ and $t\in(0,1)$, this in particular shows that $\dot r(t)>0$ for all $t\in(0,1)$. If the considered point $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in(\mathbb{R}_{>0}\cdot\mathcal{H}_s)\cap\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\mid z\in\mathbb{R}^n\}$ is also an element of $\{1\}\times\mathrm{dom}(\mathcal{H})$, then we can use estimate (5.7) for all $s\in(0,1)$. For $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in(\mathbb{R}_{>0}\cdot\mathcal{H}_s)\cap\{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\in\mathbb{R}^{n+1}\mid z\in\mathbb{R}^n\}\setminus(\{1\}\times\mathrm{dom}(\mathcal{H}))$, we want to show that (4.5) holds for all $s\in(0,1)$, i.e. that $3\langle dz,dz\rangle-9sP_3(z,dz,dz)+\langle z,dz\rangle^2>0$ for all $s\in(0,1)$. Substituting $s=1-t$ and $z=\Psi(\widetilde{z})=r(t)\widetilde{z}$ with $\widetilde{z}\in\mathrm{dom}(\mathcal{H})$, the latter is equivalent to

$$3\langle dz,dz\rangle-9(1-t)r(t)P_3(\widetilde{z},dz,dz)+r^2(t)\langle\widetilde{z},dz\rangle^2>0.\tag{5.9}$$

Since $\mathcal{H}$ is a CCPSR manifold by assumption, we already know that

$$3\langle dz,dz\rangle-9P_3(\widetilde{z},dz,dz)+\langle\widetilde{z},dz\rangle^2>0$$

for all $\widetilde{z}\in\mathrm{dom}(\mathcal{H})$, cf. Lemma 4.8. Since $r^2(t)>1$ for all $t\in(0,1)$, proving

$$3\langle dz,dz\rangle-9(1-t)r(t)P_3(\widetilde{z},dz,dz)+\langle\widetilde{z},dz\rangle^2>0$$

for all $t\in(0,1)$ and all $\widetilde{z}\in\mathrm{dom}(\mathcal{H})$ will in particular prove (5.9). Since the estimate $3\langle dz,dz\rangle+\langle\widetilde{z},dz\rangle^2>0$ holds true for all $\widetilde{z}\in\mathbb{R}^n$, it suffices to show that $(1-t)r(t)\leq 1$ for all $t \in (0,1)$. The function $(1-t)r(t)$ is non-negative and continuous on $[0,1]$, and positive and smooth on $(0,1)$. For $t=0$, $(1-t)r(t)|_{t=0} = r(0) = 1$. Using (5.8) yields

$$\frac{\partial}{\partial t}((1-t)r(t)) = -r(t) + (1-t)\dot{r}(t) = \frac{-2r(t)\langle \overline{z},\overline{z}\rangle + 2(1-t)r^2(t)P_3(\overline{z})}{2\langle \overline{z},\overline{z}\rangle - 3(1-t)r(t)P_3(\overline{z})} < 0$$

for all $t \in (0,1)$. Hence, $0 \leq (1-t)r(t) \leq 1$ for all $t \in [0,1]$. This thus proves (5.9).

Summarising, we have shown that $3\langle dz, dz\rangle - 9sP_3(z,dz,dz) + \langle z, dz\rangle^2 > 0$ for all $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}_{>0}\cdot\mathcal{H}_s \cap \{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ for all $s \in [0,1]$, and thus have proven using Lemma 4.8 that $\mathcal{H}_s$ is a CCPSR manifold for all $s \in [0,1]$. □

An immediate consequence of Proposition 5.5 is that we can always find a continuous curve connecting two CCPSR manifolds of the same positive dimension that consists pointwise of CCPSR manifolds. However, we will prove a stronger result in the following Theorem 5.6, from which it will in particular follow how such an aforementioned curve can look like (see Corollary 5.10).

Theorem 5.6 (Convex compact generating set of CCPSR moduli space)**.** *Let $n \in \mathbb{N}$ and $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a cubic homogeneous polynomial of the form (3.12), that is $h = x^3 - x\langle y,y\rangle + P_3(y)$. Then the connected component $\mathcal{H}$ of the level set $\{h=1\} \subset \mathbb{R}^{n+1}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is a CCPSR manifold if and only if $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$.*

Proof. Firstly note that $P_3 : \mathbb{R}^n \to \mathbb{R}$ being a cubic homogeneous polynomial and, hence, an odd function implies that $\max\limits_{\|z\|=1} P_3(z) = \max\limits_{\|z\|=1} |P_3(z)|$. Assume that $\mathcal{H}$ is a CCPSR manifold. Then Lemma 4.2 shows that $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$.

Now assume that $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$. Lemma 4.2 only shows that this is a necessary requirement for $\mathcal{H}$ to be a CCPSR manifold. In order to show that it is also a sufficient condition, we have to show that

$$3\langle v,v\rangle - 9P_3(z,v,v) + \langle z,v\rangle^2 > 0 \tag{5.10}$$

for all $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in (\mathbb{R}_{>0}\cdot\mathcal{H}) \cap \{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ and all $v \in \mathbb{R}^n \setminus \{0\}$, cf. Lemma 4.8. For $z=0$, (5.10) is always true. For $z \neq 0$ and $v = rz$, $r \neq 0$, (5.10) reads $r^2(3\langle z,z\rangle - 9P_3(z) + \langle z,z\rangle^2) > 0$. Suppose that there exists a point $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in (\mathbb{R}_{>0}\cdot\mathcal{H}) \cap \{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\} \setminus \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$, such that $3\langle z,z\rangle - 9P_3(z) + \langle z,z\rangle^2 = 0$. Observe that $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$ implies

$$\begin{aligned}3\langle z,z\rangle - 9P_3(z) + \langle z,z\rangle^2 &= \|z\|^2\left(3 - 9\|z\|P_3\left(\frac{z}{\|z\|}\right) + \|z\|^2\right)\\ &\geq \|z\|^2\left(3 - 2\sqrt{3}\|z\| + \|z\|^2\right).\end{aligned}$$

The map $\|z\| \mapsto 3 - 2\sqrt{3}\|z\| + \|z\|^2$ is non-negative and its only zero is at $\|z\| = \sqrt{3}$. Hence, for $\|z\| > 0$, $\|z\|^2(3 - 2\sqrt{3}\|z\| + \|z\|^2) = 0$ if and only if $\|z\| = \sqrt{3}$. Since by assumption $\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}_{>0}\cdot\mathcal{H} \cap \{\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\} \setminus \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$, we have $h\left(\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\right) = 1 - \langle z,z\rangle + P_3(z) > 0$. But with $\|z\| = \sqrt{3}$,

$$h\left(\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\right) = 1 - \langle z,z\rangle + P_3(z) = -2 + 3\sqrt{3}P_3\left(\frac{z}{\|z\|}\right) \leq -2 + 3\sqrt{3}\cdot\frac{2}{3\sqrt{3}} = 0,$$

which is a contradiction. We conclude that whenever $z \neq 0$ and $v \neq 0$ are linearly dependent, the estimate (5.10) holds. Note that this already finishes the proof for $n = 1$.

Now assume that $\dim(\mathcal{H}) \geq 2$ and let $\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in (\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\right\} \setminus \left\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right\}$ be arbitrary. Let $v \in \mathbb{R}^n \setminus \{0\}$, such that z and v are linearly independent. In order to show (5.10), choose an orthonormal basis $\{e_1, e_2\}$ of $\operatorname{span}\{z, v\} \subset \mathbb{R}^n$ with respect to $\langle \cdot, \cdot \rangle$ and consider the cubic homogeneous polynomial $\check{h} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\check{h}_{(z,v)}\left(\left(\begin{smallmatrix} x \\ a \\ b \end{smallmatrix}\right)\right) = h\left(\left(\begin{smallmatrix} x \\ ae_1 + be_2 \end{smallmatrix}\right)\right) = x^3 - x(a^2 + b^2) + \underbrace{P_3(ae_1 + be_2)}_{=: \check{P}_3\left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right)}. \tag{5.11}$$

Let $\check{\mathcal{H}}$ be the connected component of the level set $\left\{\check{h}_{(z,v)} = 1\right\} \subset \mathbb{R}^3$ that contains the point $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^3$ and observe that

$$\begin{aligned} &\left(\mathbb{R}_{>0} \cdot \check{\mathcal{H}}\right) \cap \left\{\left(\begin{smallmatrix} 1 \\ a \\ b \end{smallmatrix}\right) \in \mathbb{R}^3 \;\middle|\; \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{R}^2\right\} \\ &\cong \left\{\left(\begin{smallmatrix} 1 \\ w \end{smallmatrix}\right) \mid w \in \operatorname{span}\{z, v\}\right\} \cap \left((\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; z \in \mathbb{R}^n\right\}\right) \end{aligned} \tag{5.12}$$

via the linear map $\left(\begin{smallmatrix} x \\ a \\ b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x \\ ae_1 + be_2 \end{smallmatrix}\right)$. Hence, if we prove that the inequality (4.5) in Lemma 4.8 holds for all cubic homogeneous polynomials $\check{h}_{(z,v)}$ of the form (5.11) with corresponding set (5.12), we will also have proven (4.5) in Lemma 4.8 for our considered h with corresponding set $(\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \left\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\right\}$ (recall that for z and v linearly dependent, (5.10) has already been shown to hold true). Furthermore note that

$$0 \leq \max_{\left\|\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right\| = 1} \check{P}_3\left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right) \leq \max_{\|z\| = 1} P_3(z) \leq \frac{2}{3\sqrt{3}}.$$

We thus see that it suffices to prove the statement of this theorem for all considered manifolds $\mathcal{H}$ with the additional restriction $\dim(\mathcal{H}) = 2$ in order to conclude that it holds true for all $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 2$. In the following, we will use the notation used in [CDL] and consider $\mathbb{R}^3$ with linear coordinates $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$,

$$h : \mathbb{R}^3 \to \mathbb{R}, \quad h = x^3 - x(y^2 + z^2) + P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right),$$

such that

$$\max_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) \leq \frac{2}{3\sqrt{3}}.$$

As before, we consider the centro-affine surface $\mathcal{H}$ which is the connected component of the level set $\{h = 1\} \subset \mathbb{R}^3$ that contains the point $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, and we want to show that $\mathcal{H}$ is a CCPSR surface (which is equivalent to the condition (4.5) in Lemma 4.8). For $P_3 \equiv 0$, the condition (4.5) in Lemma 4.8 is immediately seen to be true. For $P_3 \not\equiv 0$, Proposition 5.5 implies that it suffices to prove that $\mathcal{H}$ is a CCPSR surface if $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$, since for all non-vanishing cubic homogeneous polynomials $P_3 : \mathbb{R}^2 \to \mathbb{R}$ with $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) < \frac{2}{3\sqrt{3}}$ we can always choose a positive real number $r > 0$, such that $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1} rP_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$. Consequently assume that $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$. We can, after a possible orthogonal transformation of the $\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)$-coordinates (which does not change the form (3.12) of h), assume that $P_3|_{\left\{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1\right\}}$ attains its maximum at $\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, so that P_3 is of the form

$$P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}} y^3 + kyz^2 + \ell z^3.$$

We immediately see that $\ell \in \mathbb{R}$ needs to fulfil $|\ell| \leq \frac{2}{3\sqrt{3}}$. Furthermore, we can without loss of generality assume that $\ell \geq 0$, which can be achieved via $z \mapsto -z$ if necessary.

Now we will show that for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$, $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$ implies

$$k \in \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]. \tag{5.13}$$

It will become clear how to use this information in the step thereafter.

First assume $\ell = 0$, so that $P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}y^3 + kyz^2$. We want to determine the positive extremal values and corresponding critical points of P_3 when restricted to the set $\left\{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1\right\}$ aside from $\frac{2}{3\sqrt{3}}$, respectively $\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Suppose that there exists $k > \frac{1}{\sqrt{3}}$ or $k < -\frac{2}{\sqrt{3}}$, such that $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$. In order to find the extremal values of P_3 on $\left\{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\| = 1\right\}$ we need to solve $dP_3|_{\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)} = r\left\langle \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right), \left(\begin{smallmatrix} dy \\ dz \end{smallmatrix}\right)\right\rangle$, $r \in \mathbb{R}$, that is

$$\begin{pmatrix} \frac{2}{\sqrt{3}}y^2 + kz^2 \\ 2kyz \end{pmatrix} = \begin{pmatrix} ry \\ rz \end{pmatrix}, \quad y^2 + z^2 = 1. \tag{5.14}$$

We already know that $\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is an extremal point with $P_3 > 0$, so we assume now that $z \neq 0$. Then by (5.14) $r = 2ky$, which implies

$$z^2 = \frac{2\sqrt{3}k - 2}{\sqrt{3}k}y^2. \tag{5.15}$$

Note that

$$\frac{2\sqrt{3}k - 2}{\sqrt{3}k} > 0 \quad \forall k \in \mathbb{R} \setminus \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right], \tag{5.16}$$

so (5.15) will always have non-trivial solutions. For $k > \frac{1}{\sqrt{3}}$ or $k < -\frac{2}{\sqrt{3}}$ consider the two points

$$\eta_\pm = \sqrt{\frac{\sqrt{3}k}{3\sqrt{3}k - 2}} \begin{pmatrix} 1 \\ \pm\sqrt{\frac{2\sqrt{3}k-2}{\sqrt{3}k}} \end{pmatrix} \in \mathbb{R}^2.$$

One quickly checks that $\|\eta_\pm\| = 1$ and that $\eta_\pm$ both solve equation (5.15). We obtain

$$P_3(\eta_\pm) = \frac{2k}{3}\sqrt{\frac{\sqrt{3}k}{3\sqrt{3}k - 2}} =: \phi(k)$$

and

$$\partial_k \phi(k) = \frac{2}{3}\sqrt{\frac{\sqrt{3}k}{3\sqrt{3}k - 2}}\left(1 - \frac{1}{3\sqrt{3}k - 2}\right).$$

Furthermore,

$$\lim_{k \to \frac{1}{\sqrt{3}},\ k > \frac{1}{\sqrt{3}}} \phi(k) = \phi\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}, \tag{5.17}$$

$$\lim_{k \to -\frac{2}{\sqrt{3}},\ k < -\frac{2}{\sqrt{3}}} \phi(k) = \phi\left(-\frac{2}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}, \tag{5.18}$$

and we see that

$$\partial_k \phi(k) > 0 \quad \forall k \in \mathbb{R} \setminus \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right].$$

This shows that for $\ell=0$ there exists no $k\in\mathbb{R}\setminus\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, such that $\max\limits_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}$.

It remains to consider the case $\ell\in\left(0,\frac{2}{3\sqrt{3}}\right]$. For $P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}y^3+kyz^2+\ell z^3$ we get

$$P_3(\eta_\pm)=\phi(k)\pm\ell\left(\frac{2\sqrt{3}k-2}{3\sqrt{3}k-2}\right)^{\frac{3}{2}}$$

(note that $\|\eta_\pm\|=1$ independent of the chosen ℓ). Since

$$\frac{2\sqrt{3}k-2}{3\sqrt{3}k-2}>0\quad\forall k\in\mathbb{R}\setminus\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right],$$

it follows that

$$\partial_\ell P_3(\eta_+)=\left(\frac{2\sqrt{3}k-2}{3\sqrt{3}k-2}\right)^{\frac{3}{2}}>0, \tag{5.19}$$

$$\partial_\ell P_3(\eta_-)=-\left(\frac{2\sqrt{3}k-2}{3\sqrt{3}k-2}\right)^{\frac{3}{2}}<0 \tag{5.20}$$

for all $k\in\mathbb{R}\setminus\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. With

$$P_3(\eta_+)|_{\ell=0}>\frac{2}{3\sqrt{3}}\quad\forall k>\frac{1}{\sqrt{3}}$$

and

$$P_3(\eta_-)|_{\ell=0}<-\frac{2}{3\sqrt{3}}\quad\forall k<-\frac{2}{\sqrt{3}}$$

we can now conclude that for all $\ell>0$, i.e. in particular for all $\ell\in\left(0,\frac{2}{3\sqrt{3}}\right]$, we have $P_3(\eta_+)>\frac{2}{3\sqrt{3}}$ and $P_3(\eta_-)<-\frac{2}{3\sqrt{3}}$.

Summarising, we have shown that for all $k\in\mathbb{R}\setminus\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$ and all $\ell\in\left[0,\frac{2}{3\sqrt{3}}\right]$

$$\max_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)>\frac{2}{3\sqrt{3}},$$

which in particular implies that for all $\ell\in\left[0,\frac{2}{3\sqrt{3}}\right]$, $\max\limits_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)>\frac{2}{3\sqrt{3}}$ implies $k\in\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$ as claimed in (5.13).

Next, we will deal with the cases where

$$k\in\left\{-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\} \tag{5.21}$$

Equations (5.18) and (5.20) (for the lower limit $k=-\frac{2}{\sqrt{3}}$) imply that for $k=-\frac{2}{\sqrt{3}}$ and all $\ell\in\left(0,\frac{2}{3\sqrt{3}}\right]$

$$\max_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)>\frac{2}{3\sqrt{3}}.$$

Hence, for $k=-\frac{2}{\sqrt{3}}$, $\ell=0$ is the only allowed value for $\ell\in\left[0,\frac{2}{3\sqrt{3}}\right]$ such that

$$\max_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}.$$

The corresponding connected component $\mathcal{H}$ of $\{h = 1\}$ is linearly equivalent to the CCPSR surface a) in Theorem 2.45, cf. equation (3.15) after a sign-flip in y and, hence, in particular a CCPSR manifold. The case $k = \frac{1}{\sqrt{3}}$ is a little more complicated since then $\eta_\pm = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, for which in particular $\partial_\ell P_3(\eta_\pm)$ vanishes, see (5.19) and (5.20). Instead of $\eta_\pm$ consider for $\ell \geq 0$ the point

$$p = \frac{1}{\sqrt{27l^2+1}}\begin{pmatrix}1\\3\sqrt{3}l\end{pmatrix}, \quad \|p\| = 1.$$

One can check that $dP_3|_p \in \mathbb{R}\langle p, \cdot\rangle$ and

$$\begin{aligned}P_3(p) &= \frac{27\ell^2+2}{3\sqrt{3}\sqrt{27\ell^2+1}},\\ \partial_\ell(P_3(p)) &= \left(\frac{3\sqrt{3}\ell}{\sqrt{27\ell^2+1}}\right)^3.\end{aligned}$$

For $\ell = 0$ we have $P_3(p) = \frac{2}{3\sqrt{3}}$ and since $\partial_\ell(P_3(p)) > 0$ for all $\ell > 0$ we deduce that

$$\forall \ell > 0: \quad P_3(p) > \frac{2}{3\sqrt{3}}.$$

This proves that for $k = \frac{1}{\sqrt{3}}$, $\ell = 0$ is the only value allowed for $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$. For $k = \frac{1}{\sqrt{3}}$, $\ell = 0$, the connected component $\mathcal{H}$ of $\{h = 1\}$ is equivalent to the CCPSR surface b) in Theorem 2.45 which follows from equation (3.16). Hence, $\mathcal{H}$ is a CCPSR manifold.

Now, as stated before, we will use (5.13). Considering (4.5) in Lemma 4.8 for points in the set

$$(\mathbb{R}_{>0}\cdot\mathcal{H}) \cap \left\{\begin{pmatrix}1\\y\\0\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \mathbb{R}\right\} = \left\{\begin{pmatrix}1\\y\\0\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)\right\}$$

yields

$$\begin{aligned}&\left(3(dy^2+dz^2) - 9P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right),\cdot,\cdot\right) + (ydy+zdz)^2\right)\Big|_{\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)=\left(\begin{smallmatrix}y\\0\end{smallmatrix}\right)}\\ &= (y^2 - 2\sqrt{3}y + 3)dy^2 + (3-3ky)dz^2\\ &= \left(y-\sqrt{3}\right)^2 dy^2 + 3(1-ky)dz^2.\end{aligned}$$

With (5.13), that is $k \in \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$, and $y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ we deduce

$$\left(y-\sqrt{3}\right)^2 dy^2 + 3(1-ky)dz^2 > 0.$$

This means that the line segment $\left\{\begin{pmatrix}1\\y\\0\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)\right\} \subset \mathbb{R}^3$ consists only of hyperbolic points of h, independently of the choice of $k \in \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$. We project the line segment $\left\{\begin{pmatrix}1\\y\\0\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)\right\} \subset \mathbb{R}^3$ to $\mathcal{H}$ via point-wise multiplication with $\frac{1}{\sqrt[3]{h\left((1,y,0)^T\right)}}$. Since being a hyperbolic point of h is an open condition in $\mathbb{R}^3$, we are in the setting of Proposition 3.18 and can transform h with linear transformations of the form (3.7) along that set[9],

[9]Note that this subset of $\mathcal{H}$ is connected and contains the point $(1,0,0)^T$. Furthermore $\partial_x h = 3x^2 - y$, which is positive at all points $(1,y,0)^T$, $y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. Hence, we can in fact transform h along these points via transformations of the form (3.7).

that is along

$$\left\{\begin{pmatrix} \frac{1}{\sqrt[3]{h\left(\left(\begin{smallmatrix}1\\y\\0\end{smallmatrix}\right)\right)}} \\ \frac{y}{\sqrt[3]{h\left(\left(\begin{smallmatrix}1\\y\\0\end{smallmatrix}\right)\right)}} \\ 0\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)\right\} \subset \mathcal{H}.$$

In order not to confuse coordinates with parametrisation of said subset of $\mathcal{H}$, we replace y in the above set with the parameter $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. We start with $E = \mathbb{1}$ in (3.7) and assign for $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$

$$A(T) = \begin{pmatrix} \frac{1}{\sqrt[3]{1-T^2+\frac{2}{3\sqrt{3}}T^3}} & \frac{2T}{\sqrt{3}T+3} & 0 \\ \frac{T}{\sqrt[3]{1-T^2+\frac{2}{3\sqrt{3}}T^3}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3). \tag{5.22}$$

We obtain

$$\begin{aligned} h\left(A(T)\cdot\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)\right) &= x^3 - x\left(\frac{3\left(1-T^2+\frac{2}{3\sqrt{3}}T^3\right)^{\frac{2}{3}}}{\left(T+\sqrt{3}\right)^2}y^2 + \frac{1-kT}{\sqrt[3]{1-T^2+\frac{2}{3\sqrt{3}}T^3}}z^2\right) \\ &\quad + \frac{2\left(1-T^2+\frac{2}{3\sqrt{3}}T^3\right)}{\left(T+\sqrt{3}\right)^3}y^3 + \left(k - \frac{2T}{\sqrt{3}T+3}\right)yz^2 + \ell z^3. \end{aligned} \tag{5.23}$$

Note that $1 - T^2 + \frac{2}{3\sqrt{3}}T^3 > 0$ and $1 - kT > 0$ for all $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ and all $k \in \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$, which is in accordance with equation (3.10). We have already shown that for $k \in \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$, $\max\limits_{\left\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right\|=1} P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right) = \frac{2}{3\sqrt{3}}$ implies $\ell = 0$ and that the corresponding surfaces $\mathcal{H}$ are indeed CCPSR manifolds. We will from here on assume that $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Before bringing h in (5.23) to the standard form (3.12) we will check that we can always solve $k - \frac{2T}{\sqrt{3}T+3} = 0$ (the left hand side of which can be viewed as the "transformed k", up to scale) for $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. We obtain

$$k - \frac{2T}{\sqrt{3}T+3} = 0 \quad \Leftrightarrow \quad T = \frac{3k}{2-\sqrt{3}k} =: T(k).$$

We have to check that for all $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $T(k) \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. For the limit points $k \in \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ we have

$$T\left(-\frac{2}{\sqrt{3}}\right) = -\frac{\sqrt{3}}{2}, \quad T\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3},$$

and

$$\partial_k T(k) = \frac{6}{(2-\sqrt{3}k)^2} > 0$$

for all $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Hence,

$$\forall k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right): \quad T(k) \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$$

as required. Considering (5.23), we rescale y and z with

$$E(T)=\begin{pmatrix}\frac{T+\sqrt{3}}{\sqrt{3}\sqrt[3]{1-T^2+\frac{2}{3\sqrt{3}}T^3}} & \\ & \frac{\sqrt[6]{1-T^2+\frac{2}{3\sqrt{3}}T^3}}{\sqrt{1-kT}}\end{pmatrix}, \tag{5.24}$$

set $T=T(k)$ to obtain that h is equivalent to

$$h\left(A(T)\cdot\left(\begin{smallmatrix}1 & \\ & E(T)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2)+\frac{2}{3\sqrt{3}}y^3+\ell\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}z^3. \tag{5.25}$$

The next question one has to ask is if $k\in\left(-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and $\max\limits_{\left\|\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right\|=1}P_3\left(\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}$ (for the P_3-term in h, i.e. $P_3\left(\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}y^3+kyz^2+\ell z^3$) imply

$$\ell\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}\leq\frac{2}{3\sqrt{3}} \tag{5.26}$$

which is a necessary requirement for

$$\max_{\left\|\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right\|=1}\left(\frac{2}{3\sqrt{3}}y^3+\ell\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}z^3\right)=\frac{2}{3\sqrt{3}}$$

and thus also a necessary requirement that the transformed cubic in (5.25) needs to fulfil so that the corresponding connected component of its level set $\left\{h\left(A(T)\cdot\left(\begin{smallmatrix}1 & \\ & E\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)\right)=1\right\}$ which contains the point $\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\ 0\\ 0\end{smallmatrix}\right)$ can be a CCPSR manifold, cf. Corollary 4.3. Instead of attempting to calculate the supremum of $\ell\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}$ with conditions $k\in\left(-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ and $\max\limits_{\left\|\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right\|=1}P_3\left(\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}$ directly, we will choose another way to prove that (5.26) does, in fact, hold true.

For $k=0$, h is of the form $h=x^3-x(y^2+z^2)+\frac{2}{3\sqrt{3}}y^3+\ell z^3$. Consider for $T\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)$ arbitrary, $A(T)$ and $E(T)$ as in (5.22) and (5.24), respectively,

$$h\left(A(T)\cdot\left(\begin{smallmatrix}1 & \\ & E(T)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)\right)=x^3-x(y^2+z^2)+\frac{2}{3\sqrt{3}}y^3-\frac{2T}{3}yz^2+\ell z^3\sqrt{1-T^2+\frac{2}{3\sqrt{3}}T^3}. \tag{5.27}$$

For the following calculations, we define

$$P_{(3,\ell,T)}\left(\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right):=\frac{2}{3\sqrt{3}}y^3-\frac{2T}{3}yz^2+\ell z^3\sqrt{1-T^2+\frac{2}{3\sqrt{3}}T^3}. \tag{5.28}$$

We will show that

$$\forall\ell>\frac{2}{3\sqrt{3}}\ \forall T\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right):\quad\max_{\left\|\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right\|=1}P_{(3,\ell,T)}\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)>\frac{2}{3\sqrt{3}} \tag{5.29}$$

holds true. To do so we will for $T\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)$ and $\ell=\frac{2}{3\sqrt{3}}$ study a critical point of

$$P_{\left(3,\frac{2}{3\sqrt{3}},T\right)}\left(\left(\begin{smallmatrix}y\\ z\end{smallmatrix}\right)\right)=\frac{2}{3\sqrt{3}}y^3-\frac{2T}{3}yz^2+\frac{2}{3\sqrt{3}}z^3\sqrt{1-T^2+\frac{2}{3\sqrt{3}}T^3}$$

on the set $\{\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\| = 1\}$, namely the point

$$\begin{pmatrix} y \\ z \end{pmatrix} = \frac{1}{T+\sqrt{3}} \begin{pmatrix} -T \\ \sqrt{2\sqrt{3}T+3} \end{pmatrix} =: \zeta. \tag{5.30}$$

Note that ζ is well-defined for all $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$, and it is indeed a critical point of $P_{\left(3,\frac{2}{3\sqrt{3}},T\right)}\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right)$. Using the factorisation $1 - T^2 + \frac{2}{3\sqrt{3}}T^3 = \frac{2}{3\sqrt{3}}\left(T-\sqrt{3}\right)^2\left(T+\frac{\sqrt{3}}{2}\right)$ and $T - \sqrt{3} < 0$ for all $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$, we find

$$\begin{aligned} dP_{\left(3,\frac{2}{3\sqrt{3}},T\right)}\Big|_{\zeta} &= \left(\left(\frac{2}{\sqrt{3}}y^2 - \frac{2T}{3}z^2\right)dy + \left(-\frac{4T}{3}yz + \frac{2}{\sqrt{3}}z^2\sqrt{1-T^2+\frac{2}{3\sqrt{3}}T^3}\right)dz\right)\Big|_{\zeta} \\ &= \frac{2}{\sqrt{3}}\left\langle \zeta, \left(\begin{smallmatrix} dy \\ dz \end{smallmatrix}\right)\right\rangle. \end{aligned}$$

The corresponding critical value is given by $P_{\left(3,\frac{2}{3\sqrt{3}},T\right)}(\zeta) = \frac{2}{3\sqrt{3}}$, independent of $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. Note that $dz(\zeta) > 0$ for all $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ and consider the derivative

$$\partial_\ell \left(P_{(3,\ell,T)}\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = z^3\sqrt{1-T^2+\frac{2}{3\sqrt{3}}}.$$

Hence, $\partial_\ell \left(P_{(3,\ell,T)}\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) > 0$ for all $T \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ and all $z > 0$, in particular for $z = dz(\zeta)$. We conclude that (5.29) holds true.

We can now use (5.29) to show that (5.26) holds true for all $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. For $\ell = 0$ equation (5.26) is automatically true independently of the chosen $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Suppose that there exist $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\ell \in \left(0, \frac{2}{3\sqrt{3}}\right]$, with corresponding polynomial $h = x^3 - x(y^2+z^2) + \frac{2}{3\sqrt{3}}y^3 + kyz^2 + \ell z^3$, fulfilling

$$\max_{\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\|=1} P_3\left(\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right) = \max_{\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\|=1} \left(\frac{2}{3\sqrt{3}}y^3 + kyz^2 + \ell z^3\right) = \frac{2}{3\sqrt{3}}, \tag{5.31}$$

such that for $T = T(k) = \frac{3k}{2-\sqrt{3}k}$

$$\ell\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}} > \frac{2}{3\sqrt{3}},$$

which precisely means that (5.26) does not hold true for the chosen k, ℓ. Combining (5.25) and (5.27), one obtains that h is equivalent to

$$\widetilde{h} = x^3 - x(y^2+z^2) + \frac{2}{3\sqrt{3}}y^3 - \frac{2\widetilde{T}}{3}yz^2 + \ell z^3\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}\sqrt{1-\widetilde{T}^2+\frac{2}{3\sqrt{3}}\widetilde{T}^3}$$

for all $\widetilde{T} \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. Furthermore, (5.29) implies that for all $\widetilde{T} \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$:

$$\max_{\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\|=1} \left(\frac{2}{3\sqrt{3}}y^3 - \frac{2\widetilde{T}}{3}yz^2 + \ell z^3\frac{\sqrt{1-T(k)^2+\frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}\sqrt{1-\widetilde{T}^2+\frac{2}{3\sqrt{3}}\widetilde{T}^3}\right) > \frac{2}{3\sqrt{3}}. \tag{5.32}$$

The above estimate (5.32) must thus in particular hold for $\widetilde{T} = -\frac{3k}{2} =: \widetilde{T}(k)$ (note that $\widetilde{T}(k) \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ for all $k \in \left(-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$). But

$$\begin{aligned}
&\frac{2}{3\sqrt{3}}y^3 - \frac{2\widetilde{T}(k)}{3}yz^2 + \ell z^3 \frac{\sqrt{1 - T(k)^2 + \frac{2}{3\sqrt{3}}T(k)^3}}{(1-kT(k))^{\frac{3}{2}}}\sqrt{1 - \widetilde{T}(k)^2 + \frac{2}{3\sqrt{3}}\widetilde{T}(k)^3} \\
&= \frac{2}{3\sqrt{3}}y^3 + kyz^2 + \ell z^3,
\end{aligned}$$

which implies that (5.32) for $\widetilde{T} = \widetilde{T}(k)$ is a contradiction to the assumption (5.31). We conclude that (5.26) holds true.

In order to complete the proof of this theorem it thus suffices to show that for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$ and corresponding polynomial $h_\ell := x^3 - x(y^2+z^2) + \frac{2}{3\sqrt{3}}y^3 + \ell z^3$, the connected component $\mathcal{H}_\ell \subset \{h_\ell = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ is a CCPSR manifold. Define the P_3-part of h_ℓ as $P_{(3,\ell)} := \frac{2}{3\sqrt{3}}y^3 + \ell z^3$. One can easily check that

$$\frac{2}{3\sqrt{3}} \leq \max_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} P_{(3,\ell)} \leq \max_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} \frac{2}{3\sqrt{3}}\left(|y|^3 + |z|^3\right) = \max\left\{\frac{2}{3\sqrt{3}}, \frac{\sqrt{2}}{3\sqrt{3}}\right\} = \frac{2}{3\sqrt{3}},$$

which shows that $\max\limits_{\left\|\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)\right\|=1} P_{(3,\ell)} = \frac{2}{3\sqrt{3}}$ as required. We use the linear transformation

$$B = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3)$$

and transform h_ℓ to

$$\breve{h}_\ell\left(\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) := h_\ell\left(B \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) = x(y^2 - z^2) + \frac{2}{3\sqrt{3}}y^3 + \ell z^3.$$

In the new coordinates, $\breve{\mathcal{H}}_\ell := B^{-1}(\mathcal{H}_\ell) \subset \left\{\breve{h} = 1\right\}$ is given by

$$\breve{\mathcal{H}} = \left\{\left.\begin{pmatrix} \frac{1 - \frac{2}{3\sqrt{3}}y^3 - \ell z^3}{y^2 - z^2} \\ y \\ z \end{pmatrix} \;\right|\; y < 0,\ y^2 > z^2\right\}.$$

This follows easily by $B \cdot \left(\begin{smallmatrix} 1 \\ -\sqrt{3} \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}_\ell$ and that $x \to \infty$ for all sequences in $\overline{\{y < 0,\ y^2 > z^2\}} = \{y \leq 0,\ y^2 \geq z^2\}$ that converge to a point in $\partial\{y < 0,\ y^2 > z^2\} = \{y \geq 0,\ y^2 = z^2\}$. The latter follows from $\frac{2}{3\sqrt{3}} + \ell \leq \frac{4}{3\sqrt{3}} < 1$ for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$. We know that $\left(\begin{smallmatrix} 1 \\ -\sqrt{3} \\ 0 \end{smallmatrix}\right) = B^{-1} \cdot \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \in \breve{\mathcal{H}}_\ell$ is always a hyperbolic point of $\breve{h}_\ell$ for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$. Hence, in order to show that $\breve{\mathcal{H}}_\ell$ consists only of hyperbolic points of $\breve{h}_\ell$, it suffices to show that

$$\begin{aligned}
\det\left(-\frac{1}{2}\partial^2\breve{h}_\ell\right) &= \det\begin{pmatrix} 0 & -y & z \\ -y & -x - \frac{2}{\sqrt{3}}y & 0 \\ z & 0 & x - 3\ell z \end{pmatrix} \\
&= \frac{1}{y^2 - z^2}\left(\underbrace{\frac{2}{3\sqrt{3}}y^5 - \ell z^5 + 3\ell y^4 z - \frac{2}{\sqrt{3}}yz^4 + \frac{4}{3\sqrt{3}}y^3z^2 - 2\ell y^2 z^3}_{=:R_\ell(y,z)} - y^2 + z^2\right) < 0
\end{aligned}$$

for all $\binom{y}{z} \in \{y < 0,\ y^2 > z^2\}$. The prefactor $\frac{1}{y^2-z^2}$ is always positive if $y^2 > z^2$, and the term $-y^2 + z^2$ is always negative. Hence, it suffices to show $R_\ell(y,z) \leq 0$ for all $\binom{y}{z} \in \{y < 0,\ y^2 > z^2\}$. We calculate

$$R_\ell(y,\pm y) = y^5 \left(\frac{2}{3\sqrt{3}} \mp \ell \pm 3\ell - \frac{2}{\sqrt{3}} + \frac{4}{3\sqrt{3}} \mp 2\ell \right) = 0,$$

which implies that $R_\ell(y,z)$ vanishes on $\partial\{y < 0,\ y^2 > z^2\}$. Since the set $\{y < 0,\ y^2 > z^2\}$ is a cone and $R_\ell(y,z)$ is for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$ a homogeneous polynomial of degree 5, it only remains to check that

$$\forall s \in (-1,1)\ \forall \ell \in \left[0, \frac{2}{3\sqrt{3}}\right]: \quad R_\ell(-1,s) \leq 0. \tag{5.33}$$

We find that $s = 1$ and $s = -1$ are roots of $R_\ell(-1,s)$ for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$, which allows us to consider

$$N_\ell(s) := \frac{R_\ell(-1,s)}{(s-1)(s+1)} = \frac{R_\ell(-1,s)}{s^2-1} = -\ell s^3 + \frac{2}{\sqrt{3}} s^2 - 3\ell s + \frac{2}{3\sqrt{3}}.$$

The condition (5.33) is equivalent to

$$\forall s \in (-1,1)\ \forall \ell \in \left[0, \frac{2}{3\sqrt{3}}\right]: \quad N_\ell(s) \geq 0. \tag{5.34}$$

This motivates checking solutions of $N_\ell(s) = 0$. We get

$$N_\ell(s) = 0 \quad \Leftrightarrow \quad \ell = \frac{2}{3\sqrt{3}} \cdot \frac{3s^2+1}{s(s^2+3)}.$$

We will show that $M(s) := \frac{3s^2+1}{s(s^2+3)} \notin [0,1]$ for all $s \in (-1,1)$, which implies that there exists no pair $(\ell, s) \in \left[0, \frac{2}{3\sqrt{3}}\right] \times (-1,1)$, such that $N_\ell(s) = 0$. Since $N_0(1) = \frac{2}{\sqrt{3}} > 0$, this will then shows that $N_\ell(s) > 0$ for all $(\ell, s) \in \left[0, \frac{2}{3\sqrt{3}}\right] \times (-1,1)$ and in particular imply (5.34). We see that

$$\operatorname{sgn}(M(s)) = \begin{cases} 1, & \forall s > 0, \\ -1, & \forall s < 0, \end{cases}$$

which implies that we can reduce our studies to $s \in [0,1)$. The first derivative of $M(s)$ is easily seen to fulfil

$$\partial_s M(s) = \frac{-3(s^4 - 2s^2 + 1)}{s^2(s^2+3)^2} < 0 \tag{5.35}$$

for all $s \in (0,1)$. Furthermore

$$\lim_{s \to 0,\ s > 0} M(s) = \infty. \tag{5.36}$$

The estimate (5.35) and the limit (5.36) imply

$$\forall s \in (0,1): \quad M(s) > M(1) = 1.$$

Hence, the equation $M(s) = 1$ has no solutions in the half-open interval $[0,1)$. We conclude that (5.34) holds true.

Summarising, we have proven that for all $\ell \in \left[0, \frac{2}{3\sqrt{3}}\right]$, $\breve{\mathcal{H}}_\ell$ is a CCPSR manifold of dimension 2, which implies the same statement for $\mathcal{H}_\ell$. This finishes the proof of Theorem 5.6. $\square$

Remark 5.7 (Direct application of Theorem 5.6)**.** Theorem 5.6 might be a little surprising, since now we have a relatively easy way of checking if a connected PSR manifolds $\mathcal{H} \subset \{h = 1\}$ is closed. We have to transform h to the form $h = x^3 - x\langle y, y\rangle + P_3(y)$ (3.12) as described in Proposition 3.18 (this involves possibly the task of diagonalising a positive definite quadratic form), calculate $\max\limits_{\|z\|=1} P_3(z)$ (which should always work with a computer algebra system like MAPLE since P_3 is a cubic polynomial, i.e. the related equations are n quadratic equations for $\dim(\mathcal{H}) = n$), and whenever $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$ we know that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed and in particular complete (cf. [CNS, Thm. 2.5] or Proposition 4.17). On the other hand, the connected component $\mathcal{H}$ of the level set $\{h = 1\}$ for any h of the form (3.12) which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is automatically a CCPSR manifold if $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$.

Together with Theorem 5.3 we have obtained the following characterisation of the moduli space of CCPSR manifolds of dimension $n \geq 1$ under the action of $\mathrm{GL}(n+1)$, cf. Definition 3.2.

Proposition 5.8 (Characterisation of the moduli space of CCPSR manifolds)**.** *For all $n \in \mathbb{N}$, the set of hyperbolic homogeneous cubic polynomials*

$$\mathcal{C}_n := \left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; \max_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}} \right\} \tag{5.37}$$

is a generating set for the moduli space of n-dimensional CCPSR manifolds under the action of $\mathrm{GL}(n+1)$*, i.e. for every CCPSR manifold $\mathcal{H}$ of dimension n there exists an element $\widetilde{h} \in \mathcal{C}_n$, such that the connected component $\widetilde{\mathcal{H}} \subset \{\widetilde{h} = 1\}$ which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \left\{\widetilde{h} = 1\right\} \subset \mathbb{R}^{n+1}$ is equivalent to $\mathcal{H}$. The set $\mathcal{C}_n \subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ is a uniformly bounded compact convex subset of the affine $\frac{n^3+3n^2+2n}{6}$-dimensional affine subspace*

$$\left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; P_3 \in \mathrm{Sym}^3\left(\mathbb{R}^n\right)^* \right\} \subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*.$$

The boundary of $\mathcal{C}_n$, that is $\partial\mathcal{C}_n$, is a continuous submanifold of $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$*. Furthermore, $\widetilde{h} \in \partial\mathcal{C}_n$ if and only if the initial $\mathcal{H}$ does not have regular boundary behaviour.*

Proof. The existence of $\widetilde{h} \in \mathcal{C}_n$ follows from Proposition 3.18 and Theorem 5.6. $\widetilde{h} \in \partial\mathcal{C}_n = \left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; \max\limits_{\|z\|=1} P_3(z) = \frac{2}{3\sqrt{3}} \right\}$ if and only if the initial $\mathcal{H}$ does not have regular boundary behaviour follows from Lemma 4.6 and Theorem 5.3. It remains to show that $\mathcal{C}_n \subset \left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; P_3 \in \mathrm{Sym}^3\left(\mathbb{R}^n\right)^* \right\} \subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ is compact and that $\partial\mathcal{C}_n \subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ is a continuous submanifold. For compactness we need to show that the condition $\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}$ automatically implies that $P_3(\cdot,\cdot,\cdot)$ viewed as a symmetric 3-tensor is bounded entry-wise, and we need to show that $\mathcal{C}_n$ is closed in the subspace topology[10]. This is equivalent to showing that all third derivatives of $P_3(z)$ are bounded on $\{\|z\| = 1\}$. This follows from the fact that for all P_3 fulfilling $\max\limits_{\|z\|=1} P_3(z) = \frac{2}{3\sqrt{3}}$, the corresponding $h = x^3 - x\langle y, y\rangle + P_3(y) \in \mathcal{C}_n$ defines a CCPSR manifold and, hence, we can use Lemma 4.10 and conclude that each entry in $P_3(\cdot,\cdot,\cdot)$ is indeed bounded. $\mathcal{C}_n$ being closed follows from the continuity of $\max\limits_{\|z\|=1} P_3(z)$ with respect to the prefactors of the monomials in

[10] With respect to the topology induced by the linear homeomorphy of $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ and $\mathbb{R}^{\frac{n^3+6n^2+11n+6}{6}}$. Note that said topology on $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ does not depend on the choice of the linear homeomorphism.

$P_3(y)$, or equivalently the prefactors in the corresponding symmetric 3-tensor $P_3(\cdot,\cdot,\cdot)$. We conclude that $\mathcal{C}_n\subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ is compact in the subspace topology. The fact that $\partial\mathcal{C}$ is a continuous hypersurface in $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ also follows from the continuity of the map $P_3\mapsto \max\limits_{\|z\|=1} P_3(z)$. However, note that this map is for $n\geq 2$ in general not smooth, or even differentiable. To see this, consider the one-parameter family $P_3^t(y)=y_1^3+ty_2^3$, $t\in\left[0,\frac{2}{3\sqrt{3}}\right]$, in $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ and observe that

$$t\mapsto \max_{\|z\|=1} P_3^t(z)=\max_{\|z\|=1}\left(z_1^3+tz_2^3\right)=\begin{cases}1, & 0\leq t\leq 1,\\ t, & 1\leq t\leq \frac{2}{3\sqrt{3}}\end{cases}$$

does depend only continuously on t and is not continuously differentiable at $t=1$. □

Remark 5.9 (Comparison of $\mathcal{C}_n$ with other bounded generating sets obtained via rescaling of the h's)**.** Note that for any compact set $C\subset \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$ of dimension $\dim(C)=\dim\left(\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*\right)$, such that C contains $0\in \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$, and any given CCPSR manifold $\mathcal{H}\subset\{h=1\}$, we can always choose $r>0$, such that $rh\in C$. Then $\mathcal{H}$ is equivalent to $r^{-\frac{1}{3}}\cdot\mathcal{H}\subset\{rh=1\}$. This shows that one can choose a generating set for the moduli space of n-dimensional CCPSR manifolds that is contained in a compact set C and, hence, bounded. It was however until now for $n\geq 2$ not known whether one can choose a compact generating set like $\mathcal{C}_n$ in Proposition 5.8. For $n=1$ it was already shown in [CHM, Cor. 4] that the moduli space of CCPSR curves is generated by the set $\{x^2y, x(x^2-y^2)\}\subset \mathrm{Sym}^3\left(\mathbb{R}^2\right)^*$, which is a compact set. One can show that x^2y is equivalent to $x^3-xy^2+\frac{2}{3\sqrt{3}}y^3$. By comparing with $\mathcal{C}_1=\left\{x^3-xy^2+Ly^3 \;\middle|\; |L|\leq\frac{2}{3\sqrt{3}}\right\}$, we see that $x(x^2-y^2)=x^3-xy^2$ is an inner point of $\mathcal{C}_1$ and $x^3-xy^2+\frac{2}{3\sqrt{3}}y^3$ is one of the two points in $\partial\mathcal{C}_1$.

Proposition 5.8 allows us to answer the initial question at the beginning of this section. The polynomial $h_\varepsilon=x^3-x\langle y,y\rangle+P_3(y)+\varepsilon V(y)$ as in 5.1 defines a CCPSR manifold $\mathcal{H}_\varepsilon\subset\{h_\varepsilon=1\}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}_\varepsilon$, if and only if $\max\limits_{\|z\|=1}\left(P_3(z)+\varepsilon V(z)\right)\leq\frac{2}{3\sqrt{3}}$. Additionally to the answer to that question, Proposition 5.8 yields a geometric result for the moduli space of n-dimensional CCPSR manifolds (cf. Definition 3.2).

Corollary 5.10 (Path-connectedness and convexity of moduli space of CCPSR manifolds)**.** *For $n\in\mathbb{N}$ fixed, let $h,\widetilde{h}\in\mathcal{C}_n$ and let $\mathcal{H}\subset\{h=x^3-x\langle y,y\rangle+P_3(y)=1\}$, respectively $\widetilde{\mathcal{H}}\subset\{\widetilde{h}=x^3-x\langle y,y\rangle+\widetilde{P}_3(y)=1\}$, denote the corresponding CCPSR manifolds containing the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Then the smooth curve*

$$\gamma:[0,1]\to\mathcal{C}_n\subset\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*,\quad \gamma(t)=(1-t)h+t\widetilde{h},$$

defines an n-dimensional CCPSR manifold $\mathcal{H}_t\subset\left\{\gamma(t)=(1-t)h+t\widetilde{h}=1\right\}$ as the connected component containing $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for all $t\in[0,1]$. Furthermore, $\mathcal{H}_0=\mathcal{H}$ and $\mathcal{H}_1=\widetilde{\mathcal{H}}$.

Proof. For all $t\in[0,1]$, $\gamma(t)=x^3-x\langle y,y\rangle+(1-t)P_3(y)+t\widetilde{P}_3(y)$. Theorem 5.6 implies that it suffice to show $\max\limits_{\|z\|=1}\left((1-t)P_3(y)+t\widetilde{P}_3(y)\right)\leq\frac{2}{3\sqrt{3}}$ for all $t\in[0,1]$. We get

$$\begin{aligned}\max_{\|z\|=1}\left((1-t)P_3(y)+t\widetilde{P}_3(y)\right)&\leq(1-t)\left(\max_{\|z\|=1}P_3(y)\right)+t\left(\max_{\|z\|=1}\widetilde{P}_3(y)\right)\\&\leq(1-t)\frac{2}{3\sqrt{3}}+t\frac{2}{3\sqrt{3}}=\frac{2}{3\sqrt{3}}\end{aligned}$$

as required. □

Remark 5.11. Two distinct points in $\gamma([0,1])$ need not be equivalent in general, and they are also not inequivalent in general. Recall that in general we have seen in Lemma 4.1 that the representative of a CCPSR manifold $\mathcal{H}$, $\dim(\mathcal{H})=n$, in $\mathcal{C}_n$ is never unique. For example for a CCPSR manifold $\mathcal{H}\subset\{h=x^3-x\langle y,y\rangle+P_3(y)=1\}$ with $\max\limits_{\|z\|=1}P_3(z)=\frac{2}{3\sqrt{3}}$, which is never equivalent to $\widetilde{\mathcal{H}}\subset\{\widetilde{h}=x^3-x\langle y,y\rangle=1\}$ (since $\mathcal{H}$ is singular at infinity and $\widetilde{\mathcal{H}}$ is a CCPSR manifold that is not singular at infinity), we define $\gamma(t)=x^3-x\langle y,y\rangle+(1-2t)P_3(y)$. Then $\mathcal{H}=\mathcal{H}_0\subset\gamma(0)$ and $\mathcal{H}_1\gamma(1)$ are equivalent (via sign-flip in the y-coordinates), but $\mathcal{H}_{\frac{1}{2}}=\widetilde{\mathcal{H}}$.

Next we will present an application of the generating set $\mathcal{C}_n$ for the scalar curvature of CCPSR surfaces. Recall that we already have an estimate of the scalar curvature $S_{\mathcal{H}}$ of n-dimensional CCPSR manifolds that does not depend on the considered CCPSR manifold, see Theorem 4.13. We will now show that (4.15) is indeed not a sharp (Remark 4.11) estimate for $\dim(\mathcal{H})=n=2$, that is for CCPSR surfaces, and give a sharp estimate in that dimension.

Proposition 5.12 (Sharp $S_{\mathcal{H}}$-bounds for CCPSR surfaces)**.** *The scalar curvature $S_{\mathcal{H}}$ of a CCPSR surface $\mathcal{H}$ fulfils the global estimate*

$$-\frac{9}{4}\leq S_{\mathcal{H}}\leq 0. \tag{5.38}$$

More specifically,

$$\begin{aligned}
S_{\mathcal{H}}&\equiv 0\quad \text{if}\quad \mathcal{H}\cong\{xyz=1,\ x>0,\ y>0\},\ \text{i.e. Thm. 2.45 a)},\\
S_{\mathcal{H}}&\equiv -\frac{9}{4}\quad \text{if}\quad \mathcal{H}\cong\{x(xy-z^2)=1,\ x>0\},\ \text{i.e. Thm. 2.45 b)},\\
-\frac{9}{4}<S_{\mathcal{H}}&<0\quad \text{if}\quad \mathcal{H}\ncong\{xyz=1,\ x>0,\ y>0\}\ \text{and}\ \mathcal{H}\ncong\{x(xy-z^2)=1,\ x>0\}.
\end{aligned}$$

Proof. We can for $\mathcal{H}\subset\{h=1\}$, $\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\in\mathcal{H}$, $h=x^3-x(y^2+z^2)+P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)$, $\max\limits_{\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\|=1}P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=r\frac{2}{3\sqrt{3}}$, $r\in[0,1]$, which covers all possible CCPSR surfaces (cf. Theorem 5.6), assume after a possible orthogonal transformation in the (y,z)-coordinates assume that

$$P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=r\left(\frac{2}{3\sqrt{3}}y^3+kyz^2+\ell z^3\right). \tag{5.39}$$

Note that Proposition 3.18 ensures that we can for any $p\in\mathcal{H}$ always choose a linear transformation of the form (3.7) which maps $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\in\mathcal{H}\subset\{h=1\}\subset\mathbb{R}^3$ to p, so that we only need to consider all cubics $P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)$ of the form (5.39) to prove the claim of this proposition. We use Proposition 3.29, equation (3.34), to obtain

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=-2+r\left(\frac{3}{4}k^2-\frac{\sqrt{3}}{2}k\right). \tag{5.40}$$

Observe that $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)$ does not depend on $\ell\in\mathbb{R}$. Hence, we only need to be concerned with the domain for $k\in\mathbb{R}$. For $r=0$, $\mathcal{H}$ is equivalent to Thm. 2.45 d) (see the proof of Theorem 5.3, part **d)**). Hence, $\mathcal{H}$ is in particular not equivalent to Thm. 2.45 a) or Thm. 2.45 b). We obtain

$$S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=-2\in\left(-\frac{9}{4},0\right). \tag{5.41}$$

With that in mind we will now assume that $r\in(0,1]$. Then

$$\max_{\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\|=1}P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=r\frac{2}{3\sqrt{3}}\quad\Leftrightarrow\quad\max_{\|\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\|=1}\left(\frac{2}{3\sqrt{3}}y^3+kyz^2+\ell z^3\right)=\frac{2}{3\sqrt{3}}. \tag{5.42}$$

We have shown in the proof of Theorem 5.6 in the part where we have proven (5.13) that k needs to be an element of $\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. The condition for (5.13) is precisely (5.42). We deduce that for all $r\in(0,1]$, the allowed domain for k is also $\left[-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. Consider the function

$$\Theta(k):=\frac{3}{4}k^2-\frac{\sqrt{3}}{2},$$

so that $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=-2+r\Theta(k)$. One can easily verify that

$$\min_{k\in\left[-2/\sqrt{3},1/\sqrt{3}\right]}\Theta(k)=\Theta\left(\frac{1}{\sqrt{3}}\right)=-\frac{1}{4},$$
$$\max_{k\in\left[-2/\sqrt{3},1/\sqrt{3}\right]}\Theta(k)=\Theta\left(-\frac{2}{\sqrt{3}}\right)=2.$$

Hence, with (5.41) we have shown that for all $r\in[0,1]$

$$\underbrace{-2-\frac{1}{4}r}_{k=1/\sqrt{3}}\leq S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\leq\underbrace{-2+2r}_{k=-2/\sqrt{3}}.$$

For $r=1$, $k\in\left\{-\frac{2}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\}$ implies with (5.42) that $\ell=0$, see also the proof of Theorem 5.6, (5.21) and the following discussion. There we have also seen that for $r=1$ and $\ell=0$, $\mathcal{H}$ is equivalent to

$$\text{Thm. 2.45 a) if } k=-\frac{2}{\sqrt{3}},$$
$$\text{Thm. 2.45 b) if } k=\frac{1}{\sqrt{3}}.$$

For Thm. 2.45 a) respectively $r=1$, $k=-\frac{2}{\sqrt{3}}$, and $\ell=0$, $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=0$. Furthermore in that case $\mathcal{H}$ is a homogeneous space. This can be seen by showing that

$$xyz\cong x(y^2-z^2)=:\widetilde{h},$$

hence the (flat) Lie group $\mathrm{SO}(1,1)\times\mathbb{R}_{>0}$ acts transitively and isometrically on the corresponding CCPSR surface, where the $\mathbb{R}_{>0}$-part acts via

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}s^{-2}x\\sy\\sz\end{pmatrix},\quad s\in\mathbb{R}_{>0},$$

and $\mathrm{SO}(1,1)$ acts on the quadratic form y^2-z^2. This shows that for $\mathcal{H}$ with $r=1$, $k=-\frac{2}{\sqrt{3}}$, and $\ell=0$, $S_{\mathcal{H}}\equiv 0$.

For Thm. 2.45 b), for which the corresponding CCPSR surface is equivalent to the case $r=1$, $k=\frac{1}{\sqrt{3}}$, and $\ell=0$, $\mathcal{H}$ is also a homogeneous space. This is a bit more difficult to find than for Thm. 2.45 a), but one can show that the Lie group corresponding to the Lie algebra

$$\mathfrak{g}:=\operatorname{span}\left\{A_1=\begin{pmatrix}1&0&0\\0&-2&0\\0&0&-\frac{1}{2}\end{pmatrix},\ A_2=\begin{pmatrix}0&0&0\\0&0&2\\1&0&0\end{pmatrix}\right\},\quad [A_1,A_2]=A_1A_2-A_2A_1=-\frac{3}{2}A_2,$$

acts transitively and isometrically on the corresponding CCPSR surface. Note that in the untransformed coordinates in Thm. 2.45 a), one can show that $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ acts simply transitively on $\mathcal{H}$ via unimodular diagonal matrices, cf. [CHM, Ex. 1]. Hence, for the case $r = 1$, $k = \frac{1}{\sqrt{3}}$, and $\ell = 0$, we have shown that $S_{\mathcal{H}} \equiv -\frac{9}{4}$.

It remains to show that $-\frac{9}{4} < S_{\mathcal{H}} < 0$ if $\mathcal{H}$ is not equivalent to either Thm. 2.45 a) or Thm. 2.45 b). Observe that for $r \in [0,1)$, $-\frac{9}{4} < S_{\mathcal{H}} < 0$ follows from $k \in \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$, (5.40), and (5.41). For $r = 1$, the existence of a point $p \in \mathcal{H}$, such that the corresponding cubic polynomial h can be brought to the form (3.12) with $k \in \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ via a linear transformation of the form (3.7) already implies that $\mathcal{H}$ is equivalent to either Thm. 2.45 a) or Thm. 2.45 b). Hence, $\mathcal{H}$ not being equivalent to either Thm. 2.45 a) or Thm. 2.45 b) implies that with respect to each $p \in \mathcal{H}$, the form (3.12) with P_3 as in (5.39) implies that $r \in [0,1)$ or $k \notin \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ and, hence, that $-\frac{9}{4} < S_{\mathcal{H}} < 0$. This finishes the proof. □

The newly acquired estimate (5.38) for the scalar curvature $S_{\mathcal{H}}$ of CCPSR surfaces is thus indeed sharp. The general estimate for $S_{\mathcal{H}}$ derived in Theorem 4.13, that is (4.15), reads for CCPSR surfaces

$$-\frac{33}{4} \leq S_{\mathcal{H}} \leq \frac{17}{4},$$

which is as we have seen not sharp, neither from above nor from below. One might be able to find a better estimates for CCPSR manifolds $\mathcal{H}$ of dimension $\dim(\mathcal{H}) \geq 3$, we leave this as an open problem for future studies.

Proposition 5.12 also shows that the sectional curvature of CCPSR surfaces, which is just a smooth function $K_{\mathcal{H}} = K_{\mathcal{H}}(\partial_y, \partial_z) \in C^\infty(\mathcal{H})$ for CCPSR surfaces, can also be sharply bounded since $K_{\mathcal{H}} = \frac{1}{2} S_{\mathcal{H}}$, cf. equation (4.18).

One application of Proposition 5.12 lies in the theory of Kähler cones. Since this is not the focus of this thesis, we refer the reader to [We] and more specifically [Ma]. For the following remark, see also [CHM, p.8, Ex.]

Remark 5.13 (Relation to geometry of Kähler cones)**.** Let X be a compact Kähler manifold of complex dimension τ. Then on the Kähler cone $\mathcal{K}$ in the $(1,1)$-cohomology $H^{1,1}(X,\mathbb{R})$ of X, one can define a homogeneous polynomial $h : \mathcal{K} \to \mathbb{R}$ of homogeneity-degree τ by

$$h(\omega) = \omega^{\cup n} = \underbrace{\omega \cup \ldots \cup \omega}_{\tau \text{ times}} = \int_X \omega \wedge \ldots \wedge \omega, \tag{5.43}$$

where $\cup$ denotes the cup product. Then

$$\mathcal{H} := \{\omega \in \mathcal{K} \mid h(\omega) = 1\} \subset H^{1,1}(X,\mathbb{R}) \tag{5.44}$$

equipped with $g_{\mathcal{H}} = -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ is a generalised projective special real manifold of dimension $\dim\left(H^{1,1}(X,\mathbb{R})\right) - 1$.

One more specific field in this area is the study of Kähler cones of Calabi-Yau manifolds and their geometry, specifically in complex dimension 3 [Wi1, Wi2, KW]. For complex manifolds of complex dimension 4, examples have been studied in [T]. One specific conjecture in this field is the following.

Conjecture 5.14 (P.M.H. Wilson, [Wi2])**.** *Let X be a complex 3-dimensional Calabi-Yau manifold and let $\mathcal{H}$ be as in (5.44). Then the sectional curvatures of $\left(\mathcal{H}, \frac{1}{2} g_{\mathcal{H}}\right)$ are bounded by $-\frac{9}{4}$ from below and 0 from above.*

Note slightly different conventional factor in $g_{\mathcal{H}}$ in comparison with our conventions. While we have not found a complete answer to the above conjecture, we have found a partial answer for the case where $\mathcal{H}$ is complete and $\dim\left(H^{1,1}(X,\mathbb{R})\right)=3$, i.e. when $\mathcal{H}$ is a closed PSR surface in $\mathbb{R}^3$.

Corollary 5.15. *Conjecture 5.14 holds true if $\mathcal{H}$ is contained in a complete PSR surface.*

Proof. This follows immediately from Proposition 5.12, where we note that there exist only one sectional curvature in the case of surfaces, and keep in mind the conventional factor of $\frac{1}{2}$ for the metric used in [Wi2]. □

At this point, to my knowledge, it is however not clear which complete PSR manifolds can be realised as in (5.44). Cubics of the form (5.43) are subject of active study [KW].

Lastly in this section we will give a second alternative completeness proof for closed PSR manifolds, additionally to Proposition 4.17. While the proof is similar in comparison with the proof of the latter proposition, it makes use of Theorem 5.6, which is a stronger statement than Lemma 4.10. However, the second alternative proof of CCPSR completeness requires much less calculating. We will make use of the following lemma.

Lemma 5.16 (Family of compactly embedded geodesic balls)**.** *Let M be manifold of dimension $n\geq 1$ with a locally finite atlas, $C\subset\mathbb{R}^N$ be a compact subset for some $N\in\mathbb{N}$, and $g(\cdot):C\to\Gamma\left(\mathrm{Sym}^2\left(T^*M\right)\right)$, $c\mapsto g(c)$, be a family of Riemannian metrics depending continuously on $c\in C$ in the sense that the map*

$$g:C\times M\to\mathrm{Sym}^2(T^*M),\quad (c,q)\mapsto g(c)_q,$$

is continuous. Let $p\in M$ be arbitrary and fixed. We denote by $B_r^{g(c)}(p)\subset M$ the geodesic ball of radius $r>0$ around $p\in M$ with respect to the Levi-Civita connection of $g(c)$. Then the following is true:

$$\inf_{c\in C}\left(\sup_{\overline{B_r^{g(c)}(p)}\subset M\text{ compactly embedded}} r\right)>0. \tag{5.45}$$

Proof. Suppose (5.45) is false. Then there exists a sequence $\{c_i,i\in\mathbb{N}\}\subset C$, such that

$$\lim_{i\to\infty}\underbrace{\sup_{\overline{B_r^{g(c_i)}(p)}\subset M\text{ compactly embedded}} r}_{=:r_i}=0.$$

Since $C\subset\mathbb{R}^N$ is compact, we can restrict to a subsequence if necessary and assume without loss of generality that $\{c_i,i\in\mathbb{N}\}$ converges to a point $\overline{c}:=\lim\limits_{i\to\infty}c_i$ in C. Then, by assumption,

$$\sup_{\overline{B_r^{g(\overline{c})}(p)}\subset M\text{ compactly embedded}} r=\lim_{i\to\infty}r_i=0.$$

But this is a contradiction to the fact that $g\left(\overline{c}\right)$ is a Riemannian metric and, hence, around every $p\in M$ there exists a positive maximal radius $r>0$, such that $\overline{B_r^{g(\overline{c})}(p)}\subset M$ is compactly embedded (recall that independent of the considered Riemannian metric on M, the induced metric topology coincides with the given topology on M). Hence, (5.45) holds true. □

Proposition 5.17 (Alternative closed PSR manifolds completeness proof №2)**.** *Closed PSR manifolds $(\mathcal{H},g_{\mathcal{H}})$ are geodesically complete.*

Proof. Let $n = \dim(\mathcal{H})$. Assume without loss of generality that $\mathcal{H}$ is connected, i.e. a CCPSR manifold. Using Theorem 5.6, we can without loss of generality assume that $\mathcal{H} = \mathcal{H}_{P_3} \subset \{h_{P_3} = x^3 - x\langle y,y\rangle + P_3(y) = 1\}$ is the connected component that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h = 1\}$ and that $P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\} \subset \mathrm{Sym}^3\left(\mathbb{R}^n\right)^*$, where we view the set $\left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}$ as a compact subset of $\mathbb{R}^N$ for $N = \dim \mathrm{Sym}^3\left(\mathbb{R}^n\right)^* = \frac{n^3+3n^2+2n}{6}$. Consider the set

$$M := \bigcap_{P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}} \mathrm{dom}\left(\mathcal{H}_{P_3}\right).$$

Lemma 4.4 implies that $M = \left\{\|z\| < \frac{\sqrt{3}}{2}\right\} \subset \mathbb{R}^n$, in particular M is a smooth submanifold of $\mathbb{R}^n$. Recall that with $\beta_{P_3}(z) := h_{P_3}\left(\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)\right)$, $\left(\mathcal{H}_{P_3}, g_{\mathcal{H}_{P_3}}\right)$ is isometric to

$$\left(\mathrm{dom}\left(\mathcal{H}_{P_3}\right), -\frac{\partial^2 \beta_{P_3}}{3\beta_{P_3}} + \frac{2d\beta_{P_3}^2}{9\beta_{P_3}^2}\right)$$

for all $P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}$, cf. (3.33). Since $M \subset \mathrm{dom}\left(\mathcal{H}_{P_3}\right)$ independent of $P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}$, we can consider the family of Riemannian metrics on M

$$g(\cdot) : P_3 \mapsto -\frac{\partial^2 \beta_{P_3}}{3\beta_{P_3}} + \frac{2d\beta_{P_3}^2}{9\beta_{P_3}^2}.$$

Since $g(\cdot)$ depends continuously on the compact subset $\left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\} \subset \mathrm{Sym}^3\left(\mathbb{R}^n\right)^*$ in the sense of Lemma 5.16 (where we identify $\mathrm{Sym}^3\left(\mathbb{R}^n\right)^*$ with $\mathbb{R}^N$ as above and note that M as an open submanifold of $\mathbb{R}^n$ is in particular equipped with a finite atlas consisting of a single chart), we can use Lemma 5.16 and obtain that there exists $r > 0$, such that

$$\overline{B_r^{g(P_3)}(0)} \subset M$$

is compactly embedded. Together with Proposition 3.18 this implies that for all $P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}$ and all $p \in \mathcal{H}_{P_3}$, $\overline{B_r^{g_{\mathcal{H}_{P_3}}}(p)} \subset \mathcal{H}_{P_3}$ is compactly embedded. Hence, Lemma 2.21 shows that $\left(\mathcal{H}_{P_3}, g_{\mathcal{H}_{P_3}}\right)$ is complete for all $P_3 \in \left\{\max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\}$. $\square$

Note that the ideas behind the proofs of Proposition 4.17 and Proposition 5.17 are very similar, but in the proof of Proposition 4.17 we needed to explicitly construct the compactly embedded geodesic balls.

6 Multi-parameter families of projective special real manifolds

One subject of this thesis is the explicit construction of a multi-parameter family of inequivalent CCPSR manifolds of dimension $n \geq 3$, see Theorem 6.1. Until now[11], only one one-parameter family of inequivalent CCPSR manifolds has been known, that is the Weierstraß cubics and the corresponding CCPSR surfaces in Theorem 2.45 f).

Let $n \geq 3$, $n \in \mathbb{N}$. We will give two examples of $(n-2)$-parameter families in $\mathrm{Sym}^3(\mathbb{R}^{n+1})^*$, each consisting of pairwise inequivalent hyperbolic cubic polynomials, of which each defines a singular-at-infinity CCPSR manifold of dimension n. We will use this result to find a curve in $\mathrm{Sym}^3(\mathbb{R}^{n+1})^*$, such that each point in the curve is a hyperbolic polynomial which defines a CCPSR manifold which is singular at infinity and that the endpoints of that curve are linearly equivalent to the polynomials *a*) and *b*) in Theorem 2.46.

In the following we will denote $z = (z_1, \ldots, z_{n-1})^T$ and by $\langle\cdot,\cdot\rangle$ the standard Euclidean scalar product on $\mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} = \left\{ \left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix} \right) \middle| \, z \in \mathbb{R}^{n-1},\ w, x \in \mathbb{R} \right\}$.

Theorem 6.1. *The $(n-2)$-parameter families*

$$\mathcal{F} := \left\{ h = x\left(-w^2 + \langle z, z\rangle\right) + w \sum_{i=1}^{n-1} b_i z_i^2 \;\middle|\; 1 = b_1 \geq \ldots \geq b_{n-1} \geq 0 \right\} \tag{6.1}$$

and

$$\mathcal{G} := \left\{ h = x\left(-w^2 + \sum_{i=1}^{n-1} b_i z_i^2\right) + w\langle z, z\rangle \;\middle|\; 1 = b_1 \geq \ldots \geq b_{n-1} \geq 0 \right\} \tag{6.2}$$

consist of pairwise inequivalent hyperbolic cubic polynomials. The corresponding projective special real manifolds

$$\mathcal{H}(h) = \left\{ h = 1 \;\middle|\; x < 0,\ w < 0,\ w^2 > \langle z, z\rangle \right\}, \quad h \in \mathcal{F}, \tag{6.3}$$

and

$$\mathcal{H}(h) = \left\{ h = 1 \;\middle|\; x < 0,\ w < 0,\ w^2 > \sum_{i=1}^{n-1} b_i z_i^2 \right\}, \quad h \in \mathcal{G}, \tag{6.4}$$

respectively, are complete.

Proof. Let $M, N \in \mathrm{Mat}((n-1)\times(n-1), \mathbb{R})$ be symmetric positive semi-definite matrices, such that $\mathrm{rk}(M) = (n-1)$ or $\mathrm{rk}(N) = (n-1)$, and denote by $M(z,z) = z^T M z$, $N(z,z) = z^T N z$. We will show that

$$h = x\left(-w^2 + N(z,z)\right) + wM(z,z)$$

is hyperbolic for any such M and N on the set $\mathcal{H} := \{h = 1 \mid x < 0,\ w < 0,\ w^2 > N(z,z)\}$. Consider the vector fields ∂_w and $w\partial_w - x\partial_x$, which are both non-vanishing along $\mathcal{H}$. One can check that they are orthogonal to each other with respect to

$$g = -\frac{1}{2}\partial^2 h = -xN(dz,dz) - wM(dz,dz) + xdw^2 - 2M(z,dz)dw - 2N(z,dz)dx + 2wdwdx,$$

and that $g(\partial_w, \partial_w) = x < 0$, $g(w\partial_w - x\partial_x, w\partial_w - x\partial_x) = -xw^2 > 0$ along $\mathcal{H}$. In the above formula dz is considered as column vector with components dz_i. We will now show that g is positive definite on the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\{\partial_w, w\partial_w - x\partial_x\}$ along $\mathcal{H}$ with

[11] That is, until [CDJL]. The results related to multi-parameter families of CCPSR manifolds in [CDJL] are part of this thesis.

respect to g and thereby prove our claim. One can easily verify that every vector field Y along $\mathcal{H}$ which is perpendicular to $\mathrm{span}_{\mathbb{R}}\{\partial_w, w\partial_w - x\partial_x\}$ can be written as

$$Y = X + \frac{N(z,X)}{w}\partial_w + \frac{wM(z,X) - xN(z,X)}{w^2}\partial_x,$$

where $X = \sum\limits_{i=1}^{n-1} X^i \partial_{z_i}$. Note that $Y = 0$ if and only if $X = 0$. We obtain

$$g(Y,Y) = \frac{1}{w^2}\left(-xw^2 N(X,X) - w^3 M(X,X) - 2wM(z,X)N(z,X) + xN(z,X)^2\right).$$

If $0 \neq X \in \ker N$ it follows by assumption that $M > 0$ and, hence, $g(Y,Y) > 0$ along $\mathcal{H}$. Assume now that $N(X,X) \neq 0$. Observe that $h = 1$ is equivalent to $-xw^2 = 1 - wM(z,z) - xN(z,z)$. Hence, along $\mathcal{H}$ we have

$$\begin{aligned} &- xw^2 N(X,X) + xN(z,X)^2 \\ = \ &\underbrace{N(X,X)}_{>0} - x\big(\underbrace{N(X,X)N(z,z) - N(z,X)^2}_{\geq 0}\big) - wM(z,z)N(X,X) \\ > \ &- wM(z,z)N(X,X). \end{aligned}$$

Using this estimate and $w^2 > N(z,z)$, we obtain

$$g(Y,Y) > \frac{1}{-w}(M(z,z)N(X,X) + 2M(z,X)N(z,X) + M(X,X)N(z,z))$$

along $\mathcal{H}$. If $z \in \ker N$, it follows that $g(Y,Y) > 0$. Assume that $z \notin \ker N$. Consider

$$Q(z,X,\widetilde{z},\widetilde{X}) := M(\widetilde{z},\widetilde{z})N(X,X) + 2M(\widetilde{z},\widetilde{X})N(z,X) + M(\widetilde{X},\widetilde{X})N(z,z).$$

One observes that $Q(z,X,\widetilde{z},\widetilde{X}) \geq 0$ for all $z,X,\widetilde{z},\widetilde{X} \in \mathbb{R}^{n-1}$ if $M(\widetilde{z},\widetilde{z})M(\widetilde{X},\widetilde{X}) \geq M(\widetilde{z},\widetilde{X})^2$ for all $\widetilde{z},\widetilde{X} \in \mathbb{R}^{n-1}$. The latter estimate is true since M is positive semi-definite. Hence, $Q(z,X,z,X) \geq 0$ for all $z,X \in \mathbb{R}^{n-1}$, which shows that $g(Y,Y) > 0$ for $Y \neq 0$. This proves that the pullback of g to $\mathcal{H}$ is a Riemannian metric, so that $\mathcal{H}$ is a projective special real manifold.

We will now show that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed in the subspace topology. Notice that $\mathcal{H}$ can be written as a graph over $U := \{w < 0,\ w^2 > N(z,z)\} \subset \mathbb{R}^n$ by rewriting the equation $h = 1$ as $x = \frac{1-wM(z,z)}{-w^2+N(z,z)}$. We need to check that $x \to -\infty$ for $(w,z) \to \partial U$. Observe that $\partial U = \{w \leq 0,\ -w^2 + N(z,z) = 0\}$. For $(z,w) \in U$ we have

$$x = \frac{1 - wM(z,z)}{-w^2 + N(z,z)} \leq \frac{1}{-w^2 + N(z,z)}$$

and the right-hand side goes to $-\infty$ for all sequences in $\{(z(j),w(j)),\ j \in \mathbb{N}\} \subset U$ with the property $\lim\limits_{j\to\infty}(-w(j)^2 + N(z(j),z(j))) = 0$. This shows that $\partial\mathcal{H}$ is empty and, hence, that $\mathcal{H}$ is closed in $\mathbb{R}^{n+1}$. By [CNS, Thm. 2.5] this implies that the projective special real manifold $\mathcal{H}$ is complete.

Summarising, we have shown that $\mathcal{H}(h)$ is a complete projective special real manifold for all $h \in \mathcal{F}$ and all $h \in \mathcal{G}$. It remains to show that $\mathcal{F}$ and $\mathcal{G}$ each consist of pairwise inequivalent polynomials.

We will start with the family $\mathcal{F}$. We define

$$K := \{x(-w^2 + \langle z,z\rangle) + wM(z,z) \mid 0 \neq M \geq 0\}$$

and see that for all $h \in K$, $\mathcal{H}(h) = \{h = 1 \mid h \in K,\ x < 0,\ w < 0,\ w^2 > \langle z, z\rangle\}$ is a complete special real manifold. This follows from setting $N(\cdot,\cdot) = \langle \cdot,\cdot\rangle$. Furthermore, $\mathcal{F} \subset K$. In order to study equivalence classes of elements of K, it turns out that we have to study the cases (i) $\dim\ker M \neq 1$ and (ii) $\dim\ker M = 1$ separately. In both cases we will make use of properties of the singularity set $\{dh = 0\}$. For a given $h \in K$ we will determine all possible $A \in \mathrm{GL}(n+1)$, such that $h \circ A \in K$. In case (i) we will see that this set of transformations is independent of the chosen h. In case (ii) it will turn out that this set of transformations will depend on the chosen h. We will then use the results to show that $\mathcal{F} \subset K$ consists of pairwise inequivalent polynomials and that for each polynomial $h \in K$ there is a unique representative in $\mathcal{F}$ of the $\mathrm{GL}(n+1)$-orbit of h.

For case (i) we will employ the following lemma.

Lemma 6.2. *Let $h \in K$ and M the corresponding positive semi-definite bilinear form, such that* $\dim\ker M \neq 1$. *Then for $A \in \mathrm{GL}(n+1)$, $h \circ A \in K$ if and only if A is of the form*

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right),\ r > 0,\ E \in \mathrm{O}(n-1).$$

Proof. (of Lemma 6.2) Observe that for all $A \in \mathrm{GL}(n+1)$, $dh_p = 0$ if and only if $d(h\circ A)_{A^{-1}p} = 0$, i.e. $\{d(h\circ A) = 0\}$ is precisely the image of $\{dh = 0\}$ under A^{-1}. First we describe $\{dh = 0\}$ explicitly. We have

$$dh = 2x\langle z, dz\rangle + 2wM(z, dz) + (-2xw + M(z,z))dw + (-w^2 + \langle z, z\rangle)dx.$$

To determine the points $p = (z, w, x)$ such that $dh_p = 0$ we distinguish the cases $w = 0$ and $w \neq 0$. If $w = 0$ then $dh_p = 0$ if and only if $z = 0$. If $w \neq 0$ then $dh_p = 0$ if and only if $w^2 = \langle z, z\rangle$, $z \in \ker M$, and $x = 0$. To see this it suffices to substitute $2xw = M(z,z)$ and $w^2 = \langle z, z\rangle$ into $2xw\langle z, dz\rangle + 2w^2 M(z, dz) = 0$ and insert the position vector z on the left hand side of the latter equation. We have thus determined the set $\{dh = 0\}$ and see that the cone $\{dh = 0\} \setminus \{0\}$ has the following components :

$$\begin{aligned}\{dh = 0\} \setminus \{0\} = \{z = 0,\ w = 0,\ x > 0\} &\dot\cup \{z = 0,\ w = 0,\ x < 0\}\\ &\dot\cup \{z \in \ker M \setminus \{0\},\ w = \sqrt{\langle z, z\rangle},\ x = 0\}\\ &\dot\cup \{z \in \ker M \setminus \{0\},\ w = -\sqrt{\langle z, z\rangle},\ x = 0\}.\end{aligned}$$

The latter two sets are either smooth manifolds of dimension $\dim\ker M$ in the case that $\dim\ker M \neq 0$, or empty if $M > 0$. By assumption they are not of dimension 1 and, hence, connected. Since A^{-1} maps connected components of $\{dh = 0\} \setminus \{0\}$ to connected components of $\{d(h \circ A) = 0\} \setminus \{0\}$, we see that if $\bar h = h \circ A$ is contained in K and, hence, associated with some $\overline{M} \geq 0$, then M and $\overline{M}$ have the same rank and A maps the line $\{z = 0,\ w = 0,\ x \in \mathbb{R}\}$ to itself. Note that it is precisely at this point that we have used the condition $\dim\ker M \neq 1$. This means that A has the following form:

$$A = \left(\begin{array}{c|c} \multicolumn{2}{c}{B} \\ \hline (\alpha^T, \beta) & r \end{array}\right),\ B \in \mathrm{Mat}(n \times n,\ \mathbb{R}),\ \alpha \in \mathbb{R}^{n-1},\ \beta \in \mathbb{R},\ r \in \mathbb{R} \setminus \{0\}.$$

By writing down $(h \circ A)(z, w, x)$, one can easily verify that $r > 0$ and $B = r^{-\frac{1}{2}}C$, $C \in \mathrm{O}(n-1, 1)$, are necessary for $h\circ A$ to be contained in K. Here $\mathrm{O}(n-1,1)$ is the automorphism group of the quadratic form $-w^2 + \langle z, z\rangle$ on $\mathbb{R}^n$. Using the notation $C\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \tilde z \\ \tilde w \end{smallmatrix}\right)$ we obtain

$$(h \circ A)(z, w, x) = x(-w^2 + \langle z, z\rangle) + r^{-\frac{3}{2}}((\langle r^{\frac{1}{2}}\alpha, z\rangle + r^{\frac{1}{2}}\beta w)(-w^2 + \langle z, z\rangle) + \tilde w M(\tilde z, \tilde z)).$$

C is of the form

$$C = \left(\begin{array}{c|c} E & \xi \\ \hline \eta^T & \mu \end{array}\right),\ E \in \mathrm{Mat}((n-1)\times(n-1),\ \mathbb{R}),\ \eta, \xi \in \mathbb{R}^{n-1},\ \mu \in \mathbb{R},$$

and fulfils

$$C^T \left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -1 \end{array}\right) C = \left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -1 \end{array}\right).$$

The left hand side of the above equation equals

$$\left(\begin{array}{c|c} E^T E - \eta \otimes \langle \eta, \cdot \rangle & E^T \xi - \mu\eta \\ \hline \xi^T E - \mu \eta^T & \langle \xi, \xi \rangle - \mu^2 \end{array}\right),$$

which in particular implies that $\mu \neq 0$ and $\mathrm{rk}\ E = n-1$. To see the latter, suppose that there exists $0 \neq v \in \ker E$. Since $E^T\xi - \mu\eta = 0$, it follows that $\eta = \mu^{-1}E^T\xi$. Hence,

$$\left(E^T E - \eta \otimes \langle \eta, \cdot \rangle\right) v = E^T E v - \mu^{-2} E^T \xi \langle E^T \xi, v\rangle = -\mu^{-2} E^T \xi \langle \xi, Ev\rangle = 0,$$

which contradicts the assumption that $E^T E - \eta \otimes \langle \eta, \cdot \rangle = \mathbb{1}$. With $\kappa := r^{\frac{1}{2}}\alpha$ and $\rho := r^{\frac{1}{2}}\beta$,

$$\begin{aligned}(h \circ A)(z, w, x) = {} & x(-w^2 + \langle z, z\rangle) \\ & + r^{-\frac{3}{2}}(w^3(\mu M(\xi,\xi) - \rho) && (6.5) \\ & + w^2(2\mu M(Ez, \xi) + \langle \eta, z\rangle M(\xi,\xi) - \langle \kappa, z\rangle) && (6.6) \\ & + w(\mu M(Ez, Ez) + 2\langle \eta, z\rangle M(Ez, \xi) + \rho\langle z, z\rangle) \\ & + \langle \eta, z\rangle M(Ez, Ez) + \langle \kappa, z\rangle \langle z, z\rangle). && (6.7)\end{aligned}$$

The requirements for $h \circ A$ to be contained in K are (6.5) = (6.6) = (6.7) = 0 and

$$\mu M(Ez, Ez) + 2\langle \eta, z\rangle M(Ez, \xi) + \rho\langle z, z\rangle \geq 0 \quad \forall z \in \mathbb{R}^{n-1}. \tag{6.8}$$

We will show that this implies $\kappa = 0$ and $\rho = 0$ and, consequently, $\alpha = 0$ and $\beta = 0$. Firstly, we will show that $\rho = 0$ implies $\kappa = 0$, and secondly that a transformation with $\rho \neq 0$ contradicts the requirement $C \in \mathrm{O}(n-1, 1)$.

Assume $\rho = 0$. Then (6.5) is equivalent to $M(\xi, \xi) = 0$. Since $M \geq 0$, this implies $\xi \in \ker M$. Equation (6.6) is thus equivalent to $\langle \kappa, z\rangle = 0$ for all $z \in \mathbb{R}^{n-1}$. This shows $\kappa = 0$.

Now assume that $\rho \neq 0$. Then by equation (6.5)

$$M(\xi, \xi) = \mu^{-1}\rho.$$

Note that this implies $\mu^{-1}\rho > 0$ and in particular $\xi \notin \ker M$. Inserting the above equation in (6.6) yields

$$2\mu M(Ez, \xi) + \langle \eta, z\rangle \mu^{-1}\rho = \langle \kappa, z\rangle.$$

Using that, (6.7) becomes

$$\langle \eta, z\rangle (M(Ez, Ez) + \mu^{-1}\rho\langle z, z\rangle) + 2\mu M(Ez, \xi)\langle z, z\rangle = 0.$$

Since $C \in \mathrm{O}(n-1, 1)$, we have $\eta = \mu^{-1}E^T\xi$ and, hence,

$$\langle z, E^T\xi\rangle (\underbrace{M(Ez, Ez) + \mu^{-1}\rho\langle z, z\rangle}_{>0\ \forall z \neq 0}) + \langle z, E^T M\xi\rangle \underbrace{2\mu^2\langle z, z\rangle}_{>0\ \forall z \neq 0} = 0.$$

An immediate consequence is that $E^T\xi$ and $E^T M\xi$ are linearly dependent. Since $\ker E^T = \{0\}$ and $\xi \notin \ker M$ this is equivalent to $E^T M\xi = sE^T\xi$ for some $s \in \mathbb{R}\setminus\{0\}$, which shows that $M\xi = s\xi$, that is ξ needs to be an eigenvector of M. This also shows $s > 0$. Hence,

$$\langle z, E^T\xi\rangle\underbrace{(M(Ez,Ez) + (\mu^{-1}\rho + 2\mu^2 s)\langle z,z\rangle)}_{>0\ \forall z\neq 0} = 0.$$

This shows that $E^T\xi = 0$ which contradicts $\ker E = \{0\}$. This proves $\rho = 0$, $\kappa = 0$, and $\xi \in \ker M$.

Summarising, we have shown that A needs to be of the form

$$A = \left(\begin{array}{c|c} r^{-\frac{1}{2}}C & \\ \hline & r \end{array}\right),\ C \in \mathrm{O}(n-1,1),\ r > 0.$$

For such A, equations (6.5) and (6.6) are automatically fulfilled, and equation (6.7) becomes

$$\langle \eta, z\rangle M(Ez,Ez) = 0. \tag{6.9}$$

Since rk $E = n-1$ we know that $M(Ez,Ez)$ is a non-vanishing quadratic polynomial. Hence, (6.9) is true if and only if $\eta = 0$. As we have seen before, $\eta = 0$ implies $\xi = 0$ since $C \in \mathrm{O}(n-1,1)$. Observe that $\xi = 0$ and $C \in \mathrm{O}(n-1,1)$ also imply $-\mu^2 = -1$. The inequality (6.8) becomes $\mu M(Ez,Ez) \geq 0$, from which we deduce that $\mu = 1$. Hence, all possible transformations such that $h \circ A \in K$ with

$$h = x(-w^2 + \langle z,z\rangle) + wM(z,z),\ M \geq 0,\ M \neq 0,\ \dim\ker M \neq 1,$$

can be written as

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right),\ E \in \mathrm{O}(n-1),\ r > 0, \tag{6.10}$$

independent of the choice of $h \in K$. $\square$

Next, we will deal with case (ii).

Lemma 6.3. *Let $A \in \mathrm{GL}(n+1)$, $h \in K$ and M the corresponding positive semi-definite bilinear form, such that* $\dim\ker M = 1$. *Then $h \circ A \in K$ if and only if M has at least* 2 *distinct positive eigenvalues and A is of the form (6.10) or, if M has precisely* 1 *positive eigenvalue, A can be written as a product of transformations of the form (6.10) and*

$$\left(\begin{array}{c|c|c|c} \mathbb{1} & & & \\ \hline & \frac{1}{2} & \frac{-1}{2} & 1 \\ \hline & \frac{-1}{2} & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right).$$

Furthermore, in the case when M has precisely 1 *positive eigenvalue the two sets*

$$\{h \circ A \mid A \in \mathrm{GL}(n+1),\ h \circ A \in K\}$$

and

$$\{h \circ A \mid A \textit{ is of the form } (6.10)\}$$

coincide.

Proof. (of Lemma 6.3) In case (ii), that is $\dim\ker M=1$, $\{dh=0\}$ consists of 3 distinct lines that intersect at $0\in\mathbb{R}^{n+1}$,

$$\begin{aligned}\{dh=0\}=&\{z=0,\ w=0,\ x\in\mathbb{R}\}\\ &\cup\{z\in\ker M,\ w=\sqrt{\langle z,z\rangle},\ x=0\}\\ &\cup\{z\in\ker M,\ w=-\sqrt{\langle z,z\rangle},\ x=0\}.\end{aligned}$$

Note that each of the latter two sets is not a line, but their union is a union of two distinct lines. Contrary to case (i) we can no longer assume that a transformation mapping $h=x(-w^2+\langle z,z\rangle)+wM(z,z)\in K$ to $\overline{h}=x(-w^2+\langle z,z\rangle)+w\overline{M}(z,z)\in K$ preserves the line $\{z=0,\ w=0,\ x\in\mathbb{R}\}$, since all connected components of $\{dh=0\}\setminus\{0\}$ are of dimension one. Note that we can, after a possible orthogonal transformation of the z-coordinates, assume that

$$M=\begin{pmatrix}\lambda_1&&&\\&\ddots&&\\&&\lambda_{n-2}&\\&&&0\end{pmatrix},\quad \overline{M}=\begin{pmatrix}\overline{\lambda}_1&&&\\&\ddots&&\\&&\overline{\lambda}_{n-2}&\\&&&0\end{pmatrix},$$

which in particular implies $\ker M=\ker\overline{M}$. Thus in addition to the transformations (6.10), considered in case (i), we need to consider transformations of the form

$$A=\left(\begin{array}{c|c|c}E&\xi&v\\\hline \eta^T&\mu&\pm\|v\|\\\hline \alpha^T&\beta&0\end{array}\right),\quad v\in\ker M\setminus\{0\},$$

which map $\{z=0,\ w=0,\ x\in\mathbb{R}\}$ to either $\{z=rv,\ w=r\|v\|,\ x=0\mid r\in\mathbb{R}\}$ or $\{z=rv,\ w=-r\|v\|,\ x=0\mid r\in\mathbb{R}\}$, and are required to preserve $\{dh=0\}=\{d\overline{h}=0\}$. By calculating $(h\circ A)(z,w,x)$, we obtain the following system of equations, which is equivalent to $h\circ A=\overline{h}$:

$$\mp2\|v\|\beta\langle\eta,z\rangle+2\beta\langle Ez,v\rangle\pm2\|v\|M(Ez,\xi)\mp2\|v\|\mu\langle\alpha,z\rangle+2\langle\xi,v\rangle\langle\alpha,z\rangle=0 \tag{6.11}$$
$$\beta(-\mu^2+\langle\xi,\xi\rangle)+\mu M(\xi,\xi)=0 \tag{6.12}$$
$$\langle\alpha,z\rangle(-\mu^2+\langle\xi,\xi\rangle)+\langle\eta,z\rangle(-2\beta\mu+M(\xi,\xi))+2\beta\langle Ez,\xi\rangle+2\mu M(Ez,\xi)=0 \tag{6.13}$$
$$-\langle\alpha,z\rangle\langle\eta,z\rangle^2+\langle\alpha,z\rangle\langle Ez,Ez\rangle+\langle\eta,z\rangle M(Ez,Ez)=0 \tag{6.14}$$
$$\mp2\|v\|\beta\mu+2\beta\langle\xi,v\rangle\pm\|v\|M(\xi,\xi)=-1 \tag{6.15}$$
$$\langle\alpha,z\rangle(\mp2\|v\|\langle\eta,z\rangle+2\langle Ez,v\rangle)\pm\|v\|M(Ez,Ez)=\langle z,z\rangle \tag{6.16}$$
$$\begin{aligned}&-2\mu\langle\alpha,z\rangle\langle\eta,z\rangle+2\langle\alpha,z\rangle\langle Ez,\xi\rangle-\beta\langle\eta,z\rangle^2\\&+\beta\langle Ez,Ez\rangle+2\langle\eta,z\rangle M(Ez,\xi)+\mu M(Ez,Ez)=\overline{M}(z,z)\end{aligned} \tag{6.17}$$

We will show that such a transformation exists if and only if $\lambda_1=\ldots=\lambda_{n-2}$.

Claim 1: $\dim\ker E\leq1$.

Proof. In general, $\dim\ker\langle\alpha,\cdot\rangle\geq n-2$. Suppose $\dim\ker E>1$. Then there exists $Y\in\mathbb{R}^{n-1}\setminus\{0\}$, such that $Y\in\ker\langle\alpha,\cdot\rangle\cap\ker E$. Hence, by equation (6.16), $0=\langle Y,Y\rangle$, which is a contradiction to $Y\neq0$. □

Claim 2: $\dim\ker E=1\Rightarrow\ker E\not\subset\ker\langle\alpha,\cdot\rangle$.

Proof. Suppose $\dim\ker E=1$ and $\ker E\subset\ker\langle\alpha,\cdot\rangle$, and let $0\neq Y\in\ker E$. Again, equation (6.16) implies $0=\langle Y,Y\rangle$ and, hence, contradicts $Y\neq0$. □

Claim 3: $\dim\ker E = 1 \Rightarrow \ker E \subset \ker\langle\eta,\cdot\rangle$.

Proof. Let $0 \neq Y \in \ker E$. Equation (6.14) reads

$$-\underbrace{\langle\alpha,Y\rangle}_{\neq 0}\langle\eta,Y\rangle^2 = 0,$$

which shows that $Y \in \ker\langle\eta,\cdot\rangle$. ☐

Claim 4: $\dim\ker E = 0$.

Proof. Suppose that $\dim\ker E \neq 0$. We have shown that the only other possible case would be $\dim\ker E = 1$. For $0 \neq Y \in \ker E$, we have also shown that $Y \in \ker\langle\eta,\cdot\rangle$. Now equation (6.16) implies $0 = \langle Y,Y\rangle$, which, again, contradicts $Y \neq 0$. Hence, we have shown that $\ker E = \{0\}$, i.e. $E \in \mathrm{GL}(n-1)$. ☐

Claim 5: $\alpha \neq 0$.

Proof. Suppose $\alpha = 0$. Equation (6.16) is now equivalent to $\pm\|v\|E^TME = \mathbb{1}$. Since $E \in \mathrm{GL}(n-1)$, this implies that M is invertible, which contradicts the assumption $\dim\ker M = 1$. ☐

Claim 6: $\eta = s\alpha,\ s \neq 0$.

Proof. If $\eta \notin \mathbb{R}\alpha \setminus \{0\}$ then there exists $Y \in \ker\langle\eta,\cdot\rangle$, such that $\langle\alpha,Y\rangle \neq 0$. Together with $E \in \mathrm{GL}(n-1)$ this implies $\langle\alpha,Y\rangle\langle EY,EY\rangle \neq 0$, which contradicts equation (6.14). ☐

Claim 7: $A\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}v\\\|v\|\\0\end{smallmatrix}\right)$.

Proof. Suppose on the contrary that $A\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}v\\-\|v\|\\0\end{smallmatrix}\right)$. Then for all $Y \in \ker\langle\alpha,\cdot\rangle$ equation (6.16) implies $-\|v\|M(EY,EY) = \langle Y,Y\rangle$. But M is positive semi-definite, hence this is a contradiction. Note that this means that in equations (6.11)–(6.17), every "$\pm$" needs to be "$+$", and every "$\mp$" needs to be "$-$". ☐

Claim 8: $\xi \in \ker M$.

Proof. By construction, A is required to map the set $\{dh = 0\} = \left\{d\bar{h} = 0\right\}$ onto itself, that is it induces a permutation of the three lines $\mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\mathbb{R}\begin{pmatrix}v\\\|v\|\\0\end{pmatrix}$, and $\mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}$. We already know that the first line is mapped to the second. Therefore, either

$$A\begin{pmatrix}v\\\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}, \tag{a}$$

or

$$A\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}v\\\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}. \tag{b}$$

In case (a), $Ev + \|v\|\xi = 0$, and, hence, using the second equation in (a), $Ev - \|v\|\xi = -2\|v\|\xi \in \mathbb{R}v = \ker M$. Similarly, in case (b) we have $Ev - \|v\|\xi = 0$, showing that $Ev + \|v\|\xi = 2\|v\|\xi \in \mathbb{R}v$. ☐

In the following we will write $\xi = kv$, $k \in \mathbb{R}$.

Claim 9: $\beta \neq 0$**.**

Proof. This follows from the previous claim and equation (6.15). $\square$

Claim 10: $\xi = -\frac{1}{4\beta\langle v,v\rangle}v$**,** $\mu = \frac{1}{4\beta\|v\|}$**,** $s = -\frac{1}{4\beta^2\|v\|}$**,** $\alpha = 4\beta^2 E^T v$**.**

Proof. We have shown that $\beta \neq 0$ and $\xi = kv \in \ker M$. Hence, (6.12) implies $\mu = \pm k\|v\|$. Furthermore the previous results imply that (6.15) is equivalent to $-2\|v\|\beta\mu + 2\beta k\langle v, v\rangle = -1$. This shows that $\mu = -k\|v\|$ and, hence,

$$k = -\frac{1}{4\beta\langle v, v\rangle}, \quad \mu = \frac{1}{4\beta\|v\|}.$$

One can easily check that equation (6.13) is equivalent to $\langle \alpha, z\rangle(-2\beta\mu s) + 2\beta\langle Ez, \xi\rangle = 0$, which shows that

$$\langle \alpha, z\rangle = -\frac{1}{s\|v\|}\langle Ez, v\rangle.$$

Using this, equation (6.11) is equivalent to

$$s = \frac{k\|v\|}{\beta} = -\frac{1}{4\beta^2\|v\|}.$$

Hence, $\langle \alpha, z\rangle = 4\beta^2\langle Ez, v\rangle$. $\square$

The restrictions derived from the equations (6.11)–(6.17) in the above series of claims already imply the equations (6.11), (6.12), (6.13), and (6.15). With the above results, one can show that the remaining equations (6.14), (6.16), and (6.17) are equivalent to

$$-\frac{1}{\langle v, v\rangle}\langle Ez, v\rangle^2 + \langle Ez, Ez\rangle - \frac{1}{4\beta^2\|v\|}M(Ez, Ez) = 0, \tag{6.18}$$

$$16\beta^2\langle Ez, v\rangle^2 + \|v\|M(Ez, Ez) = \langle z, z\rangle, \tag{6.19}$$

$$-\frac{\beta}{\langle v, v\rangle}\langle Ez, v\rangle^2 + \beta\langle Ez, Ez\rangle + \frac{1}{4\beta\|v\|}M(Ez, Ez) = \overline{M}(z, z), \tag{6.20}$$

respectively.

Claim 11: $\overline{M}(z, z) = \frac{1}{2\beta\langle v,v\rangle}\langle z, z\rangle - \frac{8\beta}{\langle v,v\rangle}\langle Ez, v\rangle^2$**.**

Proof. By multiplying both sides of equation (6.18) with $-\beta$ and adding them to (6.20) we obtain

$$\frac{1}{2\beta\|v\|}M(Ez, Ez) = \overline{M}(z, z).$$

By considering equation (6.19) we see that $\frac{1}{2\beta\|v\|}M(Ez, Ez) = \frac{1}{2\beta\langle v,v\rangle}\langle z, z\rangle - \frac{8\beta}{\langle v,v\rangle}\langle Ez, v\rangle^2$, which proves the claim. $\square$

Claim 12: E **is of the form** $E = \left(\begin{array}{c|c} B & \\ \hline & \pm\frac{1}{4\beta\|v\|} \end{array}\right)$**,** $B \in \mathrm{GL}(n-2)$**.**

Proof. By the assumption $\overline{M}(z,z) = \sum_{i=1}^{n-2} \overline{\lambda}_i z_i^2$, it follows that either $v = \|v\|\partial_{z_{n-1}}$, or $v = -\|v\|\partial_{z_{n-1}}$. Note that the sign does not depend on the cases (a) and (b) described in Claim 8. Using this, one can easily check that Claim 11 restricts E to be of the form

$$E = \left(\begin{array}{c|c} * & * \\ \hline 0 & \pm\frac{1}{4\beta\|v\|} \end{array}\right).$$

Recall that by Claim 8, $Ev = -\|v\|\xi = \frac{1}{4\beta\|v\|}v$ in case (a), or $Ev = \|v\|\xi = -\frac{1}{4\beta\|v\|}v$ in case (b). This shows that E needs to be of the form

$$E = \left(\begin{array}{c|c} * & 0 \\ \hline * & \pm\frac{1}{4\beta\|v\|} \end{array}\right),$$

where "$+$" corresponds to case (a) and "$-$" to case (b). This and the requirement $E \in \mathrm{GL}(n-1)$ show that E is of the claimed form. □

This shows that under our assumptions the equations (6.11)–(6.17) can only be satisfied if $\overline{M}$ has precisely one positive eigenvalue, i.e.

$$\overline{M}(z,z) = \frac{1}{2\beta\langle v,v\rangle}\sum_{i=1}^{n-2} z_i^2.$$

This also shows that $\beta > 0$ is a necessary requirement.

Claim 13: E **is of the form** $E = \frac{1}{2\beta\|v\|}\left(\begin{array}{c|c} C & \\ \hline & \pm\frac{1}{2} \end{array}\right)$**,** $C \in \mathrm{O}(n-2)$**.**

Proof. Observe that Claim 12 shows $E^T v = Ev$, which implies $\langle Ez, v\rangle^2 = \frac{z_{n-1}^2}{16\beta^2}$. Hence, equation (6.19) is equivalent to

$$\|v\|M(Ez,Ez) = \sum_{i=1}^{n-2} z_i^2, \tag{6.21}$$

and equation (6.18) is equivalent to

$$\|v\|M(Ez,Ez) = 4\beta^2\langle v,v\rangle \left\langle B\begin{pmatrix} z_1 \\ \vdots \\ z_{n-2}\end{pmatrix}, B\begin{pmatrix} z_1 \\ \vdots \\ z_{n-2}\end{pmatrix}\right\rangle. \tag{6.22}$$

On the right-hand side of (6.22), $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^{n-2}$. Note that, since E is invertible, (6.22) shows that

$$M(z,z) = 4\beta^2\|v\|\sum_{i=1}^{n-2} z_i^2,$$

so M also has exactly one positive eigenvalue. By comparing (6.21) and (6.22) we see that $B = \frac{1}{2\beta\|v\|}C$ for some $C \in \mathrm{O}(n-2)$. This proves that $E = \frac{1}{2\beta\|v\|}\begin{pmatrix} C & \\ & \pm\frac{1}{2}\end{pmatrix}$, $C \in \mathrm{O}(n-2)$. □

Since $M(z,z)$ is a positive scalar multiple of $\sum\limits_{i=1}^{n-2} z_i^2$, h is invariant under transformations of the form

$$\widehat{C} = \begin{pmatrix} C^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad C \in \mathrm{O}(n-2).$$

Replacing A with the matrix $\widehat{C}A$, we can assume without loss of generality that $E = \frac{1}{2\beta\|v\|}\begin{pmatrix} \mathbb{1} & \\ & \pm\frac{1}{2} \end{pmatrix}$. Summarising, we have shown that in case (a), depending on the choice of the sign of $v = \pm\|v\|\partial_{z_{n-1}}$,

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbb{1}}{2\beta\|v\|} & & & \\ \hline & \frac{1}{4\beta\|v\|} & \frac{\mp 1}{4\beta\|v\|} & \pm\|v\| \\ \hline & \frac{\mp 1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \pm\beta & \beta & 0 \end{array}\right),$$

and in case (b)

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbb{1}}{2\beta\|v\|} & & & \\ \hline & \frac{-1}{4\beta\|v\|} & \frac{\mp 1}{4\beta\|v\|} & \pm\|v\| \\ \hline & \frac{\pm 1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \mp\beta & \beta & 0 \end{array}\right),$$

which again depends on the sign of $v = \pm\|v\|\partial_{z_{n-1}}$.

Since both h and $\overline{h}$ are invariant under the transformation

$$K := \left(\begin{array}{c|c|c|c} \mathbb{1} & & & \\ \hline & -1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right),$$

we see that, up to automorphisms of h and $\bar{h}$, in each of the four possible cases we only need to consider

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbb{1}}{2\beta\|v\|} & & & \\ \hline & \frac{1}{4\beta\|v\|} & \frac{-1}{4\beta\|v\|} & \|v\| \\ \hline & \frac{-1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \beta & \beta & 0 \end{array}\right).$$

We set $\lambda := 4\beta^2\|v\|$, so that $M(z,z) = \lambda \sum\limits_{i=1}^{n-2} z_i^2$, $\overline{M}(z,z) = \frac{8\beta^3}{\lambda^2}\sum\limits_{i=1}^{n-2} z_i^2$, and

$$A = \left(\begin{array}{c|c|c|c} \frac{2\beta}{\lambda}\mathbb{1} & & & \\ \hline & \frac{\beta}{\lambda} & \frac{-\beta}{\lambda} & \frac{\lambda}{4\beta^2} \\ \hline & \frac{-\beta}{\lambda} & \frac{\beta}{\lambda} & \frac{\lambda}{4\beta^2} \\ \hline & \beta & \beta & 0 \end{array}\right).$$

We define

$$R_r := \left(\begin{array}{c|c|c|c} r\mathbb{1} & & & \\ \hline & r & & \\ \hline & & r & \\ \hline & & & \frac{1}{r^2} \end{array}\right), \quad \widehat{A} := \left(\begin{array}{c|c|c|c} \mathbb{1} & & & \\ \hline & \frac{1}{2} & \frac{-1}{2} & 1 \\ \hline & \frac{-1}{2} & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right).$$

One can now verify that $A = R_{\lambda^{-\frac{1}{3}}} \widehat{A} R_{\frac{2\beta}{\lambda^{\frac{2}{3}}}}$. Note that $\widehat{A}^2 = \mathbb{1}$ and that $\widehat{A}$ is an automorphism of the polynomial $h_1 := x(-w^2 + \langle z, z\rangle) + w \sum\limits_{i=1}^{n-2} z_i^2$.

Claim 13 shows that the additional transformations obtained in the special case that h is equivalent to h_1 when compared to the other considered cases are all conjugated to a composition of the additional automorphism $\widehat{A}$ of h_1 and transformations of the form (6.10). This shows that

$$\{h \circ A \mid A \in \mathrm{GL}(n+1),\ h \circ A \in K\} = \{h \circ A \mid A \text{ is of the form (6.10)}\}.$$

Hence, for choosing a representative of an h in $\mathcal{F}$ when h has the property that the corresponding M has exactly one positive eigenvalue and $\dim\ker M = 1$, it suffices to consider transformations of the form (6.10). This finishes the proof of Lemma 6.3. □

With the help of Lemma 6.2 and Lemma 6.3 we will now choose a unique representative in $\mathcal{F}$ for the $\mathrm{GL}(n+1)$-orbit of an element $h \in K$. For a given positive semi-definite bilinear form M there is a unique bilinear form

$$\widehat{M} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-1} \end{pmatrix},\ \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0,$$

such that there exists $E \in \mathrm{O}(n-1)$ with the property that $E^T M E = \widehat{M}$. The λ_i are the eigenvalues of M. $M \neq 0$ implies that M has at least one positive eigenvalue $\lambda_1 > 0$. Applying the corresponding transformation (6.10) with $r = \lambda_1^{\frac{2}{3}}$, we see that $h = x(-w^2 + \langle z, z\rangle) + wM(z,z)$ is equivalent to

$$\widehat{h} \in \mathcal{F},\ \widehat{h} = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} b_i z_i^2,\ b_1 = 1,\ b_1 \geq \ldots \geq b_{n-1} \geq 0,$$

and the b_i's thus uniquely determined by M. Summarising up to this point, we have shown that the $(n-2)$-parameter family $\mathcal{F}$ consists of pairwise inequivalent hyperbolic homogeneous polynomials, all of which define a complete projective special real manifold of dimension n.

We will now consider the family $\mathcal{G}$ and proceed similarly as for the family $\mathcal{F}$. Consider the set of homogeneous cubic polynomials

$$L := \{x(-w^2 + N(z,z)) + w\langle z, z\rangle \mid 0 \neq N \geq 0\}.$$

It is clear that $\mathcal{G} \subset L$ and that any element in L is contained in the $\mathrm{GL}(n+1)$-orbit of some element in $\mathcal{G}$. For a given $h = x(-w^2 + N(z,z)) + w\langle z, z\rangle$ we want to determine all possible $A \in \mathrm{GL}(n+1)$, such that $(h \circ A)(z, w, x) \in L$. We will see that the answer is independent of the chosen h.

For $\dim\ker N = 0$, h is equivalent to some $\widetilde{h} = x(-w^2 + \langle z, z\rangle) + wM(z,z) \in K$ with the property $M > 0$. In this case we know that there is a unique representative of $\widetilde{h}$ under the $\mathrm{GL}(n+1)$-action in $\mathcal{F}$ of the form

$$\widehat{h} = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} b_i z_i^2, \quad b_1 = 1, \quad b_1 \geq \ldots \geq b_{n-1} > 0,$$

which can easily be checked to be equivalent to

$$\check{h} = x\left(-w^2 + \sum_{i=1}^{n-1} \frac{b_{n-1}}{b_{n-i}} z_i^2\right) + w\langle z, z\rangle, \quad 1 = \frac{b_{n-1}}{b_{n-1}} \geq \ldots \geq \frac{b_{n-1}}{b_1} > 0.$$

Hence, $\check{h} \in \mathcal{G}$. The uniqueness property can be shown the following way. Assume that $h = x\left(-w^2 + \sum_{i=1}^{n-1} c_i z_i^2\right) + w\langle z, z\rangle \in \mathcal{G}$ and $\overline{h} = x\left(-w^2 + \sum_{i=1}^{n-1} \overline{c}_i z_i^2\right) + w\langle z, z\rangle \in \mathcal{G}$ are equivalent. The polynomials h and $\overline{h}$ are equivalent to

$$h' = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} \frac{c_{n-1}}{c_{n-i}} z_i^2 \in \mathcal{F}$$

and

$$\overline{h}' = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} \frac{\overline{c}_{n-1}}{\overline{c}_{n-i}} z_i^2 \in \mathcal{F},$$

respectively. We have shown that h' and $\overline{h}'$ are equivalent if and only if $\frac{c_{n-1}}{c_{n-i}} = \frac{\overline{c}_{n-1}}{\overline{c}_{n-i}}$ for all $1 \leq i \leq n-1$. Since $c_1 = \overline{c}_1 = 1$, this shows that $c_{n-1} = \overline{c}_{n-1}$. Hence, $c_i = \overline{c}_i$ must hold for all $1 \leq i \leq n-1$.

Thus, we can reduce this question and assume that the $h \in L$ we are starting with has the property that $N \geq 0$, $N \neq 0$, and $\dim \ker N \neq 0$.

Lemma 6.4. *Let $h \in L \setminus \{x(-w^2 + N(z,z)) + w\langle z, z\rangle \mid N > 0\}$. Then $h \circ A \in L$, $A \in \mathrm{GL}(n+1)$, if and only if*

$$A = \left(\begin{array}{c|c|c} r^{\frac{1}{4}} F & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right), \quad F \in \mathrm{O}(n-1),\ r > 0.$$

In particular the possible choices for A do not depend on h.

Proof. Let $h = x(-w^2 + N(z,z)) + w\langle z, z\rangle$. We obtain

$$dh = 2xN(z, dz) + 2w\langle z, dz\rangle + (-2wx + \langle z, z\rangle)dw + (-w^2 + N(z,z))dx.$$

We will determine the set $\{dh = 0\}$. Observe that for $w = 0$ it follows that $\langle z, z\rangle = 0$ and, hence, $z = 0$. Then all entries of dh are 0 for all $x \in \mathbb{R}$. For $w \neq 0$, substitute the equations $2wx = \langle z, z\rangle$ and $w^2 = N(z,z)$ into $2wxN(z,\cdot) + 2w^2\langle z, \cdot\rangle = 0$, which is the first equation in $dh = 0$ multiplied by w. We obtain $\langle z, z\rangle N(z,\cdot) + 2\langle z, \cdot\rangle N(z,z) = 0$, which in particular implies $3\langle z, z\rangle N(z,z) = 0$. This shows that $z \in \ker N$. But then $w^2 = N(z,z) = 0$, which is a contradiction to the assumption $w \neq 0$. Summarising, we have shown that for all $N \geq 0$

$$\{dh = 0\} = \{z = 0,\ w = 0,\ x \in \mathbb{R}\}.$$

Hence, A needs to be of the form

$$A = \left(\begin{array}{c|c} B & \\ \hline (\alpha^T, \beta) & r \end{array}\right), \quad B \in \mathrm{Mat}(n \times n,\ \mathbb{R}),\ \alpha \in \mathbb{R}^{n-1},\ \beta \in \mathbb{R},\ r \in \mathbb{R} \setminus \{0\}.$$

Let $\overline{h} = x(-w^2 + \overline{N}(z,z)) + w\langle z, z\rangle$ and assume that $h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) = \overline{h}\left(\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right)$. Denote by $\left(\begin{smallmatrix} \widetilde{z} \\ \widetilde{w} \\ \widetilde{x} \end{smallmatrix}\right) = A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)$. We obtain

$$h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) = (\langle \alpha, z\rangle + \beta w + rx)(-\widetilde{w}^2 + N(\widetilde{z}, \widetilde{z})) + \widetilde{w}\langle \widetilde{z}, \widetilde{z}\rangle.$$

Since $\widetilde{w}\langle \widetilde{z}, \widetilde{z}\rangle$ does not depend on the variable x, this shows that

$$-\widetilde{w}^2 + N(\widetilde{z}, \widetilde{z}) = r^{-1}\left(-w^2 + \overline{N}(z,z)\right).$$

Hence, $B = r^{-\frac{1}{2}}C$ with

$$C^T \left(\begin{array}{c|c} N & \\ \hline & -1 \end{array}\right) C = \left(\begin{array}{c|c} \overline{N} & \\ \hline & -1 \end{array}\right), \quad C \in \mathrm{GL}(n).$$

For $C = \left(\begin{array}{c|c} E & \xi \\ \hline \eta^T & \mu \end{array}\right)$ the above equation is equivalent to

$$\left(\begin{array}{c|c} E^T N E - \eta \otimes \langle \eta, \cdot \rangle & E^T N \xi - \mu\eta \\ \hline \xi^T N E - \mu \eta^T & N(\xi,\xi) - \mu^2 \end{array}\right) = \left(\begin{array}{c|c} \overline{N} & \\ \hline & -1 \end{array}\right).$$

Note that this shows $\mu \neq 0$. This is equivalent to

$$\mu^2 = 1 + N(\xi,\xi), \tag{6.23}$$

$$E^T N \xi = \mu\eta, \tag{6.24}$$

$$E^T N E - \eta \otimes \langle \eta, \cdot \rangle = \overline{N}. \tag{6.25}$$

In particular $\mu \neq 0$. Up to this point, we have shown that

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & r^{-\frac{1}{2}}\xi & \\ \hline r^{-\frac{1}{2}}\eta^T & r^{-\frac{1}{2}}\mu & \\ \hline \alpha & \beta & r \end{array}\right).$$

We calculate

$$\begin{aligned} h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) &= x\left(-w^2 + \overline{N}(z,z)\right) \\ &+ w^3\left(-\beta r^{-1} + r^{-\frac{3}{2}}\mu\langle \xi,\xi\rangle\right) \\ &+ w^2\left(-r^{-1}\langle \alpha, z\rangle + r^{-\frac{3}{2}}\langle \eta, z\rangle\langle \xi,\xi\rangle + 2r^{-\frac{3}{2}}\mu\langle Ez,\xi\rangle\right) \\ &+ w\left(\beta r^{-1}\overline{N}(z,z) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle\right) \\ &+ r^{-1}\langle \alpha,z\rangle\overline{N}(z,z) + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,Ez\rangle. \end{aligned}$$

By assumption, the entries of A need to fulfil the equations

$$-\beta r^{-1} + r^{-\frac{3}{2}}\mu\langle \xi,\xi\rangle = 0, \tag{6.26}$$

$$-r^{-1}\langle \alpha, z\rangle + r^{-\frac{3}{2}}\langle \eta, z\rangle\langle \xi,\xi\rangle + 2r^{-\frac{3}{2}}\mu\langle Ez,\xi\rangle = 0, \tag{6.27}$$

$$\beta r^{-1}\overline{N}(z,z) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle = \langle z,z\rangle, \tag{6.28}$$

$$r^{-1}\langle \alpha,z\rangle\overline{N}(z,z) + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,Ez\rangle = 0. \tag{6.29}$$

Claim 1: $E \in \mathrm{GL}(n-1)$**.**

Proof. Substituting (6.25) into (6.28) yields

$$\beta r^{-1}(N(Ez,Ez) - \langle \eta,z\rangle^2) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle = \langle z,z\rangle. \tag{6.30}$$

We multiply both sides of (6.30) by μ^2 and substitute (6.24) to obtain

$$\beta r^{-1}(\mu^2 N(Ez,Ez) - N(Ez,\xi)^2) + r^{-\frac{3}{2}}\mu^3\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\mu N(Ez,\xi)\langle Ez,\xi\rangle = \mu^2\langle z,z\rangle. \tag{6.31}$$

Assume $y \in \ker E$. Then (6.31) implies $0 = \mu^2\langle y,y\rangle$. Since $\mu \neq 0$ this implies $y = 0$. This proves our claim. ☐

Claim 2: $\alpha = 0$**.**

Proof. Suppose $\alpha \neq 0$. Substituting (6.25) into (6.29), we obtain

$$r^{-1}\langle \alpha, z\rangle (N(Ez, Ez) - \langle \eta, z\rangle^2) + r^{-\frac{3}{2}}\langle \eta, z\rangle \langle Ez, Ez\rangle = 0. \tag{6.32}$$

Multiply both sides of (6.32) by $r\mu^2$ and substitute (6.24) to obtain

$$\langle \alpha, z\rangle (\mu^2 N(Ez, Ez) - N(Ez, \xi)^2) + r^{-\frac{1}{2}}\mu N(Ez, \xi)\langle Ez, Ez\rangle = 0. \tag{6.33}$$

Claim 2.1: $\alpha \neq 0 \ \Rightarrow \ E^T N\xi = s\alpha$**.**

Proof. Equation (6.33) and $E \in \mathrm{GL}(n-1)$ show that $y \in \ker\langle \alpha, \cdot\rangle$ implies $N(Ey, \xi) = 0$. Hence, $N(E\cdot, \xi) = s\langle \alpha, \cdot\rangle$. □

Claim 2.2: $\alpha \neq 0 \ \Rightarrow \ s \neq 0, \ \xi \notin \ker N$**.**

Proof. Suppose that $s = 0$. Then (6.33) becomes $\langle \alpha, z\rangle N(Ez, Ez) = 0$ for all $z \in \mathbb{R}^{n-1}$. But $E \in \mathrm{GL}(n-1)$, $N \neq 0$, and $\alpha \neq 0$, so this is a contradiction. Since $E^T N\xi = s\alpha \neq 0$, it immediately follows that $\xi \notin \ker N$. □

Claim 2.3: $E^T\xi = t\alpha$**,** $t \neq 0$**.**

Proof. Equation (6.27) implies that α, η, and $E^T\xi$ are linearly dependent. Since $\eta = \mu^{-1}E^T N\xi = \mu^{-1}s\alpha$, it follows that $E^T\xi = t\alpha$. Then $t \neq 0$ follows from $E^T \in \mathrm{GL}(n-1)$ and $\xi \neq 0$. □

Claim 2.4: $\mathrm{sgn}(\mu) = \mathrm{sgn}(-s)$ **and** $\dim\ker N = 1$**.**

Proof. Observe that Claim 2.1-2.3 and $\alpha \neq 0$ show that (6.33) is equivalent to

$$\mu^2 N(Ez, Ez) - s^2\langle \alpha, z\rangle^2 + r^{-\frac{1}{2}}\mu s\langle Ez, Ez\rangle = 0.$$

Thus, for all $y \in \ker\langle \alpha, \cdot\rangle$ we have

$$\mu^2 N(Ey, Ey) + r^{-\frac{1}{2}}\mu s\langle Ey, Ey\rangle = 0.$$

$N \geq 0$ and $E \in \mathrm{GL}(n-1)$ imply that $\mu s < 0$, which shows $\mathrm{sgn}(\mu) = \mathrm{sgn}(-s)$. Since $\langle E\cdot, E\cdot\rangle|_{\ker\langle \alpha, \cdot\rangle} > 0$ it follows that $N(E\cdot, E\cdot)|_{\ker\langle \alpha, \cdot\rangle} > 0$. Hence, N is of rank $n-2$ or $n-1$, the latter being excluded by the assumption that $N \geq 0$ but not $N > 0$. □

Claim 2.5: $\mathrm{sgn}(s) = \mathrm{sgn}(t)$**.**

Proof. We have $\alpha = s^{-1}E^T N\xi$ and $\alpha = t^{-1}E^T\xi$. The invertibility of E shows $N\xi = st^{-1}\xi$. Since $\xi \notin \ker N$ and $N \geq 0$, it follows that $\mathrm{sgn}(st^{-1}) = 1$. □

To conclude the proof of Claim 2, multiply both sides of equation (6.27) by $r\mu$ and substitute (6.24) to obtain

$$-\mu\langle \alpha, z\rangle + r^{-\frac{1}{2}}\langle \xi, \xi\rangle N(Ez, \xi) + 2r^{-\frac{1}{2}}\mu^2\langle Ez, \xi\rangle = 0. \tag{6.34}$$

Claim 2.1-2.3 and $\alpha \neq 0$ show that (6.34) is equivalent to

$$-\mu + r^{-\frac{1}{2}}\langle \xi, \xi\rangle s + 2r^{-\frac{1}{2}}\mu^2 t = 0. \tag{6.35}$$

We have shown that all terms are non-vanishing and, by Claim 2.4-2.5, have the same sign. Hence, (6.35) cannot be true. This completes the proof of Claim 2, that is $\alpha = 0$. □

Claim 3: $\xi = \eta = 0$.

Proof. Since $\alpha = 0$, using (6.23) and (6.24) shows that equation (6.29) is equivalent to $N(E\cdot, \xi) = 0$. But $E \in \mathrm{GL}(n-1)$, thus it follows that $\xi \in \ker N$ and $\eta = 0$. Equation (6.27) and $E \in \mathrm{GL}(n-1)$ now show that $\xi = 0$. $\square$

Claim 4: $\beta = 0$, $\mu = 1$, **and** $E = r^{\frac{3}{4}} F$, $F \in \mathrm{O}(n-1)$.

Proof. Equation (6.26), $\xi = 0$, and $r > 0$ imply $\beta = 0$. Using $\xi = 0$ we see that equation (6.31) is equivalent to

$$r^{-\frac{3}{2}} \mu \langle Ez, Ez \rangle = \langle z, z \rangle. \tag{6.36}$$

Equations (6.23) and (6.36) are satisfied if and only if $\mu = 1$ and $r^{-\frac{3}{4}} E \in O(n-1)$, that is $E = r^{\frac{3}{4}} F$ with $F \in \mathrm{O}(n-1)$. $\square$

This finishes the proof of Lemma 6.4. $\square$

Now one can show in the exact same way as for the family $\mathcal{F}$ that each element of L has a unique representative in $\mathcal{G}$. Hence, the $(n-2)$-parameter family $\mathcal{G}$ consists of pairwise inequivalent hyperbolic homogeneous cubic polynomials, each defining a complete projective special real manifold of dimension n. This concludes the proof of Theorem 6.1. $\square$

A consequence of the Lemmata 6.2, 6.3, and 6.4 is the following corollary.

Corollary 6.5. *The automorphism groups of elements $h \in \mathcal{G}$ and $h \in \mathcal{F}$, $h \neq h_1 := x(-w^2 + \langle z, z \rangle) + w \sum\limits_{i=1}^{n-2} z_i^2$, are of the form*

$$\mathrm{Aut}(h) = \mathrm{O}(m_1) \times \ldots \times \mathrm{O}(m_k), \; 1 \leq k \leq n-1, \; \sum_{j=1}^{k} m_k = n-1.$$

The automorphism group of h_1 is generated by $\mathrm{O}(n-2)$ *and $\widehat{A}$ defined as*

$$\widehat{A} := \left(\begin{array}{c|c|c|c} \mathbb{1} & & & \\ \hline & \frac{1}{2} & \frac{-1}{2} & 1 \\ \hline & \frac{-1}{2} & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right),$$

i.e.

$$\mathrm{Aut}(h_1) \cong \mathrm{O}(n-2) \rtimes \mathbb{Z}_2.$$

In view of Corollary 6.5 also recall that the CCPSR manifolds associated to the polynomials *a*) and *b*) in Theorem 2.46 are homogeneous spaces (cf. equations (6.46) and (6.48), respectively (6.47) and (6.49) for a more detailed description). Next, we will show that the CCPSR manifolds defined by elements of $\mathcal{F}$ and $\mathcal{G}$ of the form (6.3) and (6.4), respectively, are always singular-at-infinity CCPSR manifolds, cf. Definition 3.16.

Proposition 6.6 (Singular-at-infinity property of $\mathcal{F}$ and $\mathcal{G}$). *Each CCPSR manifold $\mathcal{H}(h)$ for $h \in \mathcal{F}$ and $h \in \mathcal{G}$ as in (6.3) and (6.4), respectively, is singular at infinity.*

Proof. Let $h = x(-w^2+N(z,z))+wM(z,z)$, $\mathcal{H}(h) = \{h = 1 \mid x < 0, w < 0, -w^2+N(z,z) < 0\}$, where $M > 0$ or $N > 0$, and let $(z_1, \ldots, z_{n-1}, w, x)^T$ denote the linear coordinates of $\mathbb{R}^{n+1}$. We can assume without loss of generality that M and N are diagonal matrices, i.e.

$$M = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{n-1} \end{pmatrix}, \quad N = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_{n-1} \end{pmatrix}, \quad \mu_i \geq 0,\ \eta_i \geq 0\ \forall 1 \leq i \leq n-1. \tag{6.37}$$

We want to transform h so that it is of the form $\widetilde{h} = x^3 - x\langle y, y\rangle + P_3(y)$ and that $\mathcal{H}(h)$ is equivalent to the connected component $\widetilde{\mathcal{H}} \subset \left\{\widetilde{h} = 1\right\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. We start[12] with the linear transformation $A \in \mathrm{GL}(n+1)$ of the form

$$A = \left(\begin{array}{c|cc} 2^{-\frac{1}{3}}3^{\frac{1}{2}}\mathbb{1} & & \\ \hline & -2^{-\frac{1}{3}} & -2^{\frac{1}{6}} \\ & 2^{\frac{1}{6}} & -2^{-\frac{1}{3}} \end{array}\right), \quad A \cdot \begin{pmatrix} z \\ w \\ x \end{pmatrix} = \begin{pmatrix} 2^{-\frac{1}{3}}3^{\frac{1}{2}}z \\ -2^{-\frac{1}{3}}w - 2^{\frac{1}{6}}x \\ 2^{\frac{1}{6}}w - 2^{-\frac{1}{3}}x \end{pmatrix}. \tag{6.38}$$

Then

$$\begin{aligned} h\left(A \cdot \left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) &= x^3 \\ &\quad - x\left(2^{-1}3w^2 + 2^{-\frac{1}{2}}3M(z,z) + 2^{-1}3N(z,z)\right) \\ &\quad + \left(-2^{-\frac{1}{2}}w^3 + w\left(-2^{-1}3M(z,z) + 2^{-\frac{1}{2}}3N(z,z)\right)\right) \\ &= x^3 \\ &\quad - x\left(2^{-1}3w^2 + \sum_{i=1}^{n-1}\left(2^{-\frac{1}{2}}3\mu_i + 2^{-1}3\eta_i\right)z_i^2\right) \\ &\quad + \left(-2^{-\frac{1}{2}}w^3 + w\sum_{i=1}^{n-1}\left(-2^{-1}3\mu_i + 2^{-\frac{1}{2}}3\eta_i\right)z_i^2\right). \end{aligned}$$

After rescaling and relabelling the coordinates

$$\begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ w \\ x \end{pmatrix} \to \begin{pmatrix} \left(2^{-\frac{1}{2}}3\mu_1 + 2^{-1}3\eta_1\right)^{-\frac{1}{2}} y_1 \\ \vdots \\ \left(2^{-\frac{1}{2}}3\mu_{n-1} + 2^{-1}3\eta_{n-1}\right)^{-\frac{1}{2}} y_{n-1} \\ -2^{\frac{1}{2}}3^{-\frac{1}{2}}y_n \\ x \end{pmatrix}, \tag{6.39}$$

we see that h is equivalent to

$$\begin{aligned} \widetilde{h} &= x^3 - x\langle y, y\rangle + \widetilde{P_3}(y) \\ &= x^3 - x\langle y, y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 + \frac{\sqrt{2}}{\sqrt{3}}\sum_{i=1}^{n-1}\frac{\mu_i - \sqrt{2}\eta_i}{\sqrt{2}\mu_i + \eta_i}y_i^2\right). \end{aligned} \tag{6.40}$$

Since A maps $\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \in \widetilde{\mathcal{H}} \subset \left\{\widetilde{h} = 1\right\}$ to $\left(\begin{smallmatrix} 0 \\ -2^{\frac{1}{6}} \\ -2^{-\frac{1}{3}} \end{smallmatrix}\right) \in \mathcal{H}(h)$ (note: the linear map described in equation (6.39) maps the point $(z, w, x)^T = (0, 0, 1)^T$ to itself), we see that $\mathcal{H}(h)$

[12] The matrix A has been obtained with equation (3.7) in Proposition 3.18, up to the ordering and names of the coordinates.

is in fact equivalent to $\widetilde{\mathcal{H}}$ as required. Observe that

$$\max_{\|y\|=1} \widetilde{P_3}(y) \geq \widetilde{P_3}\Big(\underbrace{(0,\ldots,0,1)^T}_{=(y_1,\ldots,y_n)^T}\Big) = \frac{2}{3\sqrt{3}}$$

independently of the considered M and N, and since we already know that $\widetilde{\mathcal{H}}$ is a CCPSR manifold we conclude that $\max\limits_{\|y\|=1} \widetilde{P_3}(y) = \frac{2}{3\sqrt{3}}$, cf. Lemma 4.2. Now we use Lemma 4.6 and deduce that $\widetilde{\mathcal{H}}$ and, hence, $\mathcal{H}(h)$ are singular-at-infinity CCPSR manifolds independent of the considered M and N. In particular this shows that $\mathcal{H}(h)$ is singular at infinity for all $h \in \mathcal{F} \cup \mathcal{G}$. □

Theorem 6.1 and Proposition 6.6 now imply the following.

Corollary 6.7. *For $n \geq 3$, $n \in \mathbb{N}$, there exists a smooth curve $\gamma : [0,1] \to \mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$, such that $\gamma(0) = x(-w^2 + \langle z, z\rangle)$, that is the polynomial a) in Theorem 2.46, and $\gamma(1) = x(-w^2) + w\langle z, z\rangle$, which is equivalent to the polynomial b) in Theorem 2.46, with the property that for each $t \in (0,1)$, the level set $\{\gamma(t) = 1\}$ contains a CCPSR manifold that is singular at infinity.*

The above corollary is also true for $n = 1$ and $n = 2$. For $n = 1$, the polynomials a) and b) in Theorem 2.46 are equivalent, cf. [CHM, Cor. 4]. Furthermore, the corresponding CCPSR curve is equivalent to the connected component $\mathcal{H} \subset \left\{x^3 - xy^2 + \frac{2}{3\sqrt{3}}y^3 = 1\right\}$ containing $\binom{x}{y} = \binom{1}{0}$. Hence, they are both singular at infinity. For $n = 2$, one choice for γ is

$$\gamma(t) = x\left(-w^2 + (1-t)z^2\right) + twz^2.$$

If we compare these polynomials with [CDL, Thm. 1], we see that $\gamma(0)$ is equivalent to a), that is xyz, $\gamma(1)$ is equivalent to b), that is $x(xy - z^2)$, and $\gamma(t)$ for all $t \in (0,1)$ is equivalent to e), that is $x(y^2 - z^2) + y^3$. The corresponding CCPSR surfaces a), b), and e) are all singular at infinity, cf. the proof of Theorem 5.3 where this is shown.

Remark 6.8. Note that Proposition 5.8 automatically implies that there exists a continuous curve of singular-at-infinity CCPSR manifolds connecting the singular-at-infinity CCPSR manifolds corresponding to Theorem 2.46 a) and b), respectively. However, in Corollary 6.7 we show that such a curve can be chosen such that it is smooth and not only continuous.

Another important question about the structure of CCPSR manifolds defined by elements of $\mathcal{F}$ and $\mathcal{G}$ is whether they are homogeneous spaces or not. We will show that they are, in fact, never homogeneous spaces.

Proposition 6.9 (Inhomogeneity of $\mathcal{H}(h)$, $h \in \mathcal{F} \cup \mathcal{G}$)**.** *Let $h \in \mathcal{F} \cup \mathcal{G}$ and $\mathcal{H}(h)$ be the corresponding CCPSR manifold as in 6.3, respectively 6.4. Then $\mathcal{H}(h)$ is inhomogeneous as a Riemannian manifold*[13].

Proof. Recall that for h equivalent to an element in $\mathcal{F} \cup \mathcal{G}$, we have shown In the proof of Proposition 6.6 that the corresponding CCPSR manifold $\mathcal{H}(h) = \{h = 1 \mid x < 0, w < 0, -w^2 + N(z,z) < 0\}$ is equivalent to the connected component $\widetilde{\mathcal{H}} \subset \left\{\widetilde{h} = 1\right\}$ containing the point $\binom{x}{y} = \binom{1}{0} \in \left\{\widetilde{h} = 1\right\} \subset \mathbb{R}^{n+1}$ with

$$\widetilde{h} = x^3 - x\langle y, y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 + \frac{\sqrt{2}}{\sqrt{3}}\sum_{i=1}^{n-1}\frac{\mu_i - \sqrt{2}\eta_i}{\sqrt{2}\mu_i + \eta_i}y_i^2\right), \tag{6.41}$$

[13] And $\mathcal{H}(h)$ is in particular also inhomogeneous in the sense of Definition 3.9.

for fittingly chosen $\mu_k \geq 0$ and $\eta_k \geq 0$, $1 \leq k \leq n-1$, cf. (6.40). Note that the result corresponding to the formula 6.41 hold also true if h corresponds, up to equivalence, to the cases $M > 0$ and $N = 0$, respectively $M = 0$ and $N > 0$. For $M > 0$ and $N = 0$, h is equivalent to

$$h_{1,n} := x(-w^2) + w\langle z, z\rangle,$$

and its equivalent polynomial of the form (6.40) is given by

$$\widetilde{h}_{1,n} := x^3 - x\langle y, y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 + \frac{1}{\sqrt{3}}\sum_{i=1}^{n-1} y_i^2\right). \tag{6.42}$$

For $M = 0$ and $N > 0$, h is equivalent to

$$h_{2,n} := x(-w^2 + \langle z, z\rangle),$$

and its corresponding form (6.40) is

$$\widetilde{h}_{2,n} := x^3 - x\langle y, y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 - \frac{2}{\sqrt{3}}\sum_{i=1}^{n-1} y_i^2\right). \tag{6.43}$$

The CCPSR manifolds

$$\mathcal{H}_{1,n} \subset \left\{\widetilde{h}_{1,n} = 1\right\}, \quad \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{1,n}, \tag{6.44}$$

and

$$\mathcal{H}_{2,n} \subset \left\{\widetilde{h}_{2,n} = 1\right\}, \quad \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{2,n}, \tag{6.45}$$

are homogeneous spaces. By applying the supergravity q-map to each of them, $\mathcal{H}_{1,n}$ yields a symmetric quaternionic Kähler manifold and $\mathcal{H}_{2,n}$ yields a homogeneous non-symmetric quaternionic Kähler manifold[14], see [DV, C]. In fact, one can show with the notation

$$\left\{h_{1,n} = x(-w^2) + w\langle z, z\rangle = 1,\ x < 0,\ w < 0\right\} \cong \mathcal{H}_{1,n}, \tag{6.46}$$

$$\left\{h_{2,n} = x(-w^2 + \langle z, z\rangle) = 1,\ x < 0,\ w < 0\right\} \cong \mathcal{H}_{2,n}, \tag{6.47}$$

that

$$\mathcal{H}_{1,n} \cong \mathbb{R}_{>0} \ltimes \mathbb{R}^{n-1}, \tag{6.48}$$

$$\mathcal{H}_{2,n} \cong \frac{\mathbb{R}_{>0} \times \mathrm{SO}^+(n-1,1)}{\mathrm{SO}(n-1)} = \mathbb{R}_{>0} \times H^{n-1}, \tag{6.49}$$

where $(\lambda, v) \in \mathbb{R}_{>0} \ltimes \mathbb{R}^{n-1}$ acts on points in $\{h_{1,n} = 1,\ x < 0,\ w < 0\}$ via

$$(\lambda, v) \cdot \begin{pmatrix} z \\ w \\ x \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\lambda}}(z + wv) \\ \lambda w \\ \frac{1}{\lambda^2}(x + \langle v, v\rangle w + 2\langle z, v\rangle) \end{pmatrix},$$

$(\lambda_1, v_1)\cdot(\lambda_2, v_2) = \left(\lambda_1\lambda_2, v_1 + \lambda_1^{\frac{3}{2}} v_2\right)$ for all $(\lambda_1, v_1), (\lambda_2, v_2) \in \mathbb{R} \ltimes \mathbb{R}^{n-1}$, and H^{n-1} denotes the $(n-1)$-dimensional oriented hyperbolic space. In the following we will use the abbreviation

$$\sigma_k := \frac{\mu_k - \sqrt{2}\eta_k}{\sqrt{2}\mu_k + \eta_k}. \tag{6.50}$$

[14] We only consider $n \geq 3$, the corresponding space $\mathcal{T}(p)$ for $p = n-1 \geq 2$ is always not symmetric, cf. [C].

In particular, the P_3-term of $\widetilde{h}$ as in (6.41) is of the form

$$P_3(y) = y_n \left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \sigma_i y_i^2 \right) \tag{6.51}$$

and fulfils

$$P_3(\partial_i, \partial_j, \partial_k) = \begin{cases} \frac{\sqrt{2}}{3\sqrt{3}} \sigma_i \delta_i^j \delta_k^n, & 1 \leq i \leq n-1,\ 1 \leq j \leq n-1,\ 1 \leq k \leq n, \\ 0, & 1 \leq i \leq n-1,\ j = k = n, \\ \frac{2}{3\sqrt{3}}, & i = j = k = n. \end{cases} \tag{6.52}$$

In order to show that for all $h \in \mathcal{F} \cup \mathcal{G}$ the CCPSR manifolds $\mathcal{H}(h) = \{h = 1 \mid x < 0, w < 0, -w^2 + N(z,z) < 0\}$ are inhomogeneous as Riemannian manifolds, that is are never isometric to some Riemannian homogeneous space, we will study the first derivative of the scalar curvature of the corresponding CCPSR manifolds $\widetilde{\mathcal{H}}$ at the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \widetilde{\mathcal{H}}$, i.e. $dS_{\widetilde{\mathcal{H}}}\big|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}$. We use Proposition 3.30, equation (3.40), and obtain

$$dS_{\widetilde{\mathcal{H}}}\Big|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} (\partial_i) = 0 \tag{6.53}$$

for all $1 \leq i \leq n-1$, and

$$\begin{aligned} dS_{\widetilde{\mathcal{H}}}\Big|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} (\partial_n) = {} & \frac{n-1}{\sqrt{3}} + \frac{n-5}{\sqrt{6}} \left(\sum_{i=1}^{n-1} \sigma_i \right) + \frac{1}{2\sqrt{3}} \left(\sum_{i=1}^{n-1} \sigma_i^2 \right) + \frac{\sqrt{3}}{\sqrt{2}} \left(\sum_{i=1}^{n-1} \sigma_i^3 \right) \\ & - \frac{1}{2\sqrt{3}} \left(\sum_{i,j=1}^{n-1} \sigma_i \sigma_j \right) - \frac{1}{\sqrt{6}} \left(\sum_{i,j=1}^{n-1} \sigma_i \sigma_j^2 \right). \end{aligned} \tag{6.54}$$

Note that (6.53) and (6.54) also hold for $\widetilde{\mathcal{H}}_0$ and $\widetilde{\mathcal{H}}_1$, and actually vanish identically for $\widetilde{\mathcal{H}}_0$ and $\widetilde{\mathcal{H}}_1$ as one would expect. Next, consider (6.54) as a symbolic equation in the variables $(\sigma_1, \ldots, \sigma_{n-1})$, so that

$$\frac{\partial}{\partial \sigma_k} \left(dS_{\widetilde{\mathcal{H}}}\Big|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} (\partial_n) \right) = \sqrt{6} \sigma_k^2 + \left(-\frac{2}{\sqrt{6}} \sum_{i \neq k} \sigma_i \right) \sigma_k + \frac{n-5}{\sqrt{6}} - \frac{1}{\sqrt{6}} \sum_{i \neq k} \left(\sqrt{2} \sigma_i + \sigma_i^2 \right) \tag{6.55}$$

for all $1 \leq k \leq n-1$. We will treat the cases $N > 0$ (corresponding to $\mathcal{F}$) and $M > 0$ (corresponding to $\mathcal{G}$) separately.

Case 1: $N > 0$ **and** $M \geq 0$, $M \neq 0$.
After a possible linear transformation we can assume without loss of generality that $N(z,z) = \langle z, z \rangle$. Then $h = x(-w^2 + \langle z, z \rangle) + wM(z,z)$ is equivalent to $h_r := x(-w^2 + \langle z, z \rangle) + rwM(z,z)$ for all $r > 0$. This can be seen by considering the rescaling

$$\begin{pmatrix} z \\ w \\ x \end{pmatrix} \to \begin{pmatrix} r^{\frac{1}{3}} z \\ r^{\frac{1}{3}} w \\ r^{-\frac{2}{3}} x \end{pmatrix}. \tag{6.56}$$

Hence, $\mathcal{H}(h)$ is equivalent to $\mathcal{H}(h_r)$ for all $r > 0$ (since this is a continuous family of transformations with $r = 1$ associated to the identity transformation $\mathbb{1}$). Thus with

$$\widetilde{h}_r := x^3 - x\langle y, y \rangle + y_n \left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \frac{r\mu_i - \sqrt{2}}{r\sqrt{2}\mu_i + 1} y_i^2 \right), \tag{6.57}$$

(cf. (6.41)) we see that $\mathcal{H}(h)$ is equivalent to the CCPSR manifold $\widetilde{\mathcal{H}}_r \subset \left\{\widetilde{h}_r = 1\right\}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \widetilde{\mathcal{H}}_r$, for all $r > 0$. We define

$$\widetilde{\sigma}_k(r) := \frac{r\mu_k - \sqrt{2}}{r\sqrt{2}\mu_k + 1} \tag{6.58}$$

and obtain for all $1 \leq k \leq n-1$

$$\left.\frac{\partial}{\partial r}(\widetilde{\sigma}_k(r))\right|_{r=0} = 3\mu_k \tag{6.59}$$

and

$$\left.\frac{\partial^2}{\partial r^2}(\widetilde{\sigma}_k(r))\right|_{r=0} = -6\sqrt{2}\mu_k^2. \tag{6.60}$$

For the limit $r \to 0$ of $\widetilde{h}_r$, we obtain $\widetilde{h}_0 = \widetilde{h}_{2,n}$, see (6.43), which corresponds to $N = \mathbb{1}$ and $M = 0$. Note that equation (6.54) shows that $\left.dS_{\widetilde{\mathcal{H}}_r}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)$ is an analytic function in r around $r = 0$. Equation (6.55) at $\mathcal{H}_{2,n}$, that is at $r = 0$ respectively $\sigma_1 = \ldots = \sigma_{n-1} = -\sqrt{2}$, reads

$$\left.\frac{\partial}{\partial \sigma_k}\left(\left.dS_{\widetilde{\mathcal{H}}}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)\right)\right|_{\sigma_1=\ldots=\sigma_{n-1}=-\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}(-n+5) \tag{6.61}$$

for all $1 \leq k \leq n-1$. Thus, for $n \neq 5$, $\mu_k \geq 0$ for all $1 \leq k \leq n-1$ and the existence of at least one such $\mu_k > 0$ imply

$$\left.\frac{\partial}{\partial r}\left(\left.dS_{\widetilde{\mathcal{H}}_r}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)\right)\right|_{r=0} = \sum_{k=1}^{n-1} \frac{3\sqrt{3}}{\sqrt{2}}(-n+5)\mu_k \begin{cases} > 0, & n \in \{3,4\} \\ < 0, & n > 5. \end{cases} \tag{6.62}$$

Since $\left.dS_{\mathcal{H}_{2,n}}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n) = 0$, this shows that for all $n \neq 5$, $n \geq 3$, and for all $M \geq 0$, $M \neq 0$, M of the form (6.37), we can choose $r > 0$ small enough, such that $\left.dS_{\widetilde{\mathcal{H}}_r}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n) \neq 0$. Since $\widetilde{\mathcal{H}}_r$ is equivalent to $\mathcal{H}(h)$ for all $r > 0$, we conclude that $\mathcal{H}(h)$ is inhomogeneous.

It remains to take care of the cases with $\dim(\mathcal{H}(h)) = n = 5$. With the definitions above, Proposition 3.29 yields

$$S_{\widetilde{\mathcal{H}}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = n(1-n) - \frac{1}{\sqrt{2}}\left(\sum_{i=1}^{n-1}\sigma_i\right) + \frac{1}{2}\left(\sum_{i=1}^{n-1}\sigma_i^2\right) - \frac{1}{4}\sum_{\substack{i,j=1\\i\neq j}}^{n-1}\sigma_i\sigma_j. \tag{6.63}$$

We use the above equation (6.63) and obtain with $\sigma_k = \widetilde{\sigma}_k(r)$ for $1 \leq k \leq n-1$ (6.58) and the equations (6.59) and (6.60) for the first r-derivative

$$\left.\frac{\partial}{\partial r}S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\right|_{r=0} = \frac{3}{\sqrt{2}}(n-5)\sum_{i=1}^{n-1}\mu_i \overset{n=5}{=} 0$$

and for the second r-derivative

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial r^2}S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\right|_{r=0} \\
&= 3(-n+11)\left(\sum_{i=1}^{n-1}\mu_i^2\right) - \frac{9}{4}\sum_{\substack{i,j=1\\i\neq j}}^{n-1}\mu_i\mu_j \\
&\overset{n=5}{=} \frac{9}{4}\left(8\left(\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2\right) - 2\left(\mu_1\mu_2+\mu_1\mu_3+\mu_1\mu_4+\mu_2\mu_3+\mu_2\mu_4+\mu_3\mu_4\right)\right).
\end{aligned}$$

We want to prove $\frac{\partial^2}{\partial r^2} S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\Big|_{r=0} > 0$. One sees that

$$\frac{\partial^2}{\partial r^2} S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\Big|_{r=0} = \frac{9}{4} V\left(\left(\begin{smallmatrix}\mu_1\\\mu_2\\\mu_3\\\mu_4\end{smallmatrix}\right), \left(\begin{smallmatrix}\mu_1\\\mu_2\\\mu_3\\\mu_4\end{smallmatrix}\right)\right)$$

with

$$V = \begin{pmatrix} 8 & -1 & -1 & -1 \\ -1 & 8 & -1 & -1 \\ -1 & -1 & 8 & -1 \\ -1 & -1 & -1 & 8 \end{pmatrix}$$

viewed as a bilinear form. It thus suffices to show that all eigenvalues of V are positive, and indeed one finds that V has precisely one simple eigenvalue $\lambda_1 = 5$ and an eigenvalue of multiplicity 3, namely $\lambda_2 = 9$. Recall that, independent of $n \geq 3$, the map $r \mapsto S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$ is analytic around $r = 0$. Since all $\mu_1, \ldots, \mu_4$ are non-negative and at least one value of $\mu_1, \ldots, \mu_4$ is positive ($M \geq 0$, $M \neq 0$), we have shown that

$$\frac{\partial^2}{\partial r^2} S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\Big|_{r=0} = \frac{9}{4} V\left(\left(\begin{smallmatrix}\mu_1\\\mu_2\\\mu_3\\\mu_4\end{smallmatrix}\right), \left(\begin{smallmatrix}\mu_1\\\mu_2\\\mu_3\\\mu_4\end{smallmatrix}\right)\right) > 0.$$

Hence, there exists $\varepsilon > 0$, such that $r \mapsto S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$ is strictly monotonously increasing for all $r \in (0, \varepsilon)$, and in particular $S_{\widetilde{\mathcal{H}}_r}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) > S_{\mathcal{H}_{2,n}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$. Since $\widetilde{\mathcal{H}}_r$ and $\mathcal{H}(h)$ are equivalent for all $r > 0$, we have shown that $S_{\mathcal{H}(h)}$ is not constant for any allowed initial choices of $\mu_1, \ldots, \mu_4$. This proves that the CCPSR manifold $\mathcal{H}(h)$ cannot be homogeneous.

Case 2: $M > 0$ and $N \geq 0$, $N \neq 0$.
We proceed similarly to **Case 1**. There is no special case for the dimension of $\mathcal{H}(h)$ in comparison with **Case 1** for $\dim(\mathcal{H}(h)) = 5$. As in **Case 1**, we can assume without loss of generality that $M = \mathbb{1}$. Then for all $r > 0$, $h = x(-w^2 + N(z,z)) + wM(z,z)$ is equivalent to

$$\overline{h}_r = x^3 - x\langle y, y\rangle + y_n\left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \frac{1 - r\sqrt{2}\eta_i}{\sqrt{2} + r\eta_i} y_i^2\right),$$

and $\mathcal{H}(r)$ is equivalent to the CCPSR manifold $\overline{\mathcal{H}}_r \subset \left\{\overline{h}_r = 1\right\}$, $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \overline{\mathcal{H}}_r$. Similar to (6.58) we define for $1 \leq k \leq n-1$

$$\overline{\sigma}_k(r) := \frac{1 - r\sqrt{2}\eta_k}{\sqrt{2} + r\eta_k}, \tag{6.64}$$

so that

$$\frac{\partial}{\partial r}\left(\overline{\sigma}_k(r)\right)\Big|_{r=0} = -\frac{3}{2}\eta_k. \tag{6.65}$$

Equation (6.55) for the limit $r \to 0$, i.e. at $\mathcal{H}_{1,n} \subset \left\{\overline{h}_0 = \widetilde{h}_{1,n} = 1\right\}$ respectively $\sigma_1 = \ldots = \sigma_{n-1} = \frac{1}{\sqrt{2}}$, reads

$$\frac{\partial}{\partial \sigma_k}\left(dS_{\widetilde{\mathcal{H}}}\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)\right)\Big|_{\sigma_1 = \ldots = \sigma_{n-1} = 1/\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}}(-n+2) < 0 \quad \forall n \geq 3.$$

Hence,

$$\frac{\partial}{\partial r}\left(dS_{\overline{\mathcal{H}}_r}\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)\right)\Big|_{r=0} = \sum_{k=1}^{n-1} \frac{3\sqrt{3}}{4\sqrt{2}}(n-2)\eta_k > 0$$

since $n \geq 3$ and at least one η_k is positive, and the others are always non-negative. Since $dS_{\overline{\mathcal{H}}_r}\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n)$ is analytic around $r=0$, there thus exists $r>0$ small enough, such that $dS_{\overline{\mathcal{H}}_r}\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}(\partial_n) \neq 0$. This proves that $\mathcal{H}(h)$ is inhomogeneous.

Summarising, we have shown that for all $h \in \mathcal{F} \cup \mathcal{G}$ the corresponding CCPSR manifold $\mathcal{H}(h)$ as in (6.3) respectively (6.4) is inhomogeneous. □

Lemma 6.10 (Scalar curvature of $\mathcal{H}_{1,n}$ and $\mathcal{H}_{2,n}$)**.** *For $n \geq 3$,*

$$\begin{aligned} S_{\mathcal{H}_{1,n}} &\equiv -\frac{9}{8}n(n-1),\\ S_{\mathcal{H}_{2,n}} &\equiv \left(-\frac{3}{2}n+3\right)(n-1). \end{aligned}$$

In particular $S_{\mathcal{H}_{1,n}} < 0$ and $S_{\mathcal{H}_{2,n}} < 0$.

Proof. This easily follows from $n \geq 3$ and equation (6.63) with $\sigma_1 = \ldots = \sigma_{n-1} = \frac{1}{\sqrt{2}}$ for $\mathcal{H}_{1,n}$, and $\sigma_1 = \ldots = \sigma_{n-1} = -\sqrt{2}$ for $\mathcal{H}_{2,n}$. □

Remark 6.11. We have seen in Proposition 6.9 that the limits for $r \to 0$ of $\overline{\mathcal{H}}_r$ and $\widetilde{\mathcal{H}}_r$ are $\mathcal{H}_{1,n}$ and $\mathcal{H}_{2,n}$, respectively. For elements $h \in \mathcal{F} \cup \mathcal{G}$ with $\mathrm{rk}(M) = \mathrm{rk}(N) = n-1$, we can thus interpret $\mathcal{H}(h)$ as an inhomogeneous CCPSR manifold "interpolating" the scalar curvature between the homogeneous CCPSR manifolds manifolds $\mathcal{H}_{1,n}$ and $\mathcal{H}_{2,n}$ along the curve in $\mathcal{H}(h)$ generated by the curve

$$r \mapsto \begin{pmatrix} 0 \\ -2^{\frac{1}{6}} r^{\frac{1}{3}} \\ -2^{-\frac{1}{3}} r^{-\frac{2}{3}} \end{pmatrix} \in \mathcal{H}(h),$$

cf. (6.38) and (6.56).

Remark 6.12 (Candidates for sharp $S_{\mathcal{H}}$-bounds for $\dim(\mathcal{H}) \geq 3$)**.** Note that setting $n=2$ in the formulas for $S_{\mathcal{H}_{1,n}}$ and $S_{\mathcal{H}_{2,n}}$ in Lemma 6.10 yields $-\frac{9}{4}$ and 0, respectively. Recall that these are precisely the sharp $S_{\mathcal{H}}$-bounds for CCPSR surfaces $\mathcal{H}$ as we have seen in Proposition 5.12. Thus, the values $-\frac{9}{8}n(n-1)$ and $\left(-\frac{3}{2}n+3\right)(n-1)$ in Lemma 6.10, or one of them, might provide general bounds for the scalar curvature of $n \geq 3$-dimensional CCGPSR manifolds, but as mentioned before after the proof of Proposition 5.12 we will leave this as a problem for future studies.

Recall the definition of $\mathcal{C}_n$ in (5.37), and Definition 3.2. With Theorem 5.6, the existence of the pair-wise inequivalent $(n-2)$-parameter families of CCPSR manifolds $\mathcal{F}$, respectively $\mathcal{G}$, and Propositions 6.6 and 6.9, we have gained following information about the moduli space of CCPSR manifolds.

Corollary 6.13 (Lower bound of minimal number of parameters of "maximal" parameter families)**.** *For $n \geq 3$ there exists an $(n-2)$-parameter family of pairwise inequivalent singular-at-infinity inhomogeneous CCPSR manifolds $\mathcal{H}_{(t_1,\ldots,t_{n-2})}$ of dimension n. In particular, for each such $\mathcal{H}_{(t_1,\ldots,t_{n-2})}$ there exists $P_3 \in \mathrm{Sym}^3\left(\mathbb{R}^n\right)^*$, such that $h = x^3 - x\langle y,y\rangle + P_3(y) \in \partial\mathcal{C}_n$ and $\left[\mathcal{H}_{(t_1,\ldots,t_{n-2})}\right] = [\mathcal{H}]$, where $\mathcal{H} \subset \{h=1\}$ is the CCPSR manifold containing the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \{h=1\} \subset \mathbb{R}^{n+1}$. This also means that a maximal multi-parameter family of pairwise inequivalent $n \geq 3$-dimensional CCPSR manifolds depending on $m \in \mathbb{N}$ parameters in the sense that there exists no multi-parameter family of pairwise inequivalent $n \geq 3$-dimensional CCPSR manifolds that depends on $m+1$ parameters fulfils $m \geq n-2$.*

Remark 6.14 (Implications and for a possible topology on the moduli-space of CCPSR manifolds in dimension $n \geq 3$)**.** We can use the results of this section to describe one problem in finding a meaningful topology on the moduli space of $n \geq 3$-dimensional CCPSR manifolds as defined in Definition 3.2. Recall that we have always viewed that moduli space as a set, see Remark 3.3. Consider for $r > 0$ the cubic polynomial $\widetilde{h}_r$ in (6.57) and the corresponding CCPSR manifold $\widetilde{\mathcal{H}}_r \subset \{\widetilde{h}_r = 1\}$. Further assume that in (6.57) $\mu_i > 0$ for all $1 \leq i \leq n-1$. Then, when viewing $\widetilde{h}_r$ as an element of the vector space $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$,

$$\lim_{r\to 0} \widetilde{h}_r = \widetilde{h}_{2,n}, \quad \lim_{r\to\infty} \widetilde{h}_r = \widetilde{h}_{1,n},$$

cf. (6.43) and (6.42). However, we have seen in **Case 1** in the proof of Proposition 6.9 that all CCPSR manifolds $\widetilde{\mathcal{H}}_r$ are equivalent to some $\mathcal{H}(h)$ for $h \in \mathcal{F} \cup \mathcal{G}$ (cf. Theorem 6.1) independent of $r > 0$, e.g. for $h = x\left(-w^2 + \langle z, z\rangle\right) + w\langle z, z\rangle \in \mathcal{F} \cap \mathcal{G}$. Hence, if one considers the topology on the moduli space of CCGPSR manifolds of dimension $n \geq 3$ induced by the equivalence of hyperbolic cubic homogeneous polynomials, then the moduli space of CCGPSR manifolds of dimension $n \geq 3$ would not be a Hausdorff space since the then constant sequence of equivalence classes

$$\left\{\left[\widetilde{h}_{e^k}\right],\ k \in \mathbb{Z}\right\}$$

would have two distinct limits, namely $\left[\widetilde{h}_{2,n}\right]$ for $k \to -\infty$ and $\left[\widetilde{h}_{1,n}\right]$ for $k \to \infty$ (recall that $\mathcal{H}_{2,n}$ (6.45) and $\mathcal{H}_{1,n}$ (6.44) are not equivalent for $n \geq 3$).

7 Geometry and examples of quartic generalized projective special real manifolds

In this section we will be concerned with quartic CCGPSR manifolds, i.e. CCGPSR manifolds of homogeneity-degree $\tau = 4$. Proposition 3.18 implies that any n-dimensional quartic CCGPSR manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ can without loss of generality be assumed to be the connected component of

$$\left\{ h = x^4 - x^2 \langle y, y \rangle + x P_3(y) + P_4(y) = 1 \right\} \subset \mathbb{R}^{n+1}$$

that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h = 1\} \subset \mathbb{R}^{n+1}$. Recall that CCPSR manifolds are complete, which was first shown in [CNS] and for which we found two alternative proofs, see Proposition 4.17 and Proposition 5.17. An important and still open problem is the following.

Open problem 7.1 (Completeness of quartic CCGPSR manifolds)**.** *Let $\mathcal{H}$ be a quartic CCGPSR manifold of dimension n. Is $\mathcal{H}$ being closed in the ambient space $\mathbb{R}^{n+1}$ equivalent to $\mathcal{H}$ being geodesically complete with respect to its centro-affine fundamental form?*

For dimension $n = 1$, that is for quartic CCGPSR curves, we know that they are always complete, see [CNS, Thm. 2.9]. But for quartic CCGPSR manifolds of dimension $n \geq 2$, the question of geodesic completeness is not solved yet. We will provide partial results to this question, that is we will give examples of complete quartic CCGPSR manifolds for each dimension $n \in \mathbb{N}$ and we will completely classify quartic CCGPSR curves up to equivalence.

Theorem 7.2 (Classification of quartic CCGPSR curves)**.** *Any quartic CCGPSR curve is equivalent to the connected component $\mathcal{H}$ of the level set $\{h = 1\} \subset \mathbb{R}^2$ which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ for precisely one of the following polynomials h. The respective level set $\{h = 1\}$ contains the following closed connected hyperbolic subsets, and the automorphism group G^h of h has the following properties:*

a) $h = x^4 - x^2 y^2 + \frac{1}{4} y^4$, $\{h = 1\}$ *has 4 equivalent closed hyperbolic connected components, and*

$$G^h \cong \mathrm{SO}^+(1,1) \times \mathbb{Z}_4 \times \mathbb{Z}_2,$$

where the $\mathrm{SO}^+(1,1)$*-factor acts by hyperbolic rotations with respect to the metric* $-2dx^2 + dy^2$ *(7.47),*

b) $h = x^4 - x^2 y^2 + \frac{2\sqrt{2}}{3\sqrt{3}} x y^3 - \frac{1}{12} y^4 = 1$, $\{h = 1\}$ *has 2 equivalent closed hyperbolic connected components, and*

$$G^h \cong \mathbb{R} \times \mathbb{Z}_2,$$

where the $\mathbb{R}$*-factor acts on* $\mathcal{H}$ *as described in equation (7.58),*

c) $h = x^4 - x^2 y^2 + \frac{2}{3\sqrt{3}} x y^3 = 1$, $\{h = 1\}$ *has 4 equivalent closed hyperbolic connected components, and*

$$G^h \cong \mathbb{Z}_4,$$

d) $h = x^4 - x^2 y^2 + K y^4 = 1$ *for exactly one* $K < \frac{1}{4}$*. The set* $\{h = 1\}$ *has 4 equivalent closed hyperbolic connected components if* $0 < K < \frac{1}{4}$ *with*

$$G^h \cong \mathbb{Z}_4 \times \mathbb{Z}_2,$$

and 2 equivalent closed hyperbolic connected components if $K \leq 0$ *with*

$$G^h \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In the cases a) and b) the respective maximal connected subgroup $G_0^h \subset G^h$ acts simply transitively on the curve $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be a maximal quartic (not necessarily closed) connected GPSR curve. Proposition 3.18 implies that we can assume without loss of generality that $\mathcal{H} = \mathcal{H}_{L,K}$ is the maximally extended quartic GPSR curve contained in the level set $\{h_{L,K} = 1\} \subset \mathbb{R}^2$,

$$h_{L,K} = x^4 - x^2y^2 + Lxy^3 + Ky^4 \tag{7.1}$$

for some $(L,K)^T \in \mathbb{R}^2$, which contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. This means that $P_3(y) = Ly^3$ and $P_4(y) = Ky^4$ in equation (3.12), leading to (7.1). We will say that the polynomial $h_{L,K}$ "corresponds" to the point $(L,K)^T \in \mathbb{R}^2$. On the other hand, note that for all $(L,K)^T \in \mathbb{R}^2$, the maximal open connected hyperbolic subset of the connected component of $\{h_{L,K} = 1\}$ that contains $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is always a connected quartic GPSR curve. This proof primarily relies on the properties of $\delta P_3(y)$ and $\delta P_4(y)$, defined in Definition 3.27. Since $\dim(\mathcal{H}_{L,K}) = 1$, the term dB_0 in equations (3.31) and (3.32) vanishes and we calculate

$$\delta P_3(y) = \left(\frac{9}{2}L^2 + 4K - 1\right) y^3 dz, \tag{7.2}$$

$$\delta P_4(y) = L\left(6K + \frac{1}{2}\right) y^4 dz. \tag{7.3}$$

In the above formulas, z denotes the induced coordinate of $\mathrm{dom}(\mathcal{H}_{L,K})$, cf. Definition 3.22. This motivates the consideration of the vector field $\mathcal{V} \in \Gamma\left(T\mathbb{R}^2\right)$ that is given by

$$\mathcal{V} = \mathcal{V}_{\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)} := \left(\frac{9}{2}L^2 + 4K - 1\right)\partial_L + L\left(6K + \frac{1}{2}\right)\partial_K, \tag{7.4}$$

see Figure 8 for a plot[15] of $\mathcal{V}$. We denote by

$$\{\mathcal{V} = 0\} := \left\{ \begin{pmatrix} L \\ K \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \mathcal{V}_{\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)} = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{2\sqrt{2}}{3\sqrt{3}} \\ -\frac{1}{12} \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{2}}{3\sqrt{3}} \\ -\frac{1}{12} \end{pmatrix} \right\}. \tag{7.5}$$

$\mathcal{V}$ has the property that the polynomials

$$h_{\gamma_L(t),\gamma_K(t)} = x^4 - x^2y^2 + \gamma_L(t)xy^3 + \gamma_K(t)y^4 \tag{7.6}$$

associated to each integral curve[16] and in particular each maximal integral curve γ of the restricted vector field $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$,

$$t \mapsto \gamma(t) = \begin{pmatrix} \gamma_L(t) \\ \gamma_K(t) \end{pmatrix} \in \mathbb{R}^2, \quad \mathcal{V}_\gamma = \dot{\gamma}, \quad \gamma_L(0) = L,\ \gamma_K(0) = K,$$

for all initial $(L,K)^T \in \mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$, are equivalent to $h_{L,K}$. To see that this is true, let $\mathcal{H}_{L,K}$ be the maximal open connected hyperbolic subset of the connected component of $\{h_{L,K} = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. We will use the techniques of Propositions 3.18 and 3.26. First, we need to calculate $\mathcal{A} : \mathrm{dom}(\mathcal{H}_{L,K}) \to \mathrm{GL}(2)$ as in (3.23) and find

$$\mathcal{A}(T) = \left(\begin{array}{c|c} \frac{1}{\sqrt[4]{h_{L,K}\left(\left(\begin{smallmatrix} 1 \\ T \end{smallmatrix}\right)\right)}} & \frac{2T - 3LT^2 - 4KT^3}{4 - 2T^2 + LT^3} r(L,K,T) \\ \hline \frac{T}{\sqrt[4]{h_{L,K}\left(\left(\begin{smallmatrix} 1 \\ T \end{smallmatrix}\right)\right)}} & r(L,K,T) \end{array}\right)$$

with

[15] The plot was created with MAPLE, using the option `fieldstrength=average(9/16)` for better visibility.

[16] We assume in the following that integral curves are connected and parametrised over an open interval.

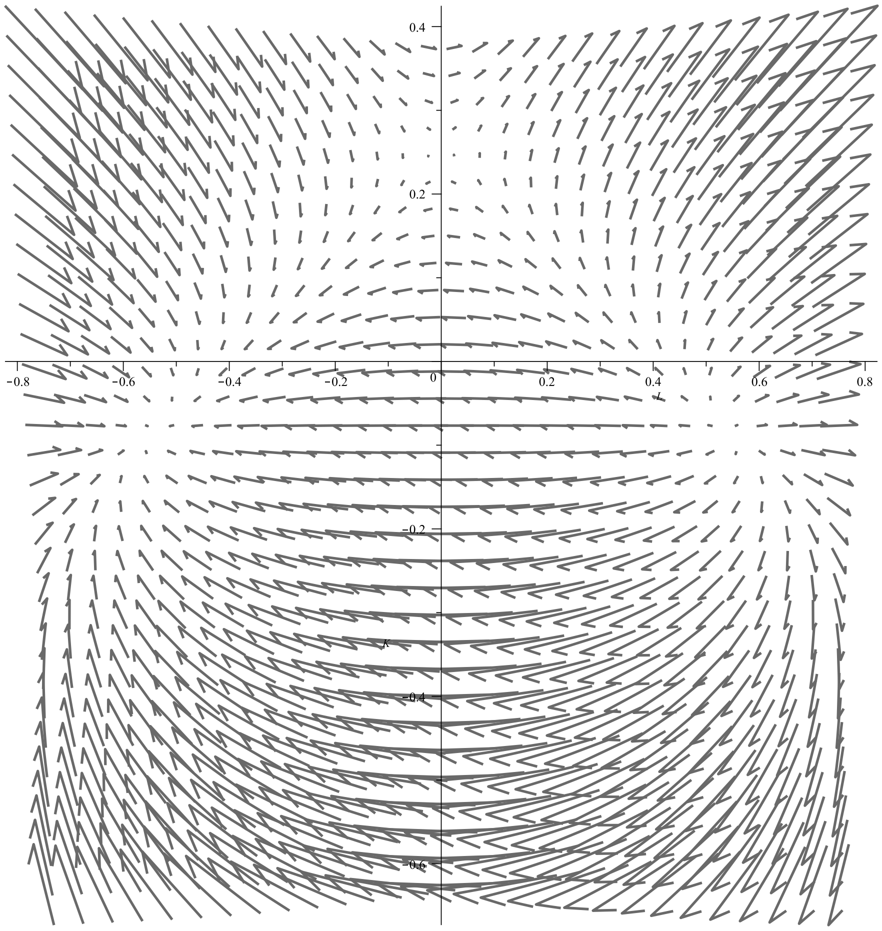


Figure 8: The vector field $\mathcal{V}$ (7.4) plotted with MAPLE. The horizontal axis is the L-axis and the vertical axis is the K-axis.

$$\begin{aligned}
r(L,K,T) = {} & \sqrt[4]{h_{L,K}\left(\left(\begin{smallmatrix}1\\T\end{smallmatrix}\right)\right)}\left(4-2T^2+LT^3\right)\\
& \cdot\Big(16-48LT+(-8-96K)T^2+56LT^3+\left(-8-42L^2+128K\right)T^4\\
& +(16-144LK)T^5+\left(-14L^2-8K-96K^2\right)T^6\\
& +\left(6L^3+8LK\right)T^7+\left(6L^2K+16K^2\right)T^8\Big)^{-\frac{1}{2}}.
\end{aligned}$$

Note that in order to see that $\mathcal{A}(T)$ is actually well-defined for all $T \in \operatorname{dom}(\mathcal{H}_{L,K})$ and not just in some neighbourhood of $0 \in \operatorname{dom}(\mathcal{H}_{L,K})$ as implied by Proposition 3.18, we need to show that $\partial_x h_{L,K}|_{\Phi(T)} > 0$ for all $T \in \operatorname{dom}(\mathcal{H}_{L,K})$, where $\Phi : \operatorname{dom}(\mathcal{H}_{L,K}) \to \mathcal{H}_{L,K} \subset \mathbb{R}^2$ denotes the diffeomorphism defined as in equation (3.14). To see that this is true, suppose that there exists a point $\overline{T} \in \operatorname{dom}(\mathcal{H}_{L,K})$, such that $\partial_x h_{L,K}|_{\Phi(\overline{T})} = 0$. We can assume without loss of generality that $\overline{T} > 0$ (after a possible sign-flip in the y-coordinate of the ambient space $\mathbb{R}^2 \supset \mathcal{H}_{L,K}$) and we can further assume that $\overline{T}$ is minimal in the sense that for all $t \in \left[0,\overline{T}\right)$ we have $\partial_x h_{L,K}|_{\Phi(t)} > 0$ (recall that $\partial_x h_{L,K}|_{\Phi(0)} = 4$ for all choices of L and K). Let $\mathrm{pr}_x : \mathbb{R}^2 \to \mathbb{R}$ and $\mathrm{pr}_y : \mathbb{R}^2 \to \mathbb{R}$ denote the projections to the x- and y-coordinate of the ambient space $\mathbb{R}^2$, respectively, and write the set $\Phi\left(\left[0,\overline{T}\right]\right) \subset \mathcal{H}_{L,K}$ as the graph of a function $\mu : \left[0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right] \to \mathbb{R}$ which is uniquely determined by the system of equations

$$h_{L,K}\left(\left(\begin{smallmatrix}0\\t\end{smallmatrix}\right)+\mu(t)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = 1 \quad \forall t \in \left[0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right], \quad \mu(0) = 1.$$

The function μ is smooth on $\left[0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right)$ and fulfils

$$\partial_t^2\mu(t)\partial_x h_{L,K}|_{\left(\begin{smallmatrix}0\\t\end{smallmatrix}\right)+\mu(t)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} + \partial^2 h_{L,K}|_{\left(\begin{smallmatrix}0\\t\end{smallmatrix}\right)+\mu(t)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}\left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)+\partial_t\mu(t)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)+\partial_t\mu(t)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) = 0 \quad (7.7)$$

for all $t \in \left[0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right)$. Note that the hyperbolicity of points in $\Phi\left(\left[0,\overline{T}\right)\right)$, the property that $\partial_x h_{L,K}|_{\Phi(t)} > 0$ for all $t \in \left[0,\overline{T}\right)$, and equation (7.7) thus imply that $\partial_t^2\mu(t) > 0$ for all $t \in \left[0,\overline{T}\right)$ and, hence, show that μ is a strictly convex function. Since the point

$$\Phi\left(\overline{T}\right) = \left(\begin{smallmatrix}0\\ \mathrm{pr}_y\Phi\left(\overline{T}\right)\end{smallmatrix}\right) + \mu\left(\mathrm{pr}_y\Phi\left(\overline{T}\right)\right)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{L,K}$$

is also a hyperbolic point of $h_{L,K}$ and furthermore the vector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is tangent to $\mathcal{H}_{L,K} \subset \mathbb{R}^2$ at the point $\Phi\left(\overline{T}\right) \in \mathcal{H}_{L,K}$ by assumption that $\partial_x h_{L,K}|_{\Phi\left(\overline{T}\right)} = 0$, there exists a positive real number $R > 0$, such that

$$h_{L,K}\left(\Phi\left(\overline{T}\right) + R\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right) \in (0,1).$$

With that in mind we now define

$$\widetilde{\mu} : \left[0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right] \to \mathbb{R}, \quad \widetilde{\mu}(t) := t\frac{\mathrm{pr}_x\Phi\left(\overline{T}\right) + R}{\mathrm{pr}_y\Phi\left(\overline{T}\right)}.$$

It follows that the graphs of μ and $\widetilde{\mu}$ must have an intersection point $\left(\begin{smallmatrix}\mu(\widehat{t})\\ \widehat{t}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\widetilde{\mu}(\widehat{t})\\ \widehat{t}\end{smallmatrix}\right)$ for some $\widehat{t} \in \left(0, \mathrm{pr}_y\Phi\left(\overline{T}\right)\right)$. But by construction $h_{L,K}\left(\left(\begin{smallmatrix}\widetilde{\mu}(\overline{T})\\ \mathrm{pr}_y(\overline{T})\end{smallmatrix}\right)\right) \in (0,1)$, which by the linearity of $\widetilde{\mu}$ and the homogeneity of $h_{L,K}$ of degree 4 implies that

$$h_{L,K}\left(\left(\begin{smallmatrix}\widetilde{\mu}(\widehat{t})\\ \widehat{t}\end{smallmatrix}\right)\right) = h_{L,K}\left(\left(\begin{smallmatrix}\mu(\widehat{t})\\ \widehat{t}\end{smallmatrix}\right)\right) \in (0,1).$$

This contradicts the assumption that $\left(\begin{smallmatrix}\mu(\widehat{t})\\ \widehat{t}\end{smallmatrix}\right) \in \mathcal{H}_{L,K}$ and, hence, proves the claim that $\partial_x h_{L,K}|_{\Phi(T)} > 0$ for all $T \in \mathrm{dom}\left(\mathcal{H}_{L,K}\right)$, which shows that $\mathcal{A}(T)$ is well-defined for all $T \in \mathrm{dom}\left(\mathcal{H}_{L,K}\right)$. With

$$h_{L(T),K(T)} = h_{L,K} \circ \mathcal{A}(T) = x^4 - x^2y^2 + L(T)xy^3 + K(T)y^4$$

we then obtain

$$\begin{aligned}
L(T) = {} & -2\sqrt{2}\sqrt{h_{L,K}\left(\left(\begin{smallmatrix}1\\T\end{smallmatrix}\right)\right)} \\
& \cdot\left(-8L + (8-32K)T - 20LT^2 + 20L^2T^3 + 40LKT^4\right. \\
& \quad \left.+\left(-2L^2 - 8K + 32K^2\right)T^5 + \left(L^3 + 4LK\right)T^6\right) \\
& \cdot\left(8 - 24LT + (4-48K)T^2 - 4LT^3 + \left(3L^2 + 8K\right)T^4\right)^{-1} \\
& \cdot\left(8 - 24LT + (-4-48K)T^2 + 28LT^3 + \left(-4 - 21L^2 + 64K\right)T^4\right. \\
& \quad +(8L - 72LK)T^5 + \left(-7L^2 - 4K - 48K^2\right)T^6 \\
& \quad \left.+\left(3L^3 + 4LK\right)T^7 + \left(3L^2K + 8K^2\right)T^8\right)^{-\frac{1}{2}}
\end{aligned} \quad (7.8)$$

and

$$K(T) = \frac{1}{4}\left(256K + 128LT + \left(-64 - 192L^2 - 256K\right)T^2 + (128L - 256LK)T^3\right.$$

$$
\begin{aligned}
&+\left(16-80L^2+384K-256K^2\right)T^4+\left(48L^3-32L-512LK\right)T^5\\
&+\left(8L^2+352L^2K-64K-256K^2\right)T^6+\left(8L^3+64LK+512LK^2\right)T^7\\
&+\left(-3L^4-16L^2K+256K^3\right)T^8\Big)\\
\cdot&\left(8-24LT+(4-48K)T^2-4LT^3+\left(3L^2+8K\right)T^4\right)^{-2},
\end{aligned}
\tag{7.9}
$$

where $L(0)=L$, $K(0)=K$. Note that

$$
\frac{\partial}{\partial T}\begin{pmatrix} L(T)\\ K(T)\end{pmatrix}\Bigg|_{T=0}=\mathcal{V}_{\left(\begin{smallmatrix} L\\ K\end{smallmatrix}\right)}
\tag{7.10}
$$

as expected. By construction we know that for all $T\in\operatorname{dom}(\mathcal{H}_{L,K})$, $h_{L(T),K(T)}$ and $h_{L,K}$ are equivalent. Since $\mathcal{A}$ depends smoothly on T, the maximal connected open hyperbolic subsets of the connected component of $\{h_{L(T),K(T)}=1\}$ that contain the point $\left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right)=\left(\begin{smallmatrix} 1\\ 0\end{smallmatrix}\right)$ are also equivalent for all $T\in\operatorname{dom}(\mathcal{H}_{L,K})$. The velocities of the considered integral curve γ of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ and the curve $T\mapsto\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)$ will in general not identically coincide[17], but we will show now that the image of γ is always contained in the image of $T\mapsto\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)$. The constant (maximally extended) integral curves of $\mathcal{V}$ are precisely those with initial data as in $\{\mathcal{V}=0\}$ described in equation (7.5). One now checks that for $\left(\begin{smallmatrix} L\\ K\end{smallmatrix}\right)\in\{\mathcal{V}=0\}$, $L(T)\equiv L(0)=L$ and $K(T)\equiv K(0)=K$. Hence, for constant integral curves the maps $T\mapsto L(T)$ in (7.8) and $T\mapsto K(T)$ in (7.9) are constant and thus the images of γ and $(L(T),K(T))^T$ in $\mathbb{R}^2$ coincide. For all $\left(\begin{smallmatrix} L\\ K\end{smallmatrix}\right)\in\mathbb{R}^2\setminus\{\mathcal{V}=0\}$ one can now verify that

$$
dL\left(\mathcal{V}_{\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)}\right)\cdot\partial_T K(T)=dK\left(\mathcal{V}_{\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)}\right)\cdot\partial_T L(T)
$$

for all $T\in\operatorname{dom}(\mathcal{H}_{L,K})$. One further shows that for all $\left(\begin{smallmatrix} L\\ K\end{smallmatrix}\right)\in\mathbb{R}^2\setminus\{\mathcal{V}=0\}$

$$
\frac{dL\left(\mathcal{V}_{\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)}\right)}{\partial_T L(T)}=\frac{dK\left(\mathcal{V}_{\left(\begin{smallmatrix} L(T)\\ K(T)\end{smallmatrix}\right)}\right)}{\partial_T K(T)}=\frac{2\sqrt{2}h_{L,K}\left(\left(\begin{smallmatrix} 1\\ T\end{smallmatrix}\right)\right)}{\sqrt{8-24LT+(4-48K)T^2-4LT^3+(3L^2+8K)T^4}}
\tag{7.11}
$$

which is well defined and positive for $T\in\operatorname{dom}(\mathcal{H}_{L,K})$ small enough (a priori we might have zeros in the denominator of (7.11), hence the restriction). Now suppose that there exists a maximal integral curve $\gamma=(\gamma_L,\gamma_K)^T: I\to\mathbb{R}^2\setminus\{\mathcal{V}=0\}$ of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$, such that at least two quartic GPSR curves, and thus also the corresponding polynomials, associated to two points in the image of γ are not equivalent. Then for any fixed $w\in\gamma(I)$ there exists $\varepsilon>0$, such that all polynomials corresponding to elements in

$$
\gamma\left(\left(\gamma^{-1}(w)-\varepsilon,\gamma^{-1}(w)+\varepsilon\right)\right)
\tag{7.12}
$$

are equivalent. This follows from

$$
\gamma(w)=:\begin{pmatrix} L_w\\ K_w\end{pmatrix}\notin\{\mathcal{V}=0\}
$$

and equation (7.11) which shows that the described ratios are locally positive and bounded, and thus implies that there exists an open interval

$$
I_w^{\mathcal{H}_{L_w,K_w}}\subset\operatorname{dom}(\mathcal{H}_{L_w,K_w}),\quad 0\in I_w^{\mathcal{H}_{L_w,K_w}},
$$

[17]Exceptions are the constant integral curves of $\mathcal{V}$, although these might not be the only exceptions.

(recall that being a hyperbolic point is an open condition and, hence, $\mathrm{dom}\,(\mathcal{H}_{L_w,K_w})$ is in all cases an open interval) such that with $L(0)=L_w$, $K(0)=K_w$,

$$\gamma(w)\in\left\{\begin{pmatrix}L\,(T)\\K\,(T)\end{pmatrix}\;\middle|\;T\in I_w^{\mathcal{H}_{L_w,K_w}}\right\}\subset\gamma(I).$$

Since the map $T\mapsto(L(T),K(T))^T$ is smooth and, by the assumption $(L_w,K_w)^T\notin\{\mathcal{V}=0\}$, non-constant locally around $T=0$ it follows that the set

$$\left\{\begin{pmatrix}L\,(T)\\K\,(T)\end{pmatrix}\;\middle|\;T\in I_w^{\mathcal{H}_{L_w,K_w}}\right\}\subset\gamma(I)$$

contains an open neighbourhood[18] of $\gamma(w)$ in the subspace topology of the submanifold $\gamma(I)\subset\mathbb{R}^2$. We can thus choose $\varepsilon>0$ as in (7.12). This in particular implies that we can choose a maximal open interval $I_w\subset I$, $w\in I_w$, such that all polynomials corresponding to points in $\gamma\,(I_w)$ are equivalent and for any $\overline{w}\in(\partial I_w)\cap I$, which is by assumption not empty, $h_{\gamma(w)}$ and $h_{\gamma(\overline{w})}$ are not equivalent. Also by assumption we have

$$\gamma\,(\overline{w})=:\left(\begin{smallmatrix}L_{\overline{w}}\\K_{\overline{w}}\end{smallmatrix}\right)\in\mathbb{R}^2\setminus\{\mathcal{V}=0\},$$

hence we can use the same procedure for $\overline{w}$ as we used for w and find that there exists a maximal open interval $I_{\overline{w}}\subset I$, $\overline{w}\in I_{\overline{w}}$, such that all polynomials corresponding to elements in $\gamma\,(I_{\overline{w}})$ are equivalent. The constructed intervals I_w and $I_{\overline{w}}$ are both open, and since $\overline{w}\in(\partial I_w)\cap I$ it follows that $I_w\cap I_{\overline{w}}\neq\emptyset$. But this implies that the polynomials corresponding to $\gamma(w)$ and $\gamma(\overline{w})$ are equivalent, which is a contradiction. Summarising, this proves the claim that for all maximally extended integral curves $\gamma:I\to\mathbb{R}^2\setminus\{\mathcal{V}=0\}$ of the restricted vector field $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ the corresponding polynomials $h_{\gamma_L(t),\gamma_K(t)}$ defined in (7.6) and the corresponding maximal quartic connected GPSR curves $\mathcal{H}_{\gamma_L(t),\gamma_K(t)}$ are equivalent for all $t\in I$.

Observing the complexity of the formulas (7.8) and (7.9), the above discussion suggests that it might be easier to be concerned with properties of the vector field $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}\in\Gamma\left(T\left(\mathbb{R}^2\setminus\{\mathcal{V}=0\}\right)\right)$ and its integral curves in order to find the desired classification result instead of studying the equations (7.8) and (7.9) directly. This is precisely what we will do from this point on in the proof of this theorem. Note that we have not shown that the set of maximal integral curves of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ is in one-to-one correspondence with equivalence classes of polynomials, but rather that for each $h_{L,K}$ with $(L,K)^T\in\mathbb{R}^2\setminus\{\mathcal{V}=0\}$ as in (7.1) there exists at least one maximal integral curve $\gamma:I\to\mathbb{R}^2\setminus\{\mathcal{V}=0\}$ of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$, such that each polynomial corresponding to a point in $\gamma(I)$ is equivalent to $h_{L,K}$. Note that since we can assume that $h_{L,K}$ corresponds to a point in $\gamma(I)$ itself, we also get that the corresponding maximal quartic GPSR curves are equivalent. This leaves us with the task of checking which maximal integral curves of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ do contain points corresponding to closed quartic GPSR curves, and then checking if pairwise different maximal integral curves might still contain points corresponding to equivalent closed quartic GPSR curves. The quartic GPSR curves corresponding to points in $\{\mathcal{V}=0\}$ need to be treated as well.

In the following we will assume that $\mathcal{H}_{L,K}$ is the maximally extended open connected subset of $\{h_{L,K}=1\}$ that consists only of hyperbolic points and contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Lemma 3.8 implies that it is a necessary requirement for $\mathcal{H}_{L,K}$ to be closed and thus possibly be a quartic CCGPSR curve that the function

$$f_{L,K}(t):=h_{L,K}\left(\left(\begin{smallmatrix}1\\t\end{smallmatrix}\right)\right)=1-t^2+Lt^3+Kt^4\tag{7.13}$$

[18] This is precisely the reason why we consider the restriction $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$.

has at least one positive and one negative real root in t. This follows from the fact that the set of these roots must coincide with $\partial\mathrm{dom}\,(\mathcal{H}_{L,K})$ since otherwise the connected component of $\{h_{L,K}=1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ would not coincide with $\mathcal{H}_{L,K}$, which would in turn not be closed. Recall that in the cubic case, that is for CCPSR curves, it turned out that this was also a sufficient condition, see Lemma 4.2 and Theorem 5.6. This is not true in the quartic case as we will see. For the following studies we will frequently need the formulas for $f_{L,K}(t)$ and its first and second derivative:

$$\begin{aligned} \dot{f}_{L,K}(t) &= -2t + 3Lt^2 + 4Kt^3, \\ \ddot{f}_{L,K}(t) &= -2 + 6Lt + 12Kt^2. \end{aligned}$$

Consider first $L=0$. Then $f_{0,K}(t) = 1 - t^2 + Kt^4$. For $K \neq 0$,

$$f_{0,K} = 0 \quad \Leftrightarrow \quad t^2 = \frac{1}{2K} \pm \sqrt{\frac{1-4K}{4K^2}}. \tag{7.14}$$

This shows that $f_{0,K}(t)$ has no real roots for $K > \frac{1}{4}$. It follows that for all $K > \frac{1}{4}$, $f_{L,K}(t)$ has no positive real root for $L > 0$ and no negative real root for $L < 0$. This shows that $K \leq \frac{1}{4}$ (see Figure 9) is a necessary requirement for $\mathcal{H}_{L,K}$ to be a quartic CCGPSR curve.

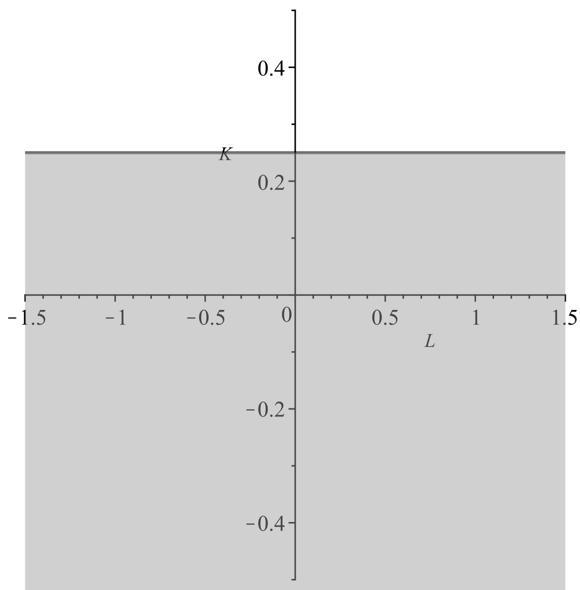


Figure 9: $\left\{K \leq \frac{1}{4}\right\} \subset \mathbb{R}^2$ marked in grey.

Next, consider $K = \frac{1}{4}$. For that specific value of K,

$$f_{0,\frac{1}{4}}(t) = 0 \quad \Leftrightarrow \quad t = \pm\sqrt{2}.$$

Since

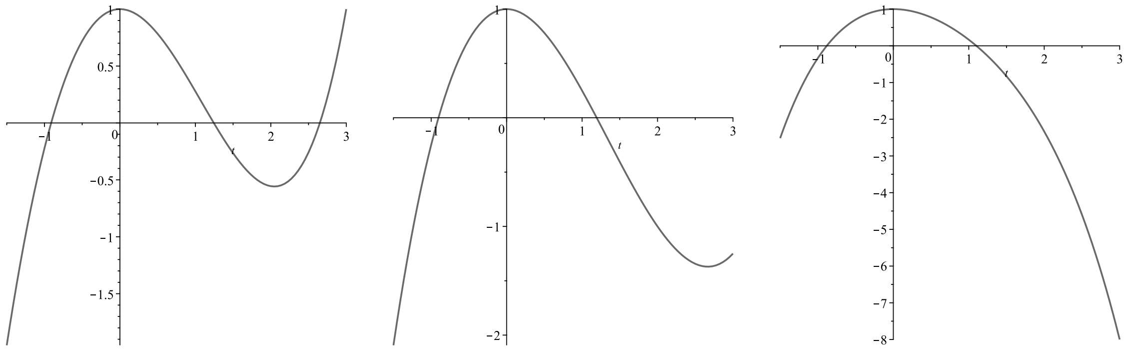
$$\dot{f}_{0,\frac{1}{4}}\left(\pm\sqrt{2}\right) = 0, \quad \ddot{f}_{0,\frac{1}{4}}\left(\pm\sqrt{2}\right) = 4 > 0,$$

it follows that $t = \sqrt{2}$ and $t = -\sqrt{2}$ are both double roots and local minima. Hence, for $L > 0$ we have $f_{L,\frac{1}{4}}(t) > 0$ for all $t > 0$ and for $L < 0$ we have $f_{L,\frac{1}{4}}(t) > 0$ for all $t < 0$. This shows that $f_{L,\frac{1}{4}}(t)$ has a positive and a negative real root if and only if $L = 0$.

Now consider $K < \frac{1}{4}$. For $L = 0$ we have shown above that $f_{0,K}(t)$ has at least one positive and one negative real root. To analyse the cases $L \neq 0$ we will study the (possibly complex) roots of $\dot{f}_{L,K}(t)$. We will without loss of generality assume that $L>0$, since $h_{L,K}$ and $h_{-L,K}$ are equivalent via $y\mapsto -y$. Then

$$\dot{f}_{L,K}(t)=0 \quad\Leftrightarrow\quad t=0 \text{ or } \begin{cases} t=\frac{2}{3L}, & K=0,\\ t=-\frac{3L}{8K}\pm\frac{1}{8K}\sqrt{9L^2+32K}, & K\neq 0. \end{cases} \tag{7.15}$$

We want to stress that for $K\neq 0$ the latter two of the above roots of $\dot{f}_{L,K}(t)$ might not be real. As an example of how the corresponding function $f_{L,K}(t)$ looks like when plotted for specific values of L and K, see the following figures (10a), (10b), and (10c):



(a) $f_{\frac{1}{4},\frac{1}{36}}(t),\ t\in\left(-\frac{3}{2},3\right)$. **(b)** $f_{\frac{1}{4},0}(t),\ t\in\left(-\frac{3}{2},3\right)$. **(c)** $f_{\frac{1}{4},-\frac{1}{12}}(t),\ t\in\left(-\frac{3}{2},3\right)$.

Figure 10: Example plots of $f_{L,K}(t)$.

We now use MAPLE to symbolically solve the system of equations

$$f_{L,K}(t)=0,\quad \dot{f}_{L,K}(t)=0 \tag{7.16}$$

for the variable L and, as one of the solutions under the restriction

$$t=t_m:=-\frac{3L}{8K}+\frac{1}{8K}\sqrt{9L^2+32K}, \tag{7.17}$$

we obtain for $K\neq 0$

$$L=\frac{\sqrt{2}}{3\sqrt{3}}\sqrt{1-36K+\sqrt{(1+12K)^3}}=:\mathbf{u}(K). \tag{7.18}$$

We will consider $\mathbf{u}$ as a function on the interval $\left(-\frac{1}{12},\frac{1}{4}\right)$, where we note that for $K=0$, $L=\mathbf{u}(0)=\frac{2}{3\sqrt{3}}$ and $t=\frac{2}{3L}=\sqrt{3}$ solve (7.16). Before explaining the reason why we choose this particular lower bound for the interval $\left(-\frac{1}{12},\frac{1}{4}\right)$ (see equation (7.21)), we will analyse $\mathbf{u}(K)$ further. We will show that $\mathbf{u}(K)>0$ for all $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$. We obtain

$$1-36K+\sqrt{(1+12K)^3}=0 \quad\Rightarrow\quad (1+12K)^3-(1-36K)^2=0 \quad\Rightarrow\quad K\in\left\{0,\frac{1}{4}\right\}.$$

So the only possible solution of $1-36K+\sqrt{(1+12K)^3}=0$ that is contained in the set $\left(-\frac{1}{12},\frac{1}{4}\right)$ is $K=0$. But $\mathbf{u}(0)=\frac{2}{3\sqrt{3}}$. We conclude

$$\forall K\in\left(-\frac{1}{12},\frac{1}{4}\right):\quad \mathbf{u}(K)>0,$$

as claimed. Note that for $K=0$, $\mathbf{u}(0)=\frac{2}{3\sqrt{3}}$ which coincides with the unique real solution of $\dot{f}_{L,0}\left(t_m|_{K=0}\right)=0$, where $t_m|_{K=0}:=\frac{2}{3L}$, cf. (7.15). Furthermore, for $K=\frac{1}{4}$, $\mathbf{u}\left(\frac{1}{4}\right)=0$, so the point $\left(\left(\mathbf{u}\left(\frac{1}{4}\right),\frac{1}{4}\right)^T\in\mathbb{R}^2\right.$ coincides with one of the fixed points of $\mathcal{V}$. Note that we have in particular shown that

$$\mathbf{u}:\left(-\frac{1}{12},\frac{1}{4}\right)\to\mathbb{R}, \tag{7.19}$$

i.e. that $\mathbf{u}$ takes only real values. Next we will show that the graph of $\mathbf{u}$ coincides with the image of a maximal integral curve of the vector field $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ defined in (7.4) (see Figure 11 for a plot of the graph of $\mathbf{u}$). We know that $\mathcal{V}$ has no zeroes in the set

$$\left\{\begin{pmatrix}\mathbf{u}(K)\\ K\end{pmatrix}\ \middle|\ K\in\left(-\frac{1}{12},\frac{1}{4}\right)\right\}\subset\mathbb{R}^2, \tag{7.20}$$

see equation (7.5).

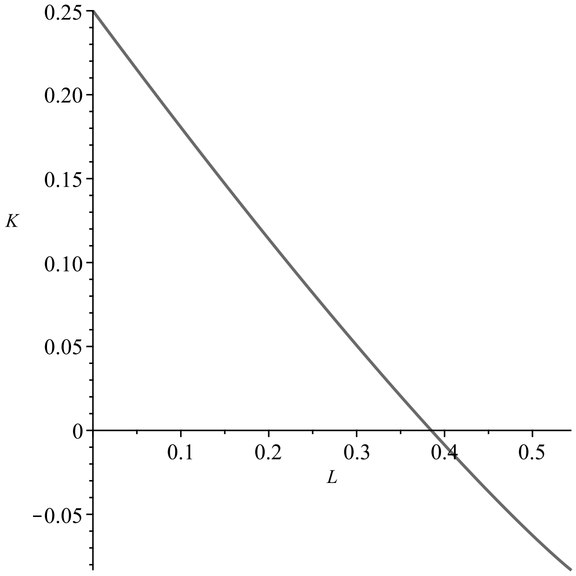


Figure 11: The graph of $\mathbf{u}$ embedded in $\mathbb{R}^2$ as in (7.20).

Furthermore $dK(\mathcal{V})=L\left(6K+\frac{1}{2}\right)$ does not vanish if $L\neq 0$ and $K\neq-\frac{1}{12}$, so in particular it does not vanish on the graph of $\mathbf{u}$. Since $\mathbf{u}$ converges for the limits $K\to\frac{1}{4}$ and $K\to-\frac{1}{12}$, $\mathbf{u}$ is continuously extensible to the set $\left[-\frac{1}{12},\frac{1}{4}\right]$, which shows that the graph of $\mathbf{u}$ is precompact in $\mathbb{R}^2$. One now verifies

$$\partial_K\mathbf{u}(K)=\frac{6\left(-2+\sqrt{1+12K}\right)}{\sqrt{6-216K+6\sqrt{(1+12K)^3}}}=\left.\frac{dL(\mathcal{V})}{dK(\mathcal{V})}\right|_{\left(\begin{smallmatrix}L\\ K\end{smallmatrix}\right)=\left(\begin{smallmatrix}\mathbf{u}(K)\\ K\end{smallmatrix}\right)}.$$

This shows that the image $K\mapsto(\mathbf{u}(K),K)^T$, $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$, is contained in the image of a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$. Since

$$\lim_{K\to\frac{1}{4}}\begin{pmatrix}\mathbf{u}(K)\\ K\end{pmatrix}=\begin{pmatrix}0\\ \frac{1}{4}\end{pmatrix}\in\{\mathcal{V}=0\},\qquad \lim_{K\to-\frac{1}{12}}\begin{pmatrix}\mathbf{u}(K)\\ K\end{pmatrix}=\begin{pmatrix}\frac{2\sqrt{2}}{3\sqrt{3}}\\ -\frac{1}{12}\end{pmatrix}\in\{\mathcal{V}=0\}, \tag{7.21}$$

we conclude that said image coincides with a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$. Note that it contains in particular the point $\left(\frac{2}{3\sqrt{3}},0\right)^T\in\mathbb{R}^2$, which corresponds to the polynomial c).

We will keep that in mind for now. Equation (7.21) also explains the lower bound $-\frac{1}{12}$ of the domain of definition of $\mathbf{u}$.

Next we will show that for all $K \in \left(-\frac{1}{12}, \frac{1}{4}\right)$ and all $\ell > 0$, the corresponding maximal quartic GPSR curves $\mathcal{H}_{L,K}$ corresponding to points of the form

$$\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbf{u}(K)+\ell \\ K \end{smallmatrix}\right) \in \mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$$

are never closed (cf. Figure 12). For $K = 0$, we need to analyse the functions

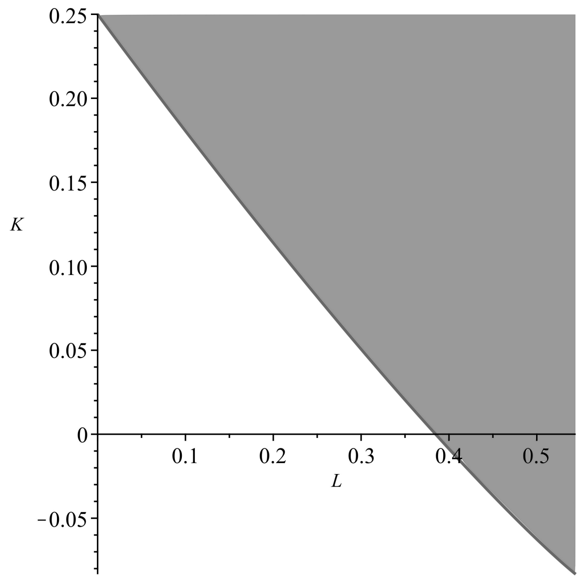


Figure 12: Part of the set $\left\{(\mathbf{u}(K)+\ell, K)^T,\ K \in \left(-\frac{1}{12}, \frac{1}{4}\right),\ \ell > 0\right\}$ marked in grey.

$$f_{\frac{2}{3\sqrt{3}}+\ell,0}(t) = 1 - t^2 + \left(\frac{2}{3\sqrt{3}} + \ell\right) t^3$$

for $\ell > 0$. Similarly to Lemma 4.2, observe that $f_{\frac{2}{3\sqrt{3}},0}(t) = 1 - t^2 + \frac{2}{3\sqrt{3}} t^3$ has precisely one positive root $t = \sqrt{3}$. Since $\dot{f}_{\frac{2}{3\sqrt{3}},0}\left(\sqrt{3}\right) = 0$ and $\ddot{f}_{\frac{2}{3\sqrt{3}},0}\left(\sqrt{3}\right) = 2$, $f_{\frac{2}{3\sqrt{3}},0}(t)$ has a local minimum at $t = \sqrt{3}$. Furthermore

$$f_{\frac{2}{3\sqrt{3}}+\ell,0}(t) > f_{\frac{2}{3\sqrt{3}},0}(t)$$

for all $\ell > 0$ and all $t > 0$. We conclude that for all $\ell > 0$, $f_{\frac{2}{3\sqrt{3}}+\ell,0}(t)$ has no positive real root and, hence, the corresponding maximal quartic GPSR curve $\mathcal{H}_{\frac{2}{3\sqrt{3}}+\ell,0}$ can not be closed. See Figure 13 for an example plot of a function of the form $f_{\frac{2}{3\sqrt{3}}+\ell,0}(t)$.

For $K \neq 0$, t_m, defined in (7.17) above, is always positive whenever $L > 0$ and $9L^2+32K \geq 0$. For $\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbf{u}(K)+\ell \\ K \end{smallmatrix}\right)$, $K \in \left(-\frac{1}{12}, \frac{1}{4}\right)$, we have

$$9L^2 + 32K = \underbrace{18\ell\mathbf{u}(K) + 9\ell^2}_{>0} + \underbrace{\frac{2}{3}(1+12K)}_{>0} + \underbrace{\frac{2}{3}\sqrt{(1+12K)^3}}_{>0} > 0. \tag{7.22}$$

Hence, t_m is real and positive along the considered points for all $K \in \left(-\frac{1}{12}, \frac{1}{4}\right) \setminus \{0\}$ and all $\ell > 0$.

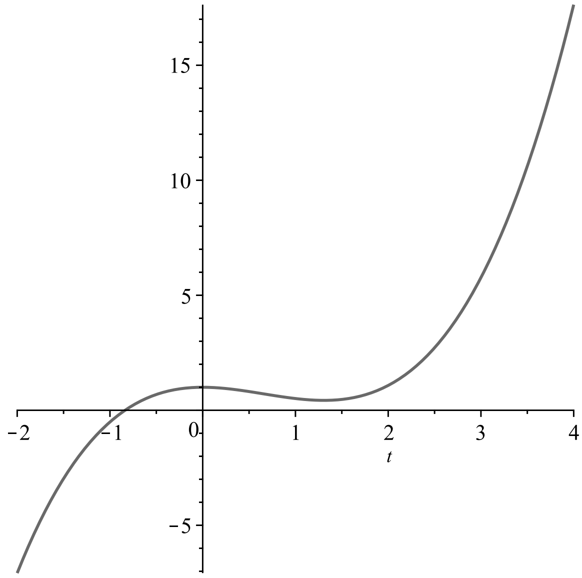


Figure 13: A plot of $f_{\mathbf{u}(K)+\ell,K}(t)$, $t \in (-2,4)$, for $K = 0$ and $\ell = \frac{1}{8}$.

For $K \in \left(0, \frac{1}{4}\right)$ and all $\ell > 0$, the function $f_{\mathbf{u}(K)+\ell,K}(t) = 1 - t^2 + (\mathbf{u}(K) + \ell)\, t^3 + Kt^4$ has a local maximum at $t = 0$ and diverges to $+\infty$ for $t \to \pm\infty$. Hence, $f_{\mathbf{u}(K)+\ell,K}(t)$ must have a local minimum at $t = t_m$. For $\ell = 0$, t_m is by construction also a root of $f_{\mathbf{u}(K),K}(t)$. Since for all $t > 0$

$$f_{\mathbf{u}(K)+\ell,K}(t) > f_{\mathbf{u}(K),K}(t)$$

and as we have seen $f_{\mathbf{u}(K),K}(t)$ has precisely one positive real root, we conclude that for all $\ell > 0$, $f_{\mathbf{u}(K)+\ell,K}(t)$ has no positive real root. Hence, the corresponding maximal quartic GPSR curve $\mathcal{H}_{\mathbf{u}(K)+\ell,K}$ can never be closed.

Next consider $K \in \left(-\frac{1}{12}, 0\right)$. In these cases $f_{\mathbf{u}(K)+\ell,K}(t)$ will always have at least one negative and one positive real root for all $\ell > 0$ since its highest-order monomial t^4 has a negative prefactor. Thus in order to prove the claim that $\mathcal{H}_{\mathbf{u}(K)+\ell,K}$ can never be closed we need to check that there exists at least one point $\widetilde{t}$ in the connected component containing $t = 0$ of the set

$$\left\{t \in \mathbb{R} \mid\ f_{\mathbf{u}(K)+\ell,K}(t) > 0\right\},$$

such that $\left(\frac{1}{t}\right)$ is not a hyperbolic point of the corresponding quartic polynomial $h_{\mathbf{u}(K)+\ell,K}$. To do so recall that we have already shown that along $L = \mathbf{u}(K) + \ell$ and $K \in \left(-\frac{1}{12}, 0\right) \subset \left(-\frac{1}{12}, \frac{1}{4}\right) \setminus \{0\}$, the term $9L^2 + 32K$ is positive, see (7.22). Hence, (7.15) implies that for all $\ell > 0$, $f_{\mathbf{u}(K)+\ell,K}(t)$ has exactly three local extrema, namely at $t = 0$, $t_m = -\frac{3(\mathbf{u}(K)+\ell)}{8K} + \frac{1}{8K}\sqrt{9\left(\mathbf{u}(K) + \ell\right)^2 + 32K}$, and with

$$t_M := -\frac{3L}{8K} - \frac{1}{8K}\sqrt{9L^2 + 32K}, \tag{7.23}$$

at $t_M = -\frac{3(\mathbf{u}(K)+\ell)}{8K} - \frac{1}{8K}\sqrt{9\left(\mathbf{u}(K) + \ell\right)^2 + 32K}$. Since $K < 0$, it follows that both t_m and t_M are positive and (7.22) implies the strict inequality $t_m < t_M$. For all $K \in \left(-\frac{1}{12}, \frac{1}{4}\right)$ and all $\ell > 0$, the function $f_{\mathbf{u}(K)+\ell,K}(t)$ is a quartic polynomial in t, which implies that it has at most three distinct local extrema. Hence, we have shown that $f_{\mathbf{u}(K)+\ell,K}(t)$ has precisely three extrema at the distinct points $t = 0$, t_m, and t_M, $f_{\mathbf{u}(K)+\ell,K}(t) \to -\infty$ for both $t \to \pm\infty$ and furthermore that $f_{\mathbf{u}(K)+\ell,K}(t)$ always has a local maximum at $t = 0$, we deduce that $f_{\mathbf{u}(K)+\ell,K}(t)$ will always have a local minimum at t_m and a local maximum at t_M. We will now show that for all $K \in \left(-\frac{1}{12},0\right)$ and all $\ell>0$, t_m is contained in the connected component that contains $t=0$ of $\left\{t\in\mathbb{R} \mid f_{\mathbf{u}(K)+\ell,K}(t)>0\right\}$ and that $\left(\begin{smallmatrix}1\\ t_m\end{smallmatrix}\right)$ is indeed not a hyperbolic point of $h_{\mathbf{u}(K)+\ell,K}$. To show the first statement, it suffices to show that always $f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)>0$ since we have seen that for all $K\in\left(-\frac{1}{12},0\right)$ and all $\ell>0$, $f_{\mathbf{u}(K)+\ell,K}(t)$ has a local maximum at $t=0$, a local minimum at $t_m>0$, and no extremal point at any $t\in(0,t_m)$. We view $f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)$ as a function in the variables K and ℓ and calculate

$$\begin{aligned}\partial_\ell\left(f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)\right)=\frac{1}{13824K^3}&\left(-9\ell-\sqrt{6-216K+6\sqrt{(1+12K)^3}}\right.\\ &\left.+\sqrt{6+72K+6\sqrt{(1+12K)^3}+18\ell\sqrt{6-216K+6\sqrt{(1+12K)^3}}+81\ell^2}\right)^3.\end{aligned}$$

At $\ell=0$,

$$\begin{aligned}&\partial_\ell\left(f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)\right)\Big|_{\ell=0}\\ &=\frac{1}{13824K^3}\left(-\sqrt{6-216K+6\sqrt{(1+12K)^3}}+\sqrt{6+72K+6\sqrt{(1+12K)^3}}\right)>0\end{aligned}$$

which is easily seen for all $K\in\left(-\frac{1}{12},0\right)$. Suppose that there exists $K\in\left(-\frac{1}{12},0\right)$ and $\ell>0$, such that $\partial_\ell\left(f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)\right)=0$. Then

$$\begin{aligned}&9\ell+\sqrt{6-216K+6\sqrt{(1+12K)^3}}\\ &=\sqrt{6+72K+6\sqrt{(1+12K)^3}+18\ell\sqrt{6-216K+6\sqrt{(1+12K)^3}}+81\ell^2}\\ \Rightarrow\quad &K=0,\end{aligned}$$

by taking the square of both sides of the first equation. This is a contradiction to $K\in\left(-\frac{1}{12},0\right)$. We conclude that for all $K\in\left(-\frac{1}{12},0\right)$ and all $\ell>0$, $\partial_\ell\left(f_{\mathbf{u}(K)+\ell,K}\left(t_m\right)\right)>0$. Since $f_{\mathbf{u}(K),K}\left(t_m\right)=0$ by construction, this shows that t_m is in fact contained in the connected component that contains $t=0$ of $\left\{t\in\mathbb{R}\mid f_{\mathbf{u}(K)+\ell,K}(t)>0\right\}$ as claimed. In order to show that $\left(\begin{smallmatrix}1\\ t_m\end{smallmatrix}\right)$ is not a hyperbolic point of $h_{\mathbf{u}(K)+\ell,K}$ we will use Lemma 3.28. In the one-dimensional case, i.e. in our case where $\dim\left(\mathcal{H}_{L,K}\right)=1$, the function $\beta:\mathbb{R}\to\mathbb{R}$ defined in (3.22) and $f_{L,K}:\mathbb{R}\to\mathbb{R}$ coincide. Using formula (3.33) yields that the pullback of the centro-affine metric at $t=t_m\in\operatorname{dom}\left(\mathcal{H}_{\mathbf{u}(K)+\ell,K}\right)$

$$\left(\Phi^* g_{\mathcal{H}_{\mathbf{u}(K)+\ell,K}}\right)_{t_m}=-\frac{\ddot{f}_{\mathbf{u}(K)+\ell,K}(t_m)}{4f_{\mathbf{u}(K)+\ell,K}(t_m)}dt^2,\tag{7.24}$$

where here t denotes the coordinate of $\operatorname{dom}\left(\mathcal{H}_{\mathbf{u}(K)+\ell,K}\right)$. But we have shown that $f_{\mathbf{u}(K)+\ell,K}(t)$ has a local minimum at t_m for all $K\in\left(-\frac{1}{12},0\right)$ and for all $\ell>0$, which implies that $-\frac{1}{4}\ddot{f}_{\mathbf{u}(K)+\ell,K}(t_m)\leq 0$. Hence, $\left(\begin{smallmatrix}1\\ t_m\end{smallmatrix}\right)$ is not a hyperbolic point of $h_{\mathbf{u}(K)+\ell,K}$ as claimed, and we deduce that for all $K\in\left(-\frac{1}{12},0\right)$ and for all $\ell>0$, the maximally extended connected quartic GPSR curve $\mathcal{H}_{\mathbf{u}(K)+\ell,K}$ is never closed in $\mathbb{R}^2$.

Summarising up to this point, we have determined for each $K>-\frac{1}{12}$ a positive lower bound for L (and by equivalence also a negative upper bound for L), such that the maximally extended connected quartic GPSR curve $\mathcal{H}_{L,K}$ is <u>not</u> a CCGPSR curve. These points form precisely the set

$$\left\{K > \frac{1}{4}\right\} \cup \left\{K = \frac{1}{4},\ |L| > 0\right\} \cup \left\{K \in \left(-\frac{1}{12}, \frac{1}{4}\right),\ |L| > \mathbf{u}(K)\right\}, \tag{7.25}$$

see Figure 14.

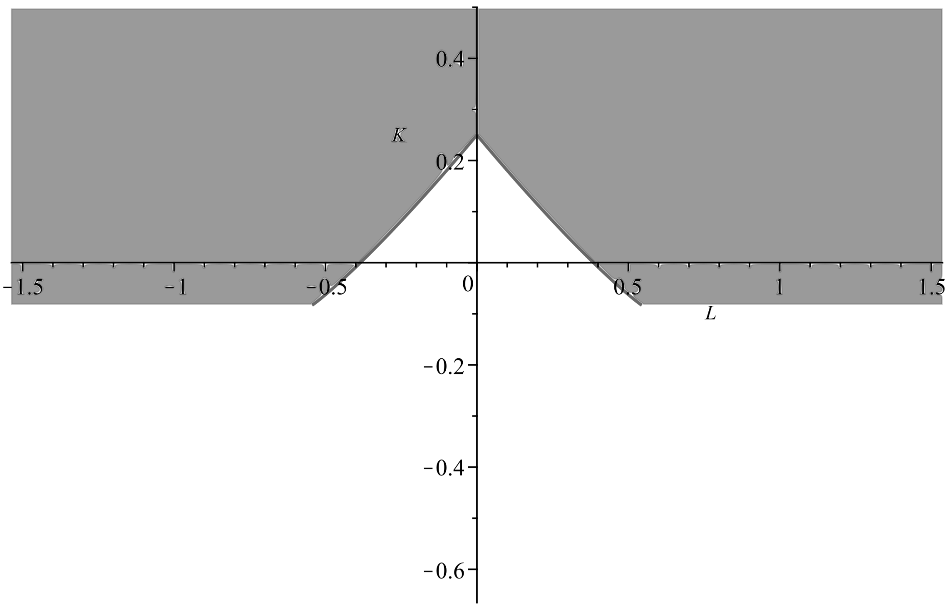


Figure 14: Part of the set (7.25) marked in grey.

Next we will deal with $K = -\frac{1}{12}$. It turns out that we can use the same strategy as for $K \in \left(-\frac{1}{12}, 0\right)$ by considering $\mathbf{u}(K)$ at the limit point $K = -\frac{1}{12}$, $\mathbf{u}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}$. For all $\ell > 0$ and the corresponding function $f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t)$, the points $t = 0$, $t_m = \frac{3}{2}\left(\frac{2\sqrt{2}}{\sqrt{3}} + \ell - \sqrt{\frac{4\sqrt{2}}{3\sqrt{3}}\ell + \ell^2}\right)$ (7.17), and $t_M = \frac{3}{2}\left(\frac{2\sqrt{2}}{\sqrt{3}} + \ell + \sqrt{\frac{4\sqrt{2}}{3\sqrt{3}}\ell + \ell^2}\right)$ (7.23), still fulfil $0 < t_m < t_M$ and are also still critical points of $f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t)$. Also, we can show similarly as for the case $K \in \left(-\frac{1}{12}, 0\right)$ that

$$\partial_\ell \left(f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t_m)\right) > 0$$

for all $\ell \geq 0$. Since $f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t_m)\Big|_{\ell=0} = 0$ and $t = 0$, t_m, and t_M are the unique critical points of $f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t)$ for all $\ell > 0$, we conclude that for all $\ell > 0$, t_m is contained in the connected component of the set

$$\left\{t \in \mathbb{R} \;\middle|\; f_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}(t) > 0\right\}$$

which contains the point $t = 0$. From equation (7.24) for the limit $K = -\frac{1}{12}$ it follows that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ t_m \end{smallmatrix}\right)$ is not a hyperbolic point of $h_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}$. Thus, for all $\ell > 0$ the maximally extended connected quartic GPSR curve $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}}+\ell,-\frac{1}{12}}$ is not closed in $\mathbb{R}^2$, i.e. not a quartic CCGPSR curve.

Lastly in this stage of the proof we will study the case $K < -\frac{1}{12}$. For $K < 0$ and $L > 0$ the points t_m (7.17) and t_M (7.23) are both real numbers if and only if $9L^2 + 32K \geq 0$, and coincide if and only if

$$9L^2 + 32K = 0 \quad \Leftrightarrow \quad L = \frac{4\sqrt{-2K}}{3} =: \mathbf{v}(K).$$

The function $\mathbf{v}$ is smooth and positive on $\{K < 0\}$, but we will consider its restriction

$$\mathbf{v} : \left(-\infty, -\frac{1}{12}\right) \to \mathbb{R}_{>0},$$

with smooth continuation to $\mathbf{v}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}$. Observe that the limits $\mathbf{u}\left(-\frac{1}{12}\right) = \mathbf{v}\left(-\frac{1}{12}\right)$ coincide (see Figure 15), and that the image of $(\mathbf{v}(K), K)^T$ for $K \in \left(-\infty, -\frac{1}{12}\right)$ is contained

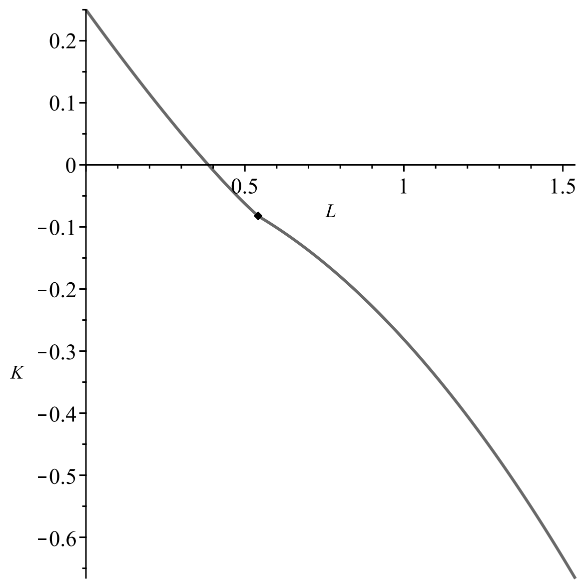


Figure 15: The respective images of $\mathbf{u}$ and (in part) $\mathbf{v}$ in $\mathbb{R}^2$, and the limit point $(L, K)^T = \left(\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{1}{12}\right)^T$ marked with a black diamond.

in $\mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$. Along points of the form $(\mathbf{v}(K), K)^T \in \mathbb{R}^2$, $f_{\mathbf{v}(K),K}(t)$ has a saddle point at $t_m = t_M$. We will now show that $f_{\mathbf{v}(K),K}(t_m) > 0$ for all $K < -\frac{1}{12}$. We view t_m as a function depending on L, K and obtain along points of the form $(\mathbf{v}(K), K)^T \in \mathbb{R}^2$

$$f_{\mathbf{v}(K),K}(t_m) = \frac{1 + 12K}{12K} > 0 \tag{7.26}$$

for all $K < -\frac{1}{12}$. Since $f_{\mathbf{v}(K),K}(t)$ is monotonously decreasing for $t > 0$, this shows that t_m is contained in the connected component of $\{f_{\mathbf{v}(K),K}(t) > 0\}$ that contains the point $t = 0$. But t_m is a saddle point of $f_{\mathbf{v}(K),K}(t)$, which implies using (3.33) that

$$\left(\Phi^* g_{\mathcal{H}_{\mathbf{v}(K),K}}\right)_{t_m} = 0$$

for all $K < -\frac{1}{12}$. Hence, for all $K < -\frac{1}{12}$ the maximally extended connected quartic GPRS curve $\mathcal{H}_{\mathbf{v}(K),K}$ is not closed.

Next we will show that for all $K < -\frac{1}{12}$ and all $\ell > 0$, the maximally extended connected quartic GPRS curve $\mathcal{H}_{\mathbf{v}(K)+\ell,K}$ is not closed. In these cases, t_m (7.17) and t_M (7.23) are both real numbers and $f_{\mathbf{v}(K)+\ell,K}(t)$ always has a local minimum at $t_m > 0$ and a local maximum at $t_M > t_m$. We will show that $f_{\mathbf{v}(K)+\ell,K}(t_m) > 0$. Since $f_{\mathbf{v}(K)+\ell,K}(t) \to -\infty$ for $t \to \pm\infty$, this means that t_m is an element of the connected component of $\{f_{\mathbf{v}(K)+\ell,K} > 0\}$ that contains $t = 0$. This will imply that $\left(\begin{smallmatrix}1\\ t_m\end{smallmatrix}\right)$ is not a hyperbolic point of the corresponding $h_{\mathbf{v}(K)+\ell,K}$ and thus that $\mathcal{H}_{\mathbf{v}(K)+\ell,K}$ is not closed for all $K < -\frac{1}{12}$ and all $\ell > 0$. We proceed similarly to the calculations for $f_{\mathbf{u}(K)+\ell,K}(t)$ with $K \in \left(-\frac{1}{12}, 0\right)$, $\ell > 0$. Along $(L,K)^T = (\mathbf{v}(K) + \ell, K)^T$ we have

$$\begin{aligned} t_m &= -\frac{3\,(\mathbf{v}(K)+\ell)}{8K} + \frac{1}{8K}\sqrt{9\,(\mathbf{v}(K)+\ell)^2 + 32K} \\ &= -\frac{4\sqrt{-2K}+3\ell}{8K} + \frac{1}{8K}\sqrt{24\sqrt{-2K}\ell + 9\ell^2} \end{aligned} \tag{7.27}$$

and

$$\partial_\ell \left(f_{\mathbf{v}(K)+\ell,K}\left(t_m\right)\right) = \dot{f}_{\mathbf{v}(K)+\ell,K}\left(t_m\right) \cdot \partial_\ell t_m + t_m^3.$$

Hence, t_m being positive, a critical point of $f_{\mathbf{v}(K)+\ell,K}\left(t\right)$, and smooth in ℓ-dependence for $\ell > 0$ implies that

$$\partial_\ell \left(f_{\mathbf{v}(K)+\ell,K}\left(t_m\right)\right) = t_m^3 > 0$$

for all $\ell > 0$. We have to be careful with the limit case $\ell = 0$ since the first ℓ-derivative of t_m (7.27) is easily seen to diverge as $\ell \to 0$, $\ell > 0$. However, (7.27) also implies that t_m can be continuously extended to $\ell = 0$ for fixed $K < -\frac{1}{12}$ (namely, t_m and t_M viewed as branches of a bifurcation behave nicely), and hence it follows with $f_{\mathbf{v}(K)+\ell,K}\left(t_m\right)\Big|_{\ell=0} > 0$ (7.26) and $\partial_\ell \left(f_{\mathbf{v}(K)+\ell,K}\left(t_m\right)\right) > 0$ for all $K < -\frac{1}{12}$ and all $\ell > 0$ that

$$f_{\mathbf{v}(K)+\ell,K}\left(t_m\right) > 0$$

for all $K < -\frac{1}{12}$ and all $\ell > 0$. We conclude that $f_{\mathbf{v}(K)+\ell,K}(t)$ has precisely one positive and one negative real root for each pair $K < -\frac{1}{12}$, $\ell > 0$, and that thus t_m is indeed contained in the (unique) connected component of $\{f_{\mathbf{v}(K)+\ell,K}(t) > 0\}$ that contains $t = 0$. Hence, $\left(\begin{smallmatrix}1\\ t_m\end{smallmatrix}\right)$ is by the local minimising property of $f_{\mathbf{v}(K)+\ell,K}(t)$ never a hyperbolic point of the corresponding cubic polynomial $h_{\mathbf{v}(K)+\ell,K}$ and the maximally extended connected quartic GPSR curve $\mathcal{H}_{\mathbf{v}(K)+\ell,K}$ can never be closed.

Up to this point we have shown for all

$$\begin{pmatrix} L \\ K \end{pmatrix} \in \left\{K > \frac{1}{4}\right\} \cup \left\{K \in \left[-\frac{1}{12}, \frac{1}{4}\right],\ |L| > \mathbf{u}(K)\right\} \cup \left\{K < -\frac{1}{12},\ |L| \geq \mathbf{v}(K)\right\} \subset \mathbb{R}^2,$$

where $\mathbf{u} : \left[-\frac{1}{12}, \frac{1}{4}\right] \to \mathbb{R}$ is the unique continuous extension of $\mathbf{u}$ (7.19), that the maximally extended connected quartic GPSR curve $\mathcal{H}_{L,K}$ is not closed (see Figure 16) . The next step might seem a little non-canonical at first. We define

$$\mathbf{w} : \left(-\infty, -\frac{1}{12}\right) \to \mathbb{R}, \quad \mathbf{w}(K) = \frac{\sqrt{6 - 216K}}{9}. \tag{7.28}$$

The function $\mathbf{w}$ is positive and can be (uniquely) smoothly extended to $K = -\frac{1}{12}$ via

$$\mathbf{w}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}} = \mathbf{u}\left(-\frac{1}{12}\right) = \mathbf{v}\left(-\frac{1}{12}\right),$$

see also Figure 17. The definition of $\mathbf{w}$ is motivated by considering (symbolic) solutions of

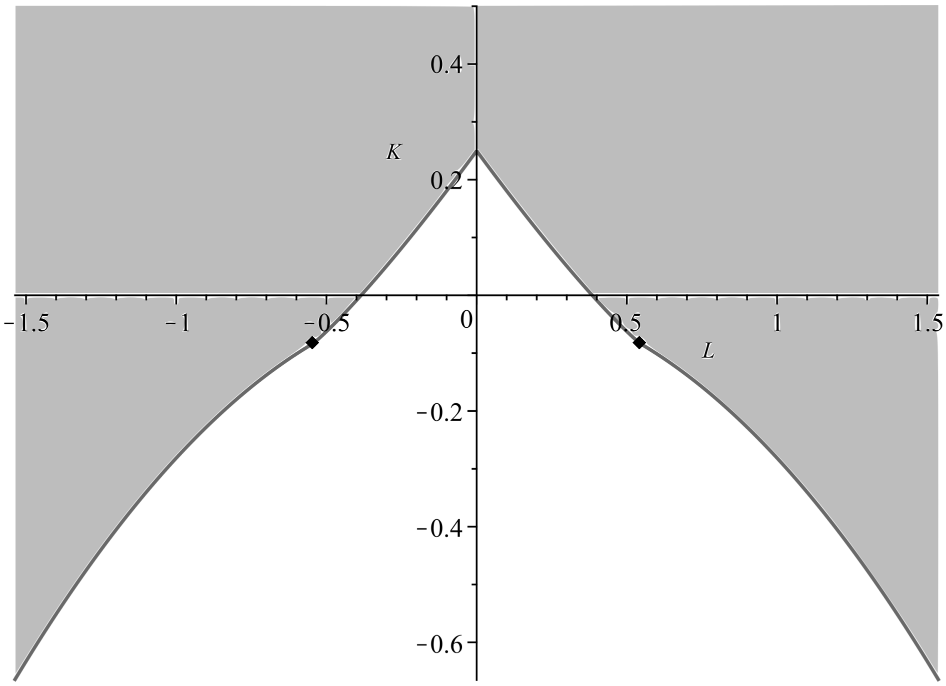


Figure 16: A part of the set $\left\{K > \frac{1}{4}\right\} \cup \left\{K \in \left[-\frac{1}{12}, \frac{1}{4}\right],\ |L| > \mathbf{u}(K)\right\} \cup \left\{K < -\frac{1}{12},\ |L| \geq \mathbf{v}(K)\right\} \subset \mathbb{R}^2$ marked in grey. Similar to Figure 15, the points $(L,K)^T = \left(\pm\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{1}{12}\right)^T$ are marked with black diamonds.

the system of equations

$$\left(\Phi^* g_{\mathcal{H}_{L,K}}\right)_t = 0, \quad \frac{\partial}{\partial t}\left(\left(\Phi^* g_{\mathcal{H}_{L,K}}\right)_t\right) = 0, \tag{7.29}$$

which can be obtained with the help of a computer algebra system like MAPLE. It turns out that the graph of $\mathbf{w}$ embedded in $\mathbb{R}^2$ via $K \mapsto \left(\begin{smallmatrix}\mathbf{w}(K)\\ K\end{smallmatrix}\right)$ consists of solutions of (7.29). Observe that

$$\mathbf{v}(K) = \mathbf{w}(K) \quad \Leftrightarrow \quad K = -\frac{1}{12}$$

when considering their continuous extension to $K = -\frac{1}{12}$, and that $\mathbf{v}\left(-\frac{1}{2}\right) = \frac{4}{3} > \frac{\sqrt{114}}{9} = \mathbf{w}\left(-\frac{1}{2}\right)$ thus implies

$$\mathbf{v}(K) > \mathbf{w}(K) \quad \forall K \in \left(-\infty, -\frac{1}{12}\right). \tag{7.30}$$

We will now show that for each $K < -\frac{1}{12}$ the maximally extended connected quartic GPSR curve $\mathcal{H}_{L,K}$ is not closed for all $L \in [\mathbf{w}(K), \mathbf{v}(K))$ (see Figure 18). The inequality (7.30) implies that for each such pair $(L,K)^T \in \mathbb{R}^2$ the corresponding function $f_{L,K}(t)$ has precisely one negative and one positive real root, and furthermore only one critical value at $t = 0$ (since the corresponding points t_m (7.17) and t_M (7.23) are not real-valued in these cases). We start by showing that the graph of $\mathbf{w}$ coincides with the image of a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2 \setminus \{\mathcal{V}=0\}}$. Firstly note that the graph of $\mathbf{w}$,

$$\left\{\left(\begin{smallmatrix}\mathbf{w}(K)\\ K\end{smallmatrix}\right) \;\middle|\; K \in \left(-\infty, -\frac{1}{12}\right)\right\} \subset \mathbb{R}^2,$$

is contained in $\mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$. Similarly to the consideration of the graph of $\mathbf{u}$, we note that $dK(\mathcal{V}) = L\left(6K + \frac{1}{2}\right)$ does not vanish if $L \neq 0$ and $K \neq -\frac{1}{12}$, hence it does not vanish along

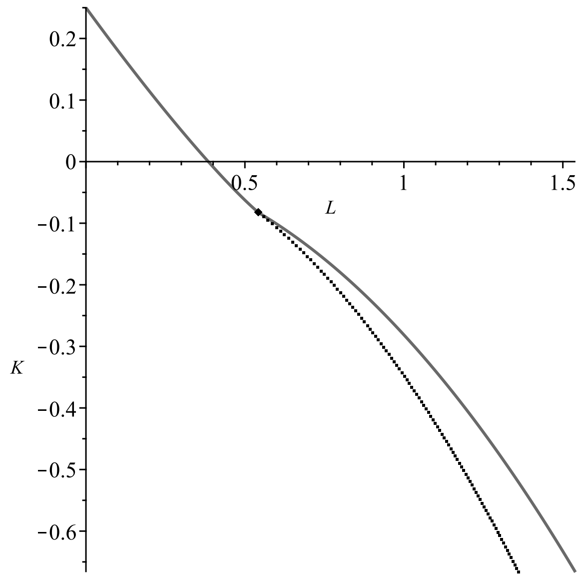


Figure 17: The images of $\mathbf{u}$, (in part) $\mathbf{v}$, and (in part) $\mathbf{w}$ in $\mathbb{R}^2$. The graph of $\mathbf{w}$ is a dotted line, the limit point $(L,K)^T = \left(\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{1}{12}\right)^T$ is marked with a black diamond.

the graph of $\mathbf{w}$. Also note that

$$\partial\left\{\left(\begin{smallmatrix}\mathbf{w}(K)\\ K\end{smallmatrix}\right)\ \middle|\ K \in \left(-\infty, -\frac{1}{12}\right)\right\} = \left\{\left(\begin{smallmatrix}L\\ K\end{smallmatrix}\right) = \left(\begin{smallmatrix}\frac{2\sqrt{2}}{3\sqrt{3}}\\ -\frac{1}{12}\end{smallmatrix}\right)\right\} \in \{\mathcal{V} = 0\}.$$

We check that

$$\partial_K \mathbf{w}(K) = -\frac{12}{\sqrt{6-216K}} = \left.\frac{dL(\mathcal{V})}{dK(\mathcal{V})}\right|_{\left(\begin{smallmatrix}L\\ K\end{smallmatrix}\right)=\left(\begin{smallmatrix}\mathbf{w}(K)\\ K\end{smallmatrix}\right)}.$$

We conclude that the graph of $\mathbf{w}$ does coincide with a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$. In order to show that each $\mathcal{H}_{\mathbf{w}(K),K}$, $K < -\frac{1}{12}$, is not closed it thus suffices to check the latter for one arbitrary $K < -\frac{1}{12}$. This follows from the fact that each pair of maximally extended quartic GPSR curves $\mathcal{H}_{\mathbf{w}(K),K}$, $K < -\frac{1}{12}$ is equivalent as we have shown in the beginning of this proof. We choose $K = -\frac{1}{6}$ with $\mathbf{w}\left(-\frac{1}{6}\right) = \frac{\sqrt{42}}{9}$. Now, we solve

$$\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}}\right)_t = 0 \tag{7.31}$$

for t and obtain as one solution $t_0 = \frac{\sqrt{21}-3}{\sqrt{2}}$, and we have

$$f_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}\left(\frac{\sqrt{21}-3}{\sqrt{2}}\right) = 6\sqrt{21} - 27 > 0. \tag{7.32}$$

Hence, t_0 is contained in the connected component of $\left\{f_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}(t) > 0\right\}$ that contains $t = 0$. But $\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}}\right)_{t_0} = 0$ implies that $\left(\begin{smallmatrix}1\\ t_0\end{smallmatrix}\right)$ is not a hyperbolic point of $h_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}$. Hence, $\mathcal{H}_{\mathbf{w}\left(-\frac{1}{6}\right),-\frac{1}{6}}$ is not closed. We conclude that for all $K < -\frac{1}{12}$, the maximally extended quartic GPSR curve $\mathcal{H}_{\mathbf{w}(K),K}$ is not closed.

Next, we will consider $K < -\frac{1}{12}$ and $L \in (\mathbf{w}(K), \mathbf{v}(K))$. With the help of MAPLE or another suited computer algebra system we can solve the equation

$$\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K),K}}\right)_t = 0 \tag{7.33}$$

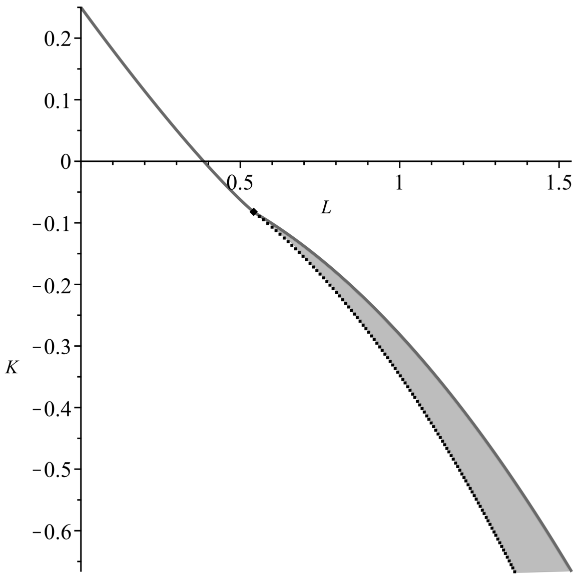


Figure 18: The area $\left\{(L,K)^T\in\mathbb{R}^2 \;\middle|\; K<-\frac{1}{12},\ L\in[\mathbf{w}(K),\mathbf{v}(K))\right\}$ between the graphs of $\mathbf{w}$ and $\mathbf{v}$ marked in grey.

for t explicitly and obtain as one solution

$$t_0=t_0(K):=\frac{\sqrt{6-216K}-3\sqrt{-2-24K}}{2}.$$

The point t_0 is real and positive for all $K<-\frac{1}{12}$ as the equation $t_0=0$ has no solutions for $K<-\frac{1}{12}$ and one can check that $t_0\left(-\frac{1}{6}\right)=\frac{\sqrt{21}-3}{\sqrt{2}}$. Thus, at $K=-\frac{1}{6}$, t_0 coincides with the point used in equation (7.32). Since $\mathbf{w}(K)<\mathbf{v}(K)$ for all $K<-\frac{1}{12}$, the corresponding functions $f_{\mathbf{w}(K),K}(t)$ have precisely one negative and one positive real root. We calculate

$$f_{\mathbf{w}(K),K}(t_0)=162K+3888K^2+23328K^3+\left(\frac{3}{2}+45K+324K^2\right)\sqrt{6-216K}\sqrt{-2-24K}$$

and obtain (again preferably by using a computer algebra system)

$$f_{\mathbf{w}(K),K}(t_0)=0\quad\Leftrightarrow\quad K=-\frac{1}{12}.$$

Equation (7.32) thus yields that

$$f_{\mathbf{w}(K),K}(t_0)>0\tag{7.34}$$

for all $K<-1/12$ and together with the uniqueness of the positive and negative real roots of $f_{\mathbf{w}(K),K}(t)$ we conclude that for all $K<-\frac{1}{12}$, the point t_0 is contained in the connected component of $\{f_{\mathbf{w}(K),K}(t)>0\}$ that contains $t=0$. This motivates studying the expression

$$f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}$$

as a function of $K<-\frac{1}{12}$ and, depending on K, $\ell\in(0,\mathbf{v}(K)-\mathbf{w}(K))$ via canonically identifying sections in $\mathrm{Sym}^2(\mathbb{R}^*)\to\mathbb{R}$ with smooth functions on $\mathbb{R}$. We obtain with the latter identification

$$f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}=\left(-\frac{27}{8}+243K+4374K^2\right)\ell^2$$

$$
\begin{aligned}
&+\left(\frac{27}{16}+\frac{243}{4}K\right)\sqrt{6-216K}\sqrt{-2-24K}\ell^2\\
&+\left(81K+972K^2\right)\sqrt{6-216K}\ell\\
&+\left(\frac{9}{2}-81K-2916K^2\right)\sqrt{-2-24K}\ell.
\end{aligned}
$$

Note that for all $K<-\frac{1}{12}$, $f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}$ is also smooth when considered for all $\ell\in\mathbb{R}$. In order to show that $\mathcal{H}_{\mathbf{w}(K)+\ell,K}$ is not closed for any $K<-\frac{1}{12}$ and, depending on K, $\ell\in(0,\mathbf{v}(K)-\mathbf{w}(K))$ it thus suffices to show that $f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}<0$ for these points. We will make use of

$$
\begin{aligned}
\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)&=\left(-\frac{27}{4}+486K+8748K^2\right)\ell\\
&\quad+\left(\frac{27}{8}+\frac{243}{2}K\right)\sqrt{6-216K}\sqrt{-2-24K}\ell\\
&\quad+\left(81K+972K^2\right)\sqrt{6-216K}\\
&\quad+\left(\frac{9}{2}-81K-2916K^2\right)\sqrt{-2-24K}.
\end{aligned}
$$

Solving $\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)=0$ for ℓ, we obtain

$$
\begin{aligned}
&\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)=0\\
\Leftrightarrow\quad&\ell=\ell_0:=\frac{4}{3}\cdot\frac{\left(-18K-216K^2\right)\sqrt{6-216K}+\left(-1+18K+648K^2\right)\sqrt{-2-24K}}{-2+144K+2592K^2+(1+36K)\sqrt{6-216K}\sqrt{-2-24K}}.
\end{aligned}
\tag{7.35}
$$

Furthermore, we get

$$
\ell_0=0\quad\Leftrightarrow\quad K\in\left\{-\frac{1}{12},\frac{1}{36}\right\}
\tag{7.36}
$$

and

$$
\ell_0=\mathbf{v}(K)-\mathbf{w}(K)\quad\Leftrightarrow\quad K=-\frac{1}{12}.
$$

This shows that ℓ_0 is not contained in the boundary of the open interval $(0,\mathbf{v}(K)-\mathbf{w}(K))$ for all $K<-\frac{1}{12}$. We check that

$$
\ell_0|_{K=-\frac{1}{6}}=\frac{2\sqrt{2}\left(3\sqrt{21}-14\right)}{15\sqrt{21}-69}
$$

and further calculate

$$
\mathbf{v}\left(-\frac{1}{6}\right)-\mathbf{w}\left(-\frac{1}{6}\right)-\ell_0|_{K=-\frac{1}{6}}=\frac{-21\sqrt{2}-92\sqrt{3}+60\sqrt{7}+5\sqrt{42}}{45\sqrt{21}-207}<0.
$$

It follows that

$$
\forall K<-\frac{1}{12}:\quad\ell_0\notin\left(0,\mathbf{v}(K)-\mathbf{w}(K)\right).
$$

Since $\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)$ can be smoothly extended to $\ell\in\mathbb{R}$ as mentioned before, we now consider $\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^*g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)\Big|_{\ell=0}$. We have seen in (7.35) and

(7.36) that the sign of $\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)\Big|_{\ell=0}$ is constant for $K<-\frac{1}{12}$, and it thus coincides with

$$\operatorname{sgn}\left(\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)\Big|_{\ell=0,\,K=-\frac{1}{6}}\right)=-1.$$

We deduce that $\partial_\ell\left(f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\right)<0$ for all $K<-\frac{1}{12}$ and correspondingly for all $\ell\in(0,\mathbf{v}(K)-\mathbf{w}(K))$. Since by construction (7.33)

$$f^2_{\mathbf{w}(K)+\ell,K}(t_0)\cdot\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}\Big|_{\ell=0}\equiv 0$$

for all $K<-\frac{1}{12}$. We conclude with (7.34) that

$$\left(\Phi^* g_{\mathcal{H}_{\mathbf{w}(K)+\ell,K}}\right)_{t_0}<0$$

for all $K<-\frac{1}{12}$ and correspondingly all $\ell\in(0,\mathbf{v}(K)-\mathbf{w}(K))$. This finally implies that for all such K and ℓ, the corresponding maximally extended connected quartic GPSR curve $\mathcal{H}_{\mathbf{w}(K)+\ell,K}$ is not closed and thereby not a quartic CCGPSR curve.

We have now, as it will turn out, identified all $(L,K)^T\in\mathbb{R}^2$, such that the corresponding maximally extended connected quartic GPSR curve $\mathcal{H}_{L,K}$ is not closed, namely

$$\begin{pmatrix}L\\K\end{pmatrix}\in\left\{K>\frac{1}{4}\right\}\cup\left\{K\in\left[-\frac{1}{12},\frac{1}{4}\right],\ |L|>\mathbf{u}(K)\right\}\cup\left\{K<-\frac{1}{12},\ |L|\geq\mathbf{w}(K)\right\},\quad(7.37)$$

see Figure 19. We will now show that every point $(L,K)^T$ not contained in the set (7.37)

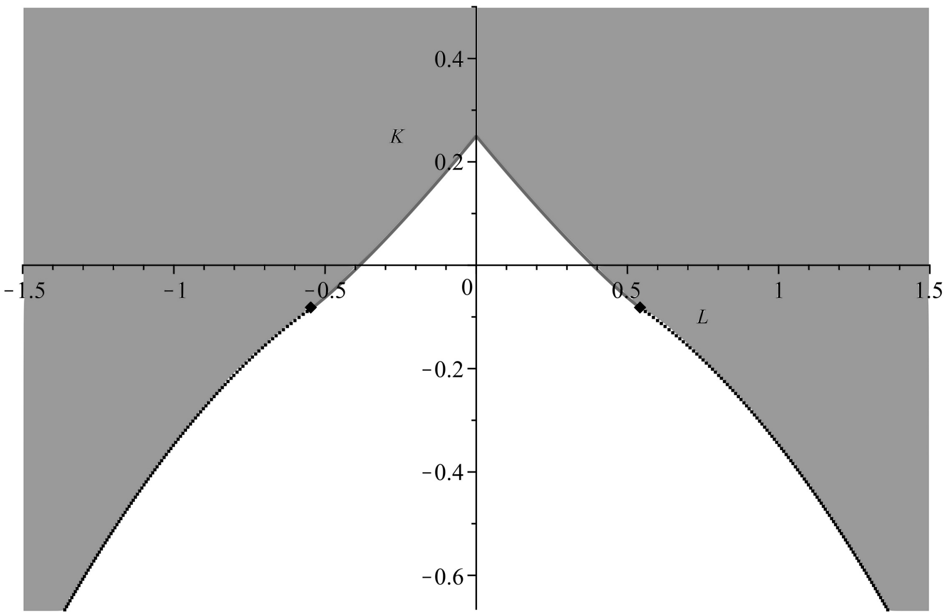


Figure 19: The set (7.37) marked in grey.

does indeed define a quartic CCGPSR curve $\mathcal{H}_{L,K}$, and we will show that we can choose for each equivalence class of such a curve a representative which is either a), b), c), or contained in the one-parameter family of quartic CCGPSR curves d).

We start with the points contained in the image of $(\mathbf{u}(K),K)^T$, $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$, cf. (7.20). We have shown that this set coincides with the image of a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$.

Thus it suffices to show that one point in that set defines a quartic CCGPSR curve to conclude that all points have that property, and furthermore that the corresponding quartic CCGPSR curves are all equivalent. We choose to check

$$\begin{pmatrix} L \\ K \end{pmatrix} = \begin{pmatrix} \mathbf{u}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3\sqrt{3}} \\ 0 \end{pmatrix}.$$

In this case, the corresponding function $\beta = \beta(t)$ as defined in (3.22) coincides with $f_{\frac{2}{3\sqrt{3}},0}(t)$ and has the form $\beta = 1 - t^2 + \frac{2}{3\sqrt{3}}t^3$. In (3.33) we have seen that

$$\left(\Phi^* g_{\mathcal{H}_{\frac{2}{3\sqrt{3}},0}}\right)_t = -\frac{\partial^2 \beta_t}{4\beta(t)} + \frac{3d\beta_t^2}{16\beta^2(t)}.$$

Note that the connected component of $\left\{f_{\frac{2}{3\sqrt{3}},0}(t) > 0\right\}$ that contains $t = 0$ is given by $\left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. The form of β and Theorem 5.6 motivate considering the cubic polynomial

$$\widetilde{h} = x^3 - xy^2 + \frac{2}{3\sqrt{3}}y^3$$

with $\widetilde{\beta}(t) = \widetilde{h}\left(\left(\begin{smallmatrix}1\\t\end{smallmatrix}\right)\right) = \beta(t)$, corresponding CCPSR curve $\widetilde{\mathcal{H}}$, and centro-affine metric

$$\left(\Phi^* g_{\widetilde{\mathcal{H}}}\right)_t = -\frac{\partial^2 \beta_t}{3\beta(t)} + \frac{2d\beta_t^2}{9\beta^2(t)}$$

on $\operatorname{dom}\left(\widetilde{\mathcal{H}}\right) = \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$. Now we see that

$$\left(\Phi^* g_{\mathcal{H}_{\frac{2}{3\sqrt{3}},0}}\right)_t = \frac{3}{4} \cdot \left(\Phi^* g_{\widetilde{\mathcal{H}}}\right)_t + \underbrace{\frac{d\beta_t^2}{48\beta^2(t)}}_{\geq 0} > 0.$$

Since the PSR curve $\left(\widetilde{\mathcal{H}}, g_{\widetilde{\mathcal{H}}}\right)$ is equivalent to the curve A) in Theorem 2.45 and, hence, is in particular a closed PSR curve, this shows that $\operatorname{dom}\left(\mathcal{H}_{\frac{2}{3\sqrt{3}},0}\right) = \operatorname{dom}\left(\widetilde{\mathcal{H}}\right) = \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$, which coincides with the connected component of $\left\{f_{\frac{2}{3\sqrt{3}},0}(t) > 0\right\}$ that contains $t = 0$ and, hence, proves that $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ is indeed also closed. Thus $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ is in fact a quartic CCGPSR curve, which proves the claim that all points in the set described in (7.20) define quartic CCGPSR curves, each equivalent to $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$. As mentioned once before in this proof, this is precisely the quartic CCGPSR curve c). It remains to determine the closed hyperbolic connected components of $\left\{h_{\frac{2}{3\sqrt{3}},0} = x^4 - x^2y^2 + \frac{2}{3\sqrt{3}}xy^3 = 1\right\}$ and show that they are equivalent as quartic CCGPSR manifolds. To do so we will determine the connected components of $\left\{h_{\frac{2}{3\sqrt{3}}} > 0\right\} \subset \mathbb{R}^2$. Since the quartic polynomial $h_{\frac{2}{3\sqrt{3}},0}$ is homogeneous, it suffices to study $\left\{h_{\frac{2}{3\sqrt{3}},0}\left(\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\right) > 0\right\}$ and $\left\{h_{\frac{2}{3\sqrt{3}},0}\left(\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\right) > 0\right\}$. We obtain

$$y \in \left\{h_{\frac{2}{3\sqrt{3}},0}\left(\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\right) > 0\right\} \quad \Leftrightarrow \quad y \in \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right) \dot{\cup} \left(\sqrt{3}, \infty\right)$$

and, using $h_{\frac{2}{3\sqrt{3}},0}\left(\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\right)=x\left(x+\frac{2}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)^2$,

$$x\in\left\{h_{\frac{2}{3\sqrt{3}}}\left(\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\right)>0\right\}\quad\Leftrightarrow\quad x\in\left(-\infty,-\frac{2}{\sqrt{3}}\right)\dot{\cup}\left(0,\frac{1}{\sqrt{3}}\right)\dot{\cup}\left(\frac{1}{\sqrt{3}},\infty\right).$$

Hence,

$$\begin{aligned}\left\{h_{\frac{2}{3\sqrt{3}},0}>0\right\}=\mathbb{R}_{>0}\cdot&\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\;\middle|\;y\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)\right\}\\ \dot{\cup}\,\mathbb{R}_{>0}\cdot&\left\{\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\;\middle|\;x\in\left(0,\frac{1}{\sqrt{3}}\right)\right\}\\ \dot{\cup}\,\mathbb{R}_{>0}\cdot&\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\;\middle|\;y\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)\right\}\\ \dot{\cup}\,\mathbb{R}_{>0}\cdot&\left\{-\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\;\middle|\;x\in\left(0,\frac{1}{\sqrt{3}}\right)\right\}.\end{aligned}\tag{7.38}$$

The quartic CCGPSR curve c) is contained in the set $\mathbb{R}_{>0}\cdot\left\{\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\in\mathbb{R}^2\;\middle|\;y\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)\right\}$, and we see that it is equivalent to the unique quartic CCGPSR curve contained in the set $\mathbb{R}_{>0}\cdot\left\{-\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\in\mathbb{R}^2\;\middle|\;y\in\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)\right\}$ via $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\mapsto-\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$. We will now show that the remaining connected components of $\left\{h_{\frac{2}{3\sqrt{3}},0}>0\right\}$ also contain a (unique) quartic CCGPSR curve that is equivalent to c). It suffices to consider the set $\mathbb{R}_{>0}\cdot\left\{\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\in\mathbb{R}^2\;\middle|\;x\in\left(0,\frac{1}{\sqrt{3}}\right)\right\}$. One can easily check that the point $\begin{pmatrix}\frac{1}{\sqrt[4]{5}}\\\frac{2\sqrt{3}}{\sqrt[4]{5}}\end{pmatrix}\in\left\{h_{\frac{2}{3\sqrt{3}},0}=1\right\}$ is a hyperbolic point of h. Consider the linear transformation of the form (3.7)

$$A\left(\begin{pmatrix}\frac{1}{\sqrt[4]{5}}\\\frac{2\sqrt{3}}{\sqrt[4]{5}}\end{pmatrix}\right)=\left(\begin{array}{c|c}\frac{1}{\sqrt[4]{5}}&\frac{1}{\sqrt[4]{5}}\\\hline\frac{2\sqrt{3}}{\sqrt[4]{5}}&\frac{1}{\sqrt{3}\sqrt[4]{5}}\end{array}\right),$$

mapping $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ to said point. Then

$$\left(h_{\frac{2}{3\sqrt{3}},0}\circ A\left(\begin{pmatrix}\frac{1}{\sqrt[4]{5}}\\\frac{2\sqrt{3}}{\sqrt[4]{5}}\end{pmatrix}\right)\right)\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=x^4-x^2y^2+\frac{4}{27}xy^3+\frac{4}{27}y^4.$$

We find that $\mathbf{u}\left(\frac{4}{27}\right)=\frac{4}{27}$ and deduce that the maximal extension of the quartic GPSR curve contained in $\left\{h_{\frac{2}{3\sqrt{3}},0}=1\right\}$ which contains the point $p=\begin{pmatrix}\frac{1}{\sqrt[4]{5}}\\\frac{2\sqrt{3}}{\sqrt[4]{5}}\end{pmatrix}$ is equivalent to the quartic CCGPSR curve c) and, in particular, closed and connected. Summarising, we have shown that $\left\{h_{\frac{2}{3\sqrt{3}},0}=1\right\}$ has 4 equivalent closed connected hyperbolic components, one of which is given by the quartic CCGPSR curve $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ which is precisely the quartic CCGPSR curve $\mathcal{H}$ in c). In order to determine $G^{h_{\frac{2}{3\sqrt{3}},0}}$, it remains to show that the only linear map $A\in\mathrm{GL}(2)$ mapping $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ to itself, that is $A\mathcal{H}_{\frac{2}{3\sqrt{3}},0}=\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$, is the identity transformation $A=\mathbb{1}\in\mathrm{GL}(2)$. Using the condition $A\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\in\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$, one can check that A must be of the form $A=A(p)$ as in (3.7), where we view $E(p)$ as an element of $\mathbb{R}\backslash\{0\}$. But then, independent of the sign $\pm$ of $E(p)$, we find with $p=\frac{1}{\sqrt[4]{h_{\frac{2}{3\sqrt{3}},0}((1,T)^T)}}\left(\begin{smallmatrix}1\\T\end{smallmatrix}\right)$, $T\in\operatorname{dom}\left(\mathcal{H}_{\frac{2}{3\sqrt{3}},0}\right)=\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)$, and with the formulas (7.8) and (7.9) and the notation

$$h_{\frac{2}{3\sqrt{3}},0}\left(A(p)\cdot\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)=x^4-x^2y^2+L(T)xy^3+K(T)y^4$$

that

$$K(T)=\frac{T\left(-T^7+4\sqrt{3}T^6+18T^5-84\sqrt{3}T^4+63T^3+432\sqrt{3}T^2-1404T+432\sqrt{3}\right)}{12\left(-T^4+2\sqrt{3}T^3-9T^2+12\sqrt{3}T-18\right)^2}. \tag{7.39}$$

Using a computer algebra system like MAPLE, one can show that the denominator of $K(T)$ in the above formula (7.39) does not have any roots in $\mathrm{dom}\left(\mathcal{H}_{\frac{2}{3\sqrt{3}},0}\right)=\left(-\frac{\sqrt{3}}{2},\sqrt{3}\right)$, and the numerator has only one root in $\mathrm{dom}\left(\mathcal{H}_{\frac{2}{3\sqrt{3}},0}\right)$, namely $T=0$. We conclude that the only linear transformations of the form (3.7) can either be the identity, or (corresponding to a possible minus sign of $E\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$) the linear transformation $\widetilde{A}=\begin{pmatrix}1&\\&-1\end{pmatrix}$. But we can quickly check that $h_{\frac{2}{3\sqrt{3}},0}\circ\widetilde{A}=x^4-x^2y^2-\frac{2}{3\sqrt{3}}\neq h_{\frac{2}{3\sqrt{3}},0}$. We can now use the previous results and obtain

$$G^{h_{\frac{2}{3\sqrt{3}},0}}\cong\mathbb{Z}_4.$$

Now we will study all maximally extended quartic GPSR curves of the form $\mathcal{H}_{0,K}$, $K\leq\frac{1}{4}$, that correspond to points of the form

$$\begin{pmatrix}L\\K\end{pmatrix}\in\left\{K\leq\frac{1}{4},\ L=0\right\}. \tag{7.40}$$

We will prove that each of these curves is a quartic CCGPSR curve, and furthermore that they are pairwise inequivalent. We have seen in (7.14) that the connected component of $\{f_{0,K}(t)>0\}$ that contains the point $t=0$ is precompact for all $K\leq\frac{1}{4}$. For each $K\leq\frac{1}{4}$, we have $f_{0,K}(t)=\beta(t)$ as in (3.22) and, hence, the pullback of the centro-affine metric $g_{\mathcal{H}_{0,K}}$ to $\mathrm{dom}\left(\mathcal{H}_{0,K}\right)$ (3.33) fulfils

$$\begin{aligned}f_{0,K}^2(t)\left(\Phi^*g_{\mathcal{H}_{0,K}}\right)_t&=\left(-\frac{f_{0,K}(t)\ddot{f}_{0,K}(t)}{4}+\frac{3\dot{f}_{0,K}^2(t)}{16}\right)dt^2\\&=\left(\frac{K}{2}t^4+\left(\frac{1}{4}-3K\right)t^2+\frac{1}{2}\right)dt^2=:\widetilde{g}_K(t)dt^2.\end{aligned}$$

First consider $K\leq 0$. Then $\ddot{f}_{0,K}(t)=-2+12Kt^2<0$ for all $t\in\mathbb{R}$. This immediately shows that $\left(\Phi^*g_{\mathcal{H}_{0,K}}\right)_t>0$ for all t contained in the connected component of $\{f_{0,K}(t)>0\}$ that contains $t=0$, which thus coincides with $\mathrm{dom}\left(\mathcal{H}_{0,K}\right)$. We deduce that for all $K\leq 0$, the corresponding maximally extended quartic GPSR curve $\mathcal{H}_{0,K}$ is closed and, hence, a quartic CCGPSR curve. Next, we will show that for all $0<K<\frac{1}{4}$ the smooth function $\widetilde{g}_K:\mathbb{R}\to\mathbb{R}$ has no real roots and is positive. For $K=\frac{1}{4}$, we will show that $\widetilde{g}_K(t)=0$ if and only if $f_{0,K}(t)=0$. This will then imply that for all $0<K\leq\frac{1}{4}$ the set $\mathrm{dom}\left(\mathcal{H}_{0,K}\right)$ and the connected component containing $t=0$ of $\{f_{0,K}(t)>0\}$ coincide, which shows that the maximally extended quartic GPSR curve $\mathcal{H}_{0,K}$ is closed, and thus a quartic CCGPSR curve. We obtain for $0<K<\frac{1}{4}$ the (symbolic) equivalence

$$\widetilde{g}_K(t)=0\quad\Leftrightarrow\quad t^2=\frac{-1+12K\pm\sqrt{144K^2-40K+1}}{4K}.$$

For t to be real, one of the two possible terms $-1+12K\pm\sqrt{144K^2-40K+1}$ must be real and non-negative (since $0<K\leq\frac{1}{4}$). We find that

$$144K^2-40K+1\geq 0\text{ and }K\in\left(0,\frac{1}{4}\right)\quad\Leftrightarrow\quad K\in\left(0,\frac{1}{36}\right],$$

and we observe that $\frac{1}{36} < \frac{1}{4}$ is a root of $144K^2 - 40K + 1$. This shows that $\widetilde{g}_K(t)$ might only have real roots in the considered domain $\left(0, \frac{1}{4}\right)$ for K if $K \in \left(0, \frac{1}{36}\right]$. One now verifies that $-1 + 12K - \sqrt{144K^2 - 40K + 1}$ has no real roots. Since $144K^2 - 40K + 1 \geq 0$ restricts K to be an element of $\left(0, \frac{1}{36}\right]$, we evaluate

$$-1 + 12K - \sqrt{144K^2 - 40K + 1}\Big|_{K=\frac{1}{72}} = -\frac{5+\sqrt{17}}{6} < 0.$$

We further obtain

$$-1 + 12K + \sqrt{144K^2 - 40K + 1} = 0 \quad \Leftrightarrow \quad K = 0$$

and deduce that the sign of the term $\frac{-1+12K+\sqrt{144K^2-40K+1}}{4K}$ is constant for $K \in \left(0, \frac{1}{36}\right]$. We evaluate

$$-1 + 12K + \sqrt{144K^2 - 40K + 1}\Big|_{K=\frac{1}{72}} = -\frac{5-\sqrt{17}}{6} < 0.$$

We conclude that there exists no $K \in \left(0, \frac{1}{4}\right)$, such that either $\frac{-1+12K-\sqrt{144K^2-40K+1}}{4K}$ or $\frac{-1+12K+\sqrt{144K^2-40K+1}}{4K}$ are positive. This and $\widetilde{g}_K(0) = \frac{1}{2}$ proves the claim that for all $K \in \left(0, \frac{1}{4}\right)$ the function $\widetilde{g}_K(t)$ is positive on $\mathbb{R}$ and, hence, that each corresponding maximally extended quartic GPSR curve $\mathcal{H}_{0,K}$ is a quartic CCGPSR curve.

Now consider the case $K = \frac{1}{4}$ and note that $\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \frac{1}{4} \end{smallmatrix}\right) \in \{\mathcal{V} = 0\}$. We calculate

$$\widetilde{g}_{\frac{1}{4}}(t) = 0 \quad \Leftrightarrow \quad t = \pm\sqrt{2},$$

which are precisely the roots of $f_{0,\frac{1}{4}}(t)$. Again, $\widetilde{g}_{\frac{1}{4}}(0) = \frac{1}{2}$ implies that $\widetilde{g}_{\frac{1}{4}}$, restricted to the connected component $\left\{f_{0,\frac{1}{4}}(t) > 0\right\}$ that contains the point $t = 0$, is positive. Similarly to $K \in \left(0, \frac{1}{4}\right)$ we conclude that the maximally extended quartic GPSR curve $\mathcal{H}_{0,\frac{1}{4}}$ is closed and, hence, a quartic CCGPSR curve. The case $K = \frac{1}{4}$ and the cases $K < \frac{1}{4}$ correspond to the polynomial *a*) and the one-parameter family of polynomials *d*), respectively. The quartic CCGPSR curve $\mathcal{H}_{0,\frac{1}{4}}$ is furthermore a homogeneous space under the action of the corresponding Lie group $G_0^{h_{0,\frac{1}{4}}}$, cf. Definition 3.13.This follows from Proposition 3.34 since $\left(0, \frac{1}{4}\right)^T \in \{\mathcal{V} = 0\}$ (7.5). Note that, using [CNS, Prop. 1.8], the homogeneity of $\mathcal{H}_{0,\frac{1}{4}}$ would also have been sufficient to prove that $\mathcal{H}_{0,\frac{1}{4}}$ is closed as a subset of $\mathbb{R}^2$, since Riemannian homogeneous spaces are always complete, cf. Remark 3.10.

It remains to prove the claim that the quartic CCGPSR curves $\mathcal{H}_{0,K}$ for $K \leq \frac{1}{4}$ are pairwise inequivalent. While proving this statement we will also determine the closed hyperbolic connected components of $\{h_{0,K} = 1\}$ and show that these are always equivalent for each fixed $K \leq \frac{1}{4}$.

Since $\left(0, \frac{1}{4}\right)^T \in \{\mathcal{V} = 0\}$ and $(0, K)^T \in \mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$ for all $K < \frac{1}{4}$, we can use Proposition 3.34 which implies that the connected component of the automorphism group of $h_{0,K}$ acts transitively on $\mathcal{H}_{0,K}$ if and only if $K = \frac{1}{4}$. In particular this shows that $\mathcal{H}_{0,\frac{1}{4}}$ is not equivalent to $\mathcal{H}_{0,K}$ for any $K < \frac{1}{4}$. It remains to show that for $K < \frac{1}{4}$ the quartic CCGPSR curves $\mathcal{H}_{0,K}$ are pairwise inequivalent. For fixed $K < \frac{1}{4}$ we want to determine every $A \in \mathrm{GL}(2)$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

such that

$$(h_{0,K}\circ A)\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)=x^4-x^2y^2+\widetilde{K}y^4=h_{0,\widetilde{K}}\tag{7.41}$$

for some $\widetilde{K}<\frac{1}{4}$ with the additional restriction that the quartic CCGPSR curves $\mathcal{H}_{0,K}$ and $\mathcal{H}_{0,\widetilde{K}}$ are required to be equivalent via $A:\mathcal{H}_{0,\widetilde{K}}\to\mathcal{H}_{0,K}$. For A to fulfil (7.41) and additionally map $\mathcal{H}_{0,\widetilde{K}}$ to $\mathcal{H}_{0,K}$ it is necessary that $h\left(\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)\right)=1$ and furthermore $\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)$ is required to be a hyperbolic point of $h_{0,K}$. We will treat the two cases $a_{11}\neq 0$ and $a_{11}=0$ separately. We start with assuming that $a_{11}\neq 0$. Then

$$(h_{0,K}\circ A)\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)=dh_{\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)}\left(\left(\begin{smallmatrix}a_{12}\\a_{22}\end{smallmatrix}\right)\right)x^3y+\text{rest},$$

where the “rest” in the above equation does not contain a non-trivial multiple of the monomial x^3y. Hence, up to a scaling factor $r\in\mathbb{R}\setminus\{0\}$,

$$\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}=r\begin{pmatrix}-\frac{\partial h}{\partial y}\Big|_{\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)}\\ \frac{\partial h}{\partial x}\Big|_{\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)}\end{pmatrix}.\tag{7.42}$$

Then, for $a_{21}=0$,

$$(h_{0,K}\circ A)\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)=a_{11}^4x^4-16r^2a_{11}^8x^2y^2+256Kr^4a_{11}^12y^4.$$

In these cases we thus immediately obtain with (7.42) that $A\in\mathrm{GL}(2)$ needs to fulfil precisely

$$A\in\left\{\mathbb{1},-\mathbb{1},\begin{pmatrix}-1&\\&1\end{pmatrix},\begin{pmatrix}1&\\&-1\end{pmatrix}\right\}.$$

to solve (7.41). Note that in these cases $\widetilde{K}=K$. For $a_{21}\neq 0$ consider

$$\begin{aligned}&(h\circ A)\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\\&=64r^3a_{11}a_{21}\\&\quad\cdot\left(-4K^3a_{21}^8+K^2\left(4a_{11}^2a_{21}^6+a_{21}^8\right)+K\left(4a_{11}^8-4a_{11}^6a_{21}^2-a_{11}^2a_{21}^6\right)-a_{11}^8+a_{11}^6a_{21}^2\right)xy^3\\&\quad+\text{rest},\end{aligned}$$

where the “rest” in the above equation does not contain any other xy^3-monomial. In order for (7.41) to be fulfilled, we thus need that (since by assumption $a_{11}\neq 0$ and $a_{21}\neq 0$)

$$-4K^3a_{21}^8+K^2\left(4a_{11}^2a_{21}^6+a_{21}^8\right)+K\left(4a_{11}^8-4a_{11}^6a_{21}^2-a_{11}^2a_{21}^6\right)-a_{11}^8+a_{11}^6a_{21}^2=0.\tag{7.43}$$

Solving equation (7.43) (symbolically) for K, we obtain

$$(7.43)\quad\Leftrightarrow\quad K\in\left\{\frac{1}{4},-\frac{a_{11}^2\left(a_{11}^2-a_{21}^2\right)}{a_{21}^4},\frac{a_{11}^4}{a_{21}^4}\right\}.$$

The value $K=\frac{1}{4}$ has already been excluded. For $K=-\frac{a_{11}^2\left(a_{11}^2-a_{21}^2\right)}{a_{21}^4}$ we get

$$(h_{0,K}\circ A)\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\equiv 0,$$

hence we can also exclude this (symbolic) solution for K. For the last possible solution for K, that is $K=\frac{a_{11}^4}{a_{21}^4}$, consider the condition

$$h_{0,\frac{a_{11}^4}{a_{21}^4}}\left(\left(\begin{smallmatrix}a_{11}\\a_{21}\end{smallmatrix}\right)\right)=1\quad\Leftrightarrow\quad a_{11}^2\left(2a_{11}^2-a_{21}^2\right)=1.$$

Hence, $a_{21}^2 = 2a_{11}^2 - a_{11}^{-2}$ and, consequently,

$$K = \frac{a_{11}^4}{a_{21}^4} = \frac{a_{11}^8}{(2a_{11}^4 - 1)^2}.$$

However

$$\frac{a_{11}^8}{(2a_{11}^4 - 1)^2} > \frac{1}{4}$$

for all $a_{11} \neq 0$. We deduce that this solution for K also does not fulfil our requirements, in this case the requirement $K < \frac{1}{4}$. Summarising, we have shown that for $a_{11} \neq 0$ the only linear transformations $A \in \mathrm{GL}(2)$ that fulfil (7.41) for $K < \frac{1}{4}$ are given by

$$A \in \left\{ \mathbb{1}, -\mathbb{1}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}, \tag{7.44}$$

and that in each case $\widetilde{K} = K$. Next consider the case $a_{11} = 0$. In this case, for (7.41) to be true, $h_{0,K}\left(\left(\begin{smallmatrix} 0 \\ a_{21} \end{smallmatrix}\right)\right) = K a_{21}^4$ must fulfil $K a_{21}^4 = 1$. We see that this can only be the case for positive K, and thus can already say that for all $K \leq 0$ the transformation A solves (7.41) if and only if $A \in \left\{ \mathbb{1}, -\mathbb{1}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$ and that in these cases $\widetilde{K} = K$. For $0 < K < \frac{1}{4}$ we obtain

$$K a_{21}^4 = 1 \quad \Leftrightarrow \quad a_{21} = \pm \frac{1}{\sqrt[4]{K}}.$$

Under the assumption that $a_{11} = 0$ and $a_{21} = \pm \frac{1}{\sqrt[4]{K}}$ we find that

$$(h_{0,K} \circ A)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = 4K a_{21}^3 a_{22} x^3 y + \text{rest} = \pm 4\sqrt[4]{K} a_{22} x^3 y + \text{rest},$$

where the above "rest" does not contain any x^3y-part. Since the x^3y-part of $(h_{0,K} \circ A)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$ must vanish for (7.41) to be fulfilled, we deduce that $a_{22} = 0$. So A must be of the form $A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$. Now for the final step we check that

$$(h_{0,K} \circ A)\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = -a_{21}^2 a_{12}^2 x^2 y^2 + \text{rest} = -\frac{a_{21}^2}{\sqrt{K}} x^2 y^2 + \text{rest},$$

the above "rest" not containing any x^2y^2-part, which shows that $a_{21} = \pm\sqrt[4]{K}$. Summarising, we have shown that the possible $A \in \mathrm{GL}(2)$ that fulfil (7.41) for $0 < K < \frac{1}{4}$ can only be of the form

$$\begin{aligned} A \in \Bigg\{ & \mathbb{1}, -\mathbb{1}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \\ & \begin{pmatrix} & \sqrt[4]{K} \\ \frac{1}{\sqrt[4]{K}} & \end{pmatrix}, \begin{pmatrix} & -\sqrt[4]{K} \\ \frac{1}{\sqrt[4]{K}} & \end{pmatrix}, \begin{pmatrix} & \sqrt[4]{K} \\ -\frac{1}{\sqrt[4]{K}} & \end{pmatrix}, \begin{pmatrix} & -\sqrt[4]{K} \\ -\frac{1}{\sqrt[4]{K}} & \end{pmatrix} \Bigg\}, \end{aligned} \tag{7.45}$$

and one can easily check that these matrices actually solve (7.41) with $\widetilde{K} = K$. We now conclude that the quartic CCGPSR curves $\mathcal{H}_{0,K}$ for $K \leq \frac{1}{4}$ are pairwise inequivalent. Now, similar to the quartic CCGPSR curve c), equation (7.38), we will determine for each $K \leq \frac{1}{4}$ the connected components of $\{h_{0,K} > 0\} \subset \mathbb{R}^2$. We have

$$h_{0,K}\left(\left(\begin{smallmatrix} 1 \\ y \end{smallmatrix}\right)\right) = 1 - y^2 + K y^4, \quad h_{0,K}\left(\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\right) = x^4 - x^2 + K,$$

and obtain

$$\underline{\text{for } K=\frac{1}{4}:}\quad \left\{h_{0,\frac{1}{4}}>0\right\}=\mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{2},\sqrt{2}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\ \middle|\ x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{2},\sqrt{2}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\ \middle|\ x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\},$$

(note: for $K=\frac{1}{4}$, $h_{0,\frac{1}{4}}\leq 0$ if and only if $h_{0,\frac{1}{4}}=0$)

$$\underline{\text{for } K\in\left(0,\frac{1}{4}\right):}$$
$$\{h_{0,K}>0\}=\mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{\frac{2\left(1-\sqrt{1-\frac{K}{4}}\right)}{K}},\sqrt{\frac{2\left(1-\sqrt{1-\frac{K}{4}}\right)}{K}}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\ \middle|\ x\in\left(-\sqrt{\frac{1-\sqrt{1-4K}}{2}},\sqrt{\frac{1-\sqrt{1-4K}}{2}}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{\frac{2\left(1-\sqrt{1-\frac{K}{4}}\right)}{K}},\sqrt{\frac{2\left(1-\sqrt{1-\frac{K}{4}}\right)}{K}}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}x\\1\end{pmatrix}\in\mathbb{R}^2\ \middle|\ x\in\left(-\sqrt{\frac{1-\sqrt{1-4K}}{2}},\sqrt{\frac{1-\sqrt{1-4K}}{2}}\right)\right\},$$

$$\underline{\text{for } K=0:}\quad \{h_{0,0}>0\}=\mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in(-1,1)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in(-1,1)\right\},$$

and

$$\underline{\text{for } K<0:}\quad \{h_{0,K}>0\}$$
$$=\mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{\frac{2\left(1+\sqrt{1-\frac{K}{4}}\right)}{K}},\sqrt{\frac{2\left(1+\sqrt{1-\frac{K}{4}}\right)}{K}}\right)\right\}$$
$$\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2\ \middle|\ y\in\left(-\sqrt{\frac{2\left(1+\sqrt{1-\frac{K}{4}}\right)}{K}},\sqrt{\frac{2\left(1+\sqrt{1-\frac{K}{4}}\right)}{K}}\right)\right\}.$$

For $K\leq 0$ we see that $\{h_{0,K}>0\}$ has exactly 2 connected components, and the corresponding unique contained quartic CCGPSR curves are equivalent via $A=-\mathbb{1}$, cf. (7.44). For $0<K<\frac{1}{4}$, $\{h_{0,K}>0\}$ has exactly 4 connected components, and the corresponding unique contained quartic CCGPSR curves are equivalent via compositions of $A=\begin{pmatrix}&-1\\1&\end{pmatrix}$ and

$A = \begin{pmatrix} & -\sqrt[4]{K} \\ \frac{1}{\sqrt[4]{K}} & \end{pmatrix}$, cf. (7.45). In the case $K = \frac{1}{4}$, $\left\{h_{0,\frac{1}{4}} > 0\right\}$ has exactly 4 connected components, and one can check that transformations $A \in \mathrm{GL}(2)$ of the form (7.45) are also automorphisms of $h_{0,\frac{1}{4}}$ as for $0 < K < \frac{1}{4}$ and map the corresponding unique contained quartic CCGPSR curves bijectively to each other. Now, for the automorphism groups $G^{h_{0,K}}$ of $h_{0,K}$ for $K \leq \frac{1}{4}$, (7.44) and (7.45) imply that

$$\forall K \leq 0: \quad G^{h_{0,K}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$\forall K \in \left(0, \frac{1}{4}\right): \quad G^{h_{0,K}} \cong \mathbb{Z}_4 \times \mathbb{Z}_2.$$

In order to explicitly find $G^{h_{0,\frac{1}{4}}}$, we need to determine $G_0^{h_{0,\frac{1}{4}}}$. To do so we will derive a suitable basis for $T_{\mathbb{1}} G_0^{h_{0,\frac{1}{4}}}$ as a Lie subalgebra of $\mathfrak{gl}(2)$. For this we will use the techniques used in Proposition 3.34 and calculate the derivative of the corresponding map $\mathcal{A} : \mathrm{dom}\left(\mathcal{H}_{0,\frac{1}{4}}\right) \to \mathrm{GL}(2)$ (3.23) at $z = 0 \in \mathrm{dom}\left(\mathcal{H}_{0,\frac{1}{4}}\right)$, cf. (3.60). Note that the corresponding $dB_0 \in \mathrm{Lin}(\mathbb{R}, \mathfrak{so}(1))$ is always zero since $\dim(\mathfrak{so}(1)) = 0$. We find

$$d\mathcal{A}_0(\partial_z) = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

and obtain

$$dh_{0,\frac{1}{4}}\Big|_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} \left(\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right) \equiv 0$$

as expected. Let $a_{\frac{1}{4}} := \sqrt{2} \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$. Then $a_{\frac{1}{4}}^2 = \mathbb{1}$ and

$$\exp\left(t a_{\frac{1}{4}}\right) = \sum_{k=0}^{\infty} \frac{t^k}{k!} a_{\frac{1}{4}}^k = \left(\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right) \mathbb{1} + \left(\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\right) a_{\frac{1}{4}} = \begin{pmatrix} \cosh(t) & \frac{\sinh(t)}{\sqrt{2}} \\ \sqrt{2}\sinh(t) & \cosh(t) \end{pmatrix} \tag{7.46}$$

for all $t \in \mathbb{R}$. Now we have an explicit description of $G_0^{h_{0,\frac{1}{4}}}$. In order to find a well known and commonly used Lie group that is isomorphic (as a Lie group) to $G_0^{h_{0,\frac{1}{4}}}$, observe that

$$a_{\frac{1}{4}}^T \begin{pmatrix} -2 & \\ & 1 \end{pmatrix} + \begin{pmatrix} -2 & \\ & 1 \end{pmatrix} a_{\frac{1}{4}} = 0. \tag{7.47}$$

This implies that the Lie algebra $T_{\mathbb{1}} G_0^{h_{0,\frac{1}{4}}}$ is isomorphic to $\mathfrak{so}(1,1)$, that is the linear automorphisms of the Lorentz vector space $(\mathbb{R}, -2dx^2 + dy^2)$, and that $G_0^{h_{0,\frac{1}{4}}} \cong \mathrm{SO}^+(1,1)$. Hence,

$$G^{h_{0,\frac{1}{4}}} \cong \mathrm{SO}^+(1,1) \times \mathbb{Z}_4 \times \mathbb{Z}_2.$$

With Proposition 3.34 and (7.45) this also shows that

$$\mathcal{H}_{0,\frac{1}{4}} \cong \mathrm{SO}^+(1,1),$$

since the only transformations in (7.45) that map $\mathcal{H}_{0,\frac{1}{4}}$ to itself are $A = \mathbb{1}$ and $A = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ and the latter is not contained in $\mathrm{SO}^+(1,1)$.

Now we will consider points

$$\begin{pmatrix} L \\ K \end{pmatrix} \in \left\{ K < -\frac{1}{12},\ |L| < \mathbf{w}(K) \right\} \subset \mathbb{R}^2$$

and the corresponding maximally extended quartic GPSR curves $\mathcal{H}_{L,K}$. We will proceed as follows. We will show that for any such point $(L,K)^T \in \left\{ K < -\frac{1}{12},\ |L| < \mathbf{w}(K) \right\}$, the image of the maximal integral curve γ of $\mathcal{V}|_{\mathbb{R}^2 \setminus \{\mathcal{V}=0\}}$ with $\gamma(0) = (L,K)^T$ contains a point of the form $\left(0, \widetilde{K}\right)^T \in \mathbb{R}^2$, $\widetilde{K} < -\frac{1}{12}$. This will thus imply that $\mathcal{H}_{L,K}$ is equivalent to $\mathcal{H}_{0,\widetilde{K}}$, and since we have already seen that $\mathcal{H}_{0,\widetilde{K}}$ is a quartic CCGPSR curve for all $\widetilde{K} \leq \frac{1}{4}$ this will show that $\mathcal{H}_{L,K}$ is also a quartic CCGPSR curve. We will without loss of generality assume that $0 < L < \mathbf{w}(K)$ for $K < -\frac{1}{12}$ fixed, as we have already dealt with the case $L = 0$ above (7.40) and since $\mathcal{H}_{L,K}$ and $\mathcal{H}_{-L,K}$ are always equivalent. Instead of checking the maximal integral curves of $\mathcal{V}|_{\mathbb{R}^2 \setminus \{\mathcal{V}=0\}}$, respectively their restriction to the set $\left\{ K < -\frac{1}{12},\ 0 < L < \mathbf{w}(K) \right\}$, directly, we will first transform this set using a suitable diffeomorphism. Recall that $\mathbf{w}(K) = \frac{\sqrt{6-216K}}{9} > 0$ for all $K < -\frac{1}{12}$ and consider the smooth map

$$F : \left\{ K < -\frac{1}{12},\ 0 < L < \mathbf{w}(K) \right\} \to \left\{ \psi < -\frac{1}{12},\ 0 < \varphi < 1 \right\},$$
$$F : \begin{pmatrix} L \\ K \end{pmatrix} \mapsto \begin{pmatrix} \frac{L}{\mathbf{w}(K)} \\ K \end{pmatrix},$$

where $\left(\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right)$ denote the coordinates of $\left\{ \psi < -\frac{1}{12},\ 0 < \varphi < 1 \right\} \subset \mathbb{R}^2$ (see Figure 20). The

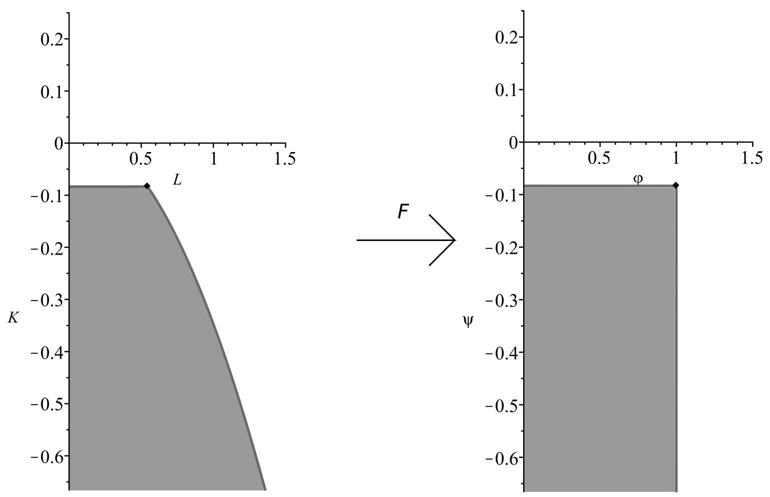


Figure 20: Parts of the domain and co-domain of F, marked in grey.

differential of F is given by

$$dF = \begin{pmatrix} \frac{1}{\mathbf{w}(K)} & \frac{162L}{(1-36K)\sqrt{6-216K}} \\ 0 & 1 \end{pmatrix}$$

and we see that F is, in fact, a diffeomorphism. We obtain for the inverse of F

$$F^{-1}\left(\left(\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \varphi \mathbf{w}(\psi) \\ \psi \end{smallmatrix}\right)$$

and for the push-forward of the vector field $\mathcal{V}$ restricted to $\{K < -1/12,\ 0 < L < \mathbf{w}(K)\}$

$$F_*\mathcal{V} = (F_*\mathcal{V})_{\left(\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right)} = \frac{9\left(1-\varphi^2\right)(-1+4\psi)}{\sqrt{6-216\psi}}\partial_\varphi + \frac{\varphi(1+12\psi)\sqrt{6-216\psi}}{18}\partial_\psi.$$

Since $d\psi\left(F_*\mathcal{V}\right) < 0$ for all $\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right) \in \left\{\psi < -\frac{1}{12},\ 0 < \varphi < 1\right\}$, we can define the smooth function

$$\begin{aligned}\mathcal{R} &: \left\{\psi < -\frac{1}{12},\ 0 < \varphi < 1\right\} \to \mathbb{R},\\ \mathcal{R}\left(\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right)\right) &:= \frac{d\varphi\left(F_*\mathcal{V}\right)}{d\psi\left(F_*\mathcal{V}\right)} = \frac{1-\varphi^2}{\varphi}\cdot\frac{27(1-4\psi)}{(1+12\psi)(-1+36\psi)}.\end{aligned}$$

Instead of studying the images of the maximal integral curves of the vector field

$$F_*\mathcal{V}|_{\left\{\psi<-\frac{1}{12},\ 0<\varphi<1\right\}},$$

we can now study the images of the maximal integral curves of the vector field

$$\mathcal{X} := \mathcal{R}\,\partial_\varphi + \partial_\psi. \tag{7.48}$$

defined on the set $\left\{\psi < -\frac{1}{12},\ 0 < \varphi < 1\right\} = F\left(\left\{K < -\frac{1}{12},\ 0 < L < \mathbf{w}(K)\right\}\right) \subset \mathbb{R}^2$, since there is a one-to-one correspondence between them, which follows from $d\psi\left(F_*\mathcal{V}\right) \neq 0$ on said set. It turns out that we can, in fact, find the general solution of the equation for integral curves of $\mathcal{X}$. For $t < -\frac{1}{12}$ and some $a < -\frac{1}{12}$, consider with $\gamma : \left(a, -\frac{1}{12}\right) \to \left\{\psi < -\frac{1}{12},\ 0 < \varphi < 1\right\}$, $\gamma(t) = \left(\begin{smallmatrix}\varphi(t)\\ t\end{smallmatrix}\right)$,

$$\begin{aligned} & \mathcal{X}_\gamma = \dot{\gamma}\\ \Leftrightarrow\quad & \frac{\varphi\dot{\varphi}}{1-\varphi^2} = \frac{27(1-4t)}{(-1+36t)(1+12t)}\\ \Leftrightarrow\quad & \varphi(t) = \sqrt{1 - \frac{c\sqrt{-(1+12t)^3}}{1-36t}},\end{aligned}$$

where $c \in \mathbb{R}_{>0}$ is chosen in such a way that the initial condition

$$\gamma(t_0) = \left(\begin{smallmatrix}\varphi(t_0)\\ t_0\end{smallmatrix}\right) \in \left\{\psi < -\frac{1}{12},\ 0 < \varphi < 1\right\}$$

is met, see Figure 21 for an example of such a curve γ (note: in our construction, the initial time t_0 fulfils $t_0 < -\frac{1}{12}$). Note that for all $t < -\frac{1}{12}$ such that $\varphi(t)$ is defined we always have $\varphi(t) < 1$, in particular arbitrarily close to $t = -\frac{1}{12}$. We will now show that $\varphi(t)$ cannot converge to the value 1 in finite negative time. Solving $\varphi(t) = 1$, we obtain as the unique negative solution $t = -\frac{1}{12}$, but this is the upper bound of the domain of definition of $\gamma(t)$ and, hence, $\varphi(t)$. This shows that for all $t < -\frac{1}{12}$, for which $\varphi(t)$ is defined, we indeed have $\varphi(t) < 1$. Thus, if we can prove that each such curve γ independent of the initial condition $(t_0, \gamma(t_0))$ converges to a point in the set $\left\{\psi < -\frac{1}{12},\ \varphi = 0\right\}$ in finite negative time[19] $t < -\frac{1}{12}$, then we will have shown using the fact $\mathcal{V}|_{\{K<-1/12,\,|L|<\mathbf{w}(K)\}} \neq 0$ that each maximal integral curve of $\mathcal{V}|_{\{K<-1/12,\,|L|<\mathbf{w}(K)\}}$ meets the set $\left\{K < -\frac{1}{12},\ L = 0\right\}$ in either finite positive or finite negative time. This means that we have to solve

$$\begin{aligned} & \sqrt{1 - \frac{c\sqrt{-(1+12t)^3}}{1-36t}} = 0,\ t < -\frac{1}{12}\\ \Leftrightarrow\quad & 1728c^2t^3 + \left(432c^2 + 1296\right) + \left(36c^2 - 72\right)t^2 + 1 = 0,\ t < -\frac{1}{12}\end{aligned}$$

[19] All possible integral curves γ move toward $\left\{\psi = -\frac{1}{12}\right\}$ in positive time-direction, hence negative time.

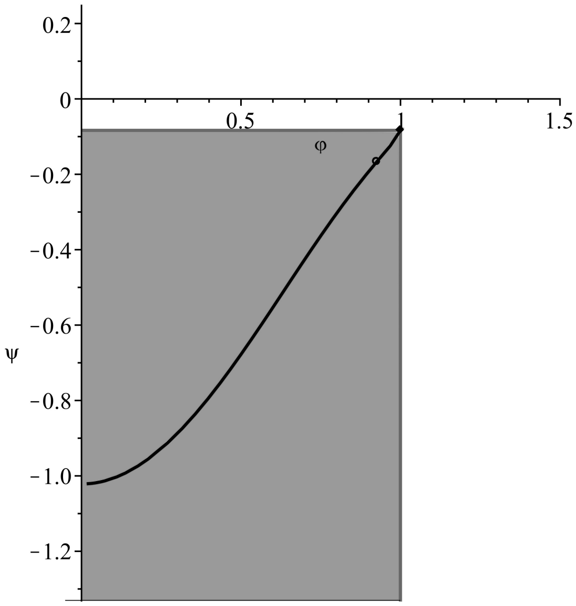


Figure 21: Example of a curve γ (in black) fulfilling $\mathcal{X}_\gamma = \dot{\gamma}$ with initial condition $\varphi\left(-\frac{1}{6}\right) = \frac{\sqrt{6}}{\sqrt{7}}$ (marked with a small cirlce).

with the restriction that we are interested in the biggest possible negative solution in t. Replacing $t = s - \frac{1}{12}$ and dividing by 16, we observe that

$$\begin{aligned} & 1728c^2t^3 + \left(432c^2 + 1296\right) + \left(36c^2 - 72\right)t^2 + 1 = 0,\ t < -\frac{1}{12} \\ \Leftrightarrow\quad & 108c^2s^3 + 81s^2 - 18s + 1 = 0,\ s < 0. \end{aligned} \tag{7.49}$$

We see that equation (7.49) always has a negative, and in particular uniquely determined biggest negative, solution[20] in s since it takes the value 1 for $s = 0$. We deduce that each maximal integral curve of $\mathcal{X}$ does meet the set $\left\{\psi < -\frac{1}{12},\ \varphi = 0\right\}$ in finite negative time and, hence, that each maximal integral curve γ of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ with initial condition $\gamma(0) \in \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\}$ meets the set $\left\{K < -\frac{1}{12},\ L = 0\right\}$ in either finite negative or finite positive time. Hence, for each maximally extended quartic GPSR curve $\mathcal{H}_{L,K}$, $(L,K)^T \in \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\}$, is equivalent to a maximally extended quartic GPSR curve of the form $\mathcal{H}_{0,\widetilde{K}}$ with $\widetilde{K} < -\frac{1}{12}$. We have already shown that the quartic CCGPSR curves $\mathcal{H}_{0,K}$, $K \leq \frac{1}{4}$, are pairwise inequivalent. Hence, the value for $\widetilde{K}$ is unique. We deduce that $\mathcal{H}_{L,K}$ is closed in $\mathbb{R}^2$ and thus a quartic CCGPSR curve as claimed.

Now consider the set $\left\{K = -\frac{1}{12},\ |L| < \mathbf{w}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}\right\}$ and the restriction of $\mathcal{V}$ to it. It turns out that this set coincides with the image of a maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$. This follows from the fact that $\left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\} \subset \mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$, and that $\mathcal{V}$ is parallel to $\left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\}$ in the sense that

$$dK\left(\mathcal{V}|_{\left\{K=-\frac{1}{12},\ |L|<\frac{2\sqrt{2}}{3\sqrt{3}}\right\}}\right) \equiv 0.$$

Hence, we only need to consider the point $(L,K)^T = \left(0, -\frac{1}{12}\right)^T \in \left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\}$

[20]This solution coincides with the minimal possible value a that was used to denote the domain of definition $\left(a, -\frac{1}{12}\right) \subset \mathbb{R}$ of $\gamma(t)$, respectively $\varphi(t)$.

and the corresponding maximally extended quartic GPSR curve $\mathcal{H}_{0,-\frac{1}{12}}$, but we have already seen that it is in fact a quartic CCGPSR curve.

Next, we will show that every maximally extended quartic connected GPSR curve $\mathcal{H}_{L,-\frac{1}{12}}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{L,-\frac{1}{12}}$, with $|L| < \mathbf{u}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}$ is equivalent to the quartic CCGPSR curve $\mathcal{H}_{0,-\frac{1}{12}}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{0,-\frac{1}{12}}$. To do so, we will study the restriction of $\mathcal{V}$ to the set

$$\left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\} \subset \mathbb{R}^2,$$

which is an embedded open connected intervall in $\mathbb{R}^2$. We find that

$$dK\left(\mathcal{V}|_{\left(\begin{smallmatrix}L\\-\frac{1}{12}\end{smallmatrix}\right)}\right) \equiv 0,$$

and since furthermore $\left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\} \cap \{\mathcal{V} = 0\} = \emptyset$ we obtain that for any point $p \in \left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\}$, every maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ through p has as image precisely the considered set $\left\{K = -\frac{1}{12},\ |L| < \frac{2\sqrt{2}}{3\sqrt{3}}\right\}$. Hence, every considered maximally extended quartic connected GPSR curve $\mathcal{H}_{L,-\frac{1}{12}}$ is equivalent to the quartic CCGPSR curve $\mathcal{H}_{0,-\frac{1}{12}}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}_{0,-\frac{1}{12}}$, as claimed and thus also a quartic CCGPSR curve.

Lastly, we have to consider the maximally extended quartic GPSR curves $\mathcal{H}_{L,K}$ with

$$\begin{pmatrix}L\\K\end{pmatrix} \in \left\{-\frac{1}{12} < K < \frac{1}{4},\ |L| < \mathbf{u}(K)\right\} \subset \mathbb{R}^2,$$

respectively the restriction of $\mathcal{V}$ to said set. Recall that

$$\mathbf{u}(K) = \frac{\sqrt{2}}{3\sqrt{3}}\sqrt{1 - 36K + \sqrt{(1+12K)^3}},$$

cf. (7.18). We proceed similarly to the case where we considered points of the form $\left(\begin{smallmatrix}L\\K\end{smallmatrix}\right) \in \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\}$. We will show that any maximal integral curve of the restricted vector field $\mathcal{V}|_{\left\{-\frac{1}{12}<K<\frac{1}{4},\ |L|<\mathbf{u}(K)\right\}}$ contains a point of the form

$$\begin{pmatrix}0\\\widetilde{K}\end{pmatrix} \in \left\{-\frac{1}{12} < K < \frac{1}{4},\ |L| < \mathbf{u}(K)\right\}.$$

To do so, it suffices to consider points in $\left\{-\frac{1}{12} < K < \frac{1}{4},\ 0 < L < \mathbf{u}(K)\right\}$ which provide the initial value for said integral curves. We define

$$\begin{aligned}
\widetilde{F} &: \left\{-\frac{1}{12} < K < \frac{1}{4},\ 0 < L < \mathbf{u}(K)\right\} \to \left\{-\frac{1}{12} < \psi < \frac{1}{4},\ 0 < \varphi < 1\right\},\\
\widetilde{F} &: \begin{pmatrix}L\\K\end{pmatrix} \mapsto \begin{pmatrix}\frac{L}{\mathbf{u}(K)}\\K\end{pmatrix},
\end{aligned}$$

see Figure 22. The map $\widetilde{F}$ is a diffeomorphism with

$$d\widetilde{F} = \begin{pmatrix}\frac{1}{\mathbf{u}(K)} & \frac{27\sqrt{3}L\left(2-\sqrt{1+12K}\right)}{\sqrt{2\left(1-36K+\sqrt{(1+12K)^3}\right)^3}}\\ 0 & 1\end{pmatrix}$$

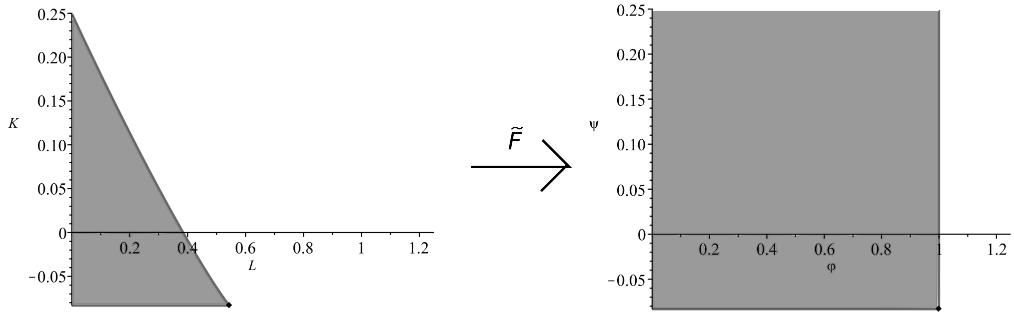


Figure 22: Parts of the domain and co-domain of $\widetilde{F}$, marked in grey.

and

$$\widetilde{F}^{-1}\left(\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right)\right)=\left(\begin{smallmatrix}\varphi\mathbf{u}(K)\\ \psi\end{smallmatrix}\right).$$

Note at this point that the term $1-36\psi+\sqrt{(1+12\psi)^3}$ is positive for all $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$, and in fact it vanishes for $K>-\frac{1}{12}$ if and only if $K=\frac{1}{4}$. The push-forward $\widetilde{F}_*\mathcal{V}$ then is of the form

$$\begin{aligned}\widetilde{F}_*\mathcal{V}=\left(\widetilde{F}_*\mathcal{V}\right)_{\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right)}&=\frac{3\sqrt{3}\,(1-\varphi^2)\left(-1+40\psi-144\psi^2+(-1-8\psi+48\psi^2)\sqrt{1+12\psi}\right)}{\sqrt{2\left(1-36\psi+\sqrt{(1+12\psi)^3}\right)^3}}\partial_\varphi\\ &\quad+\frac{1}{3\sqrt{6}}\varphi(1+12\psi)\sqrt{1-36\psi+\sqrt{(1+12\psi)^3}}\partial_\psi.\end{aligned}$$

The term $d\psi\left(\widetilde{F}_*\mathcal{V}\right)$ is positive for all $\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right)\in\left\{-\frac{1}{12}<\psi<\frac{1}{4},\ 0<\varphi<1\right\}$. Thus, the smooth function

$$\begin{aligned}&\widetilde{\mathcal{R}}:\left\{-\frac{1}{12}<\psi<\frac{1}{4},\ 0<\varphi<1\right\}\to\mathbb{R},\\ &\widetilde{\mathcal{R}}\left(\left(\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right)\right):=\frac{d\varphi\left(\widetilde{F}_*\mathcal{V}\right)}{d\psi\left(\widetilde{F}_*\mathcal{V}\right)}=\frac{1-\varphi^2}{\varphi}\cdot\frac{-27(1-4\psi)}{(1+12\psi)\left(1-36\psi+\sqrt{(1+12\psi)^3}\right)}\end{aligned}$$

is well defined. Note that $\widetilde{\mathcal{R}}$ is positive on its domain of definition. Similarly to the definition of $\mathcal{X}$ (7.48) we define the vector field $\mathcal{Y}$ on the set $\left\{-\frac{1}{12}<\psi<\frac{1}{4},\ 0<\varphi<1\right\}\subset\mathbb{R}^2$ as

$$\mathcal{Y}:=\widetilde{\mathcal{R}}\,\partial_\varphi+\partial_\psi.$$

Since $d\psi\left(\widetilde{F}_*\mathcal{V}\right)$ and, hence, $\widetilde{F}_*\mathcal{V}$ do not vanish on the set $\left\{-\frac{1}{12}<\psi<\frac{1}{4},\ 0<\varphi<1\right\}$, it follows that the images of the maximal integral curves of $\mathcal{Y}$ and $\widetilde{F}_*\mathcal{V}$ are in one-to-one correspondence. As for the vector field $\mathcal{X}$ we can give a formula for the integral curves of $\mathcal{Y}$ (although not as explicit as for the $\mathcal{X}$-case). For an open interval $(a,b)\subset\left(-\frac{1}{12},\frac{1}{4}\right)$,

$$\begin{aligned}&\mathcal{Y}_\gamma=\dot{\gamma},\quad\gamma(t)=\left(\begin{smallmatrix}\varphi(t)\\ t\end{smallmatrix}\right),\quad\gamma:(a,b)\to\left\{-\frac{1}{12}<\psi<\frac{1}{4},\ 0<\varphi<1\right\},\ \gamma(t_0)=\varphi_0\\ \Leftrightarrow\quad&\frac{\varphi\dot{\varphi}}{1-\varphi^2}=\underbrace{\frac{-27(1-4t)}{(1+12t)\left(1-36t+\sqrt{(1+12t)^3}\right)}}_{=:J(t)}\\ \Leftrightarrow\quad&\varphi(t)=\sqrt{1-c\exp\left(-2\int_{t_0}^{t}J(s)ds\right)},\end{aligned}\tag{7.50}$$

where $c > 0$ is chosen such that the initial condition $\gamma(t_0) = (\varphi(t_0), t_0)^T$ is met (for an example of such a γ see Figure 23). Observe that $\varphi(t) < 1$ for all $t \in (a, b)$, and that

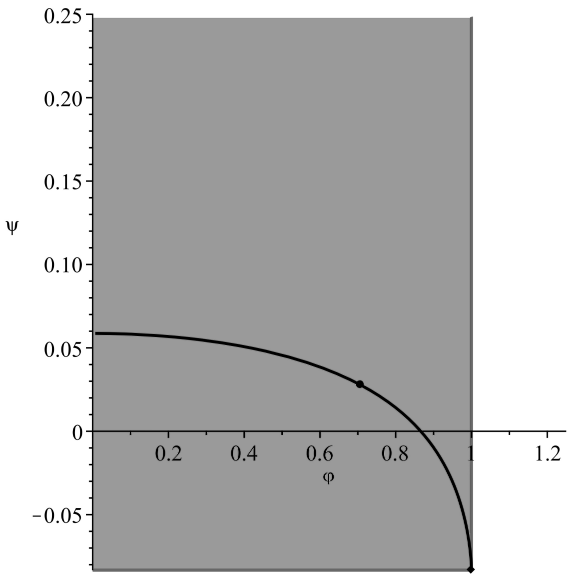


Figure 23: Example of a curve γ fulfilling $\mathcal{Y}_\gamma = \dot{\gamma}$ with initial condition $\varphi\left(\frac{1}{36}\right) = \frac{1}{\sqrt{2}}$.

$$J : \left(-\frac{1}{12}, \frac{1}{4}\right) \to \mathbb{R}, \quad J(t) = \frac{-27(1-4t)}{(1+12t)\left(1-36t+\sqrt{(1+12t)^3}\right)}, \tag{7.51}$$

is a well-defined negative function[21]. We will now show that γ being maximal implies $b < \frac{1}{4}$ and that $\lim\limits_{t\to b,\, t<b} \varphi(t) = 0$. This will imply that the corresponding maximal integral curve of $\mathcal{V}|_{\left\{-\frac{1}{12}<K<\frac{1}{4},\, 0<L<\mathbf{u}(K)\right\}}$ has a limit point in $\left\{-\frac{1}{12} < K < \frac{1}{4},\ L = 0\right\}$ bounded away from $K = \frac{1}{4}$ and, hence, that each maximal integral curve of $\mathcal{V}|_{\left\{-\frac{1}{12}<K<\frac{1}{4},\, |L|<\mathbf{u}(K)\right\}}$ meets $\left\{-\frac{1}{12} < K < \frac{1}{4},\ L = 0\right\}$ in one point (recall for this step that $\left\{-\frac{1}{12} < K < \frac{1}{4},\ |L| < \mathbf{u}(K)\right\}$ is contained in $\mathbb{R}^2 \setminus \{\mathcal{V} = 0\}$). To prove $b < \frac{1}{4}$ it suffices to show that for all $c > 0$, the equation

$$\varphi(t) = \sqrt{1 - c\exp\left(-2\int\limits_{t_0}^{t} J(s)ds\right)} = 0 \tag{7.52}$$

is fulfilled for some $t \in \left(a, \frac{1}{4}\right)$. Since $J(s) < 0$ for all $s \in \left(-\frac{1}{12}, \frac{1}{4}\right)$ and the term $-2\int\limits_{t_0}^{t} J(s)ds$ is thus strictly monotonously increasing in t, it is sufficient to show that for all $t_0 \in \left(-\frac{1}{12}, \frac{1}{4}\right)$

$$\lim_{t\to 1/4,\, t<1/4} \int\limits_{t_0}^{t} J(s)ds = -\infty. \tag{7.53}$$

[21]Note that we can actually give an explicit formula for $\gamma(t)$ as in equation (7.50). By setting $t_0 = \frac{1}{36}$ one can show that $\int\limits_{t_0}^{t} J(s)ds = -\frac{3}{4}\ln(1+12t) + \frac{1}{2}\ln\left(1-36t+\sqrt{(1+12t)^3}\right)$, but we will not need an explicit formula for this proof.

Since J is smooth on $\left(-\frac{1}{12},\frac{1}{4}\right)$ we can without restriction of generality assume $t_0=0$. Observe that $(1+12t)\in(1,4)$ for $t\in\left[0,\frac{1}{4}\right]$ implies that (7.53) is equivalent to

$$\lim_{t\to 1/4,\,t<1/4}\int\limits_0^t \frac{1-4s}{1-36s+\sqrt{(1+12s)^3}}ds=\infty. \tag{7.54}$$

We replace s by $-r+\frac{1}{4}$ in (7.54) and see that it is equivalent to

$$\lim_{t\to 0,\,t>0}\int\limits_t^{\frac{1}{4}} \frac{r}{9r-2+(-6r+2)\sqrt{1-3r}}dr=\infty. \tag{7.55}$$

Note that both the numerator and denominator of the integrand in (7.55) are positive and smooth on the interval $\left(0,\frac{1}{4}\right)$ and both converge to 0 as $r\to 0$. To prove (7.55) it is enough to show that there exists $\varepsilon\in\left(0,\frac{1}{4}\right)$ and $A>0$, such that

$$\forall r\in(0,\varepsilon):\quad 9r-2+(-6r+2)\sqrt{1-3r}\leq Ar^2. \tag{7.56}$$

The condition (7.56) on the other hand can be proven by showing that

$$\lim_{r\to 0,\,r>0}\frac{9r-2+(-6r+2)\sqrt{1-3r}}{r^2} \tag{7.57}$$

exists and is positive. Using L'Hôpitals rule for limits yields

$$\lim_{r\to 0,\,r>0}\frac{9r-2+(-6r+2)\sqrt{1-3r}}{r^2}=\lim_{r\to 0,\,r>0}\frac{9\left(1-\sqrt{1-3r}\right)}{2r}=\frac{9}{2}.$$

Hence, (7.57) holds true, and since (7.57) $\Rightarrow$ (7.56) $\Rightarrow$ (7.55) $\Rightarrow$ (7.54) $\Rightarrow$ (7.53), we have proven that for all initial values $\varphi_0\in(0,1)$ and corresponding $c>0$, there exists $t=\widetilde{t}\in(a,b)$, $\widetilde{t}>t_0$, such that equation (7.52) is fulfilled. Summarising, we have shown that each maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ starting in $\left\{-\frac{1}{12}<K<\frac{1}{4},\ |L|<\mathbf{u}(K)\right\}$ meets the set $\left\{-\frac{1}{12}<K<\frac{1}{4},\ L=0\right\}$ in one point. We have already shown that the quartic CCGPSR curves $\mathcal{H}_{0,K}$ for $K\leq\frac{1}{4}$ are pairwise inequivalent and can thus deduce that this point is unique. This proves that every maximally extended quartic GPSR curve $\mathcal{H}_{L,K}$ with $(L,K)^T\in\left\{-\frac{1}{12}<K<\frac{1}{4},\ |L|<\mathbf{u}(K)\right\}$ is equivalent to a uniquely determined quartic CCGPSR curve $\mathcal{H}_{0,\widetilde{K}}$, $\widetilde{K}\in\left(-\frac{1}{12},\frac{1}{4}\right)$, and thus in particular itself a quartic CCGPSR curve.

We have shown up to this point that the maximally extended quartic GPSR curves corresponding to $a)$, $c)$, and $d)$ are closed and, hence, quartic CCGPSR curves. The remaining case $b)$ corresponds to the point

$$\begin{pmatrix}L\\K\end{pmatrix}=\begin{pmatrix}\frac{2\sqrt{2}}{3\sqrt{3}}\\-\frac{1}{12}\end{pmatrix}\in\{\mathcal{V}=0\}\subset\mathbb{R}^2,$$

and we will now show that the corresponding maximally extended quartic GPSR curve $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ is closed. Note at this point that $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ and $\mathcal{H}_{-\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ are equivalent, and so this is indeed the last remaining maximally extended quartic GPSR curve we have to study. To do so, it suffices to show that $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ is homogeneous in the sense of Definition 3.9. Riemannian homogeneous spaces are automatically geodesically complete and we can then use [CNS, Prop. 1.8] to conclude that $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} \subset \mathbb{R}^2$ is closed and thus a quartic CCGPSR curve. In fact, $\left(\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}\right)^T \in \{\mathcal{V}=0\}$ and the formulas (7.4), (7.2), and (7.3) show that for $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ we have $\delta P_3 = \delta P_4 \equiv 0$, which implies with Proposition 3.34 the homogeneity of $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ as claimed. It remains to determine the closed hyperbolic connected components of $\left\{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}=1\right\}$ and show that they are, as quartic CCGPSR curves, equivalent. To achieve that we will determine the hyperbolic connected components of $\left\{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}>0\right\}$. We find

$$h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\right) = \frac{1}{12}\left(y-\sqrt{6}\right)^3\left(y+\frac{\sqrt{2}}{\sqrt{3}}\right)$$

and

$$h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)\right) = \left(x-\frac{1}{\sqrt{6}}\right)^3\left(x+\frac{\sqrt{3}}{\sqrt{2}}\right).$$

Hence,

$$\begin{aligned}\left\{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}>0\right\} &= \mathbb{R}_{>0}\cdot\left\{\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2 \;\middle|\; y\in\left(-\frac{\sqrt{2}}{\sqrt{3}},\sqrt{6}\right)\right\}\\ &\quad\dot\cup\ \mathbb{R}_{>0}\cdot\left\{-\begin{pmatrix}1\\y\end{pmatrix}\in\mathbb{R}^2 \;\middle|\; y\in\left(-\frac{\sqrt{2}}{\sqrt{3}},\sqrt{6}\right)\right\},\end{aligned}$$

that is $\left\{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}>0\right\}$ has precisely two connected components, both of which only contain hyperbolic points and each a unique quartic CCGPSR curve. These two curves are the connected components of $\left\{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}=1\right\}$, and they are equivalent via $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\mapsto-\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$. Note that $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} \subset \mathbb{R}_{>0}\cdot\left\{\left(\begin{smallmatrix}1\\y\end{smallmatrix}\right)\in\mathbb{R}^2 \;\middle|\; y\in\left(-\frac{\sqrt{2}}{\sqrt{3}},\sqrt{6}\right)\right\}$. In order to find the automorphism group G^h of $h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$, we now only need to determine $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ and check that there are no additional discrete symmetries of $h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ mapping $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} \subset \mathbb{R}^2$ to itself. For $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ consider, similar to $h_{0,\frac{1}{4}}$ respectively $G_0^{h_{0,\frac{1}{4}}}$, Proposition 3.34 and calculate the derivative of the corresponding map $\mathcal{A}:\mathrm{dom}\left(\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\right)\to\mathrm{GL}(2)$ (3.23) at $z=0\in\mathrm{dom}\left(\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\right)$, cf. (3.60). Again, the corresponding linear map $dB_0\in\mathrm{Lin}(\mathbb{R},\mathfrak{so}(1))$ automatically vanishes since $\dim(\mathfrak{so}(1))=0$. We obtain

$$d\mathcal{A}_0(\partial_z) = \begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{\sqrt{2}}{\sqrt{3}}\end{pmatrix}.$$

With $\widetilde{a} := \sqrt{2}\begin{pmatrix}0 & \frac{1}{2}\\ 1 & \frac{\sqrt{2}}{\sqrt{3}}\end{pmatrix}$, one can check that

$$dh_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\Big|_{\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}\left(\widetilde{a}\cdot\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right)\equiv 0$$

as expected. Let $\{c_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers, defined as follows:

$$c_1=1,\quad c_2=\frac{2}{\sqrt{3}},\quad c_i=c_{i-2}+\frac{2}{\sqrt{3}}c_{i-1}\ \forall i\geq 3.$$

If we further set $c_{-1} := 1$ and $c_0 := 0$, we get the identity $c_i = c_{i-2} + \frac{2}{\sqrt{3}} c_{i-1}$ for all $i \geq 1$. Now for the exponential map of $\widetilde{a} \in \mathfrak{gl}(2)$, one can verify that

$$\exp(t\widetilde{a}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \widetilde{a}^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(c_{k-1}\mathbb{1} + c_k \widetilde{a}\right) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} c_{k-1}\right) \mathbb{1} + \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} c_k\right) \widetilde{a}. \tag{7.58}$$

Note that $\widetilde{a}$, when viewed as an element of $\mathfrak{gl}(2)$, has eigenvalues $\sqrt{3}$ and $-\frac{1}{\sqrt{3}}$ and is thus (as an endomorphism of $\mathbb{R}^2$) equivalent to

$$\widehat{a} := \begin{pmatrix} \sqrt{3} & \\ & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

and

$$\exp(t\widehat{a}) = \begin{pmatrix} e^{\sqrt{3}t} & \\ & e^{-\frac{t}{\sqrt{3}}} \end{pmatrix}.$$

It is clear that $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ and $(\mathbb{R},+)$ are isomorphic. More precisely, we find that $\widehat{a}$ can be written as

$$\widehat{a} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \\ & \frac{1}{\sqrt{3}} \end{pmatrix} + \begin{pmatrix} \frac{2}{\sqrt{3}} & \\ & -\frac{2}{\sqrt{3}} \end{pmatrix}$$

which shows that we can view the action generated by $\widehat{a}$ as a one-parameter subgroup of the conformal group $\mathrm{CO}(1,1)$ of the Lorentz vector space $(\mathbb{R}^2, dxdy)$. The quartic CCGPSR curve thus fulfils $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} \cong \mathbb{R}$ as Riemannian homogeneous spaces via the corresponding action of $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$. Now, again similar to the cases $\mathcal{H}_{0,K}$ for $K \leq \frac{1}{4}$, we need to find all $A \in G^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}} \subset \mathrm{GL}(2)$ which are not contained in $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$, such that $A\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} = \mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$. With

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

it is immediate that $a_{11} \neq 0$, since otherwise

$$h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right) x\right) = h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\left(\begin{smallmatrix} 0 \\ a_{21} \end{smallmatrix}\right) x\right) = -\frac{1}{12} a_{21}^4 x^4,$$

but $h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right) x\right) = x^4$ is a necessary requirement for A to be an automorphism of $h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$. Furthermore, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ is mapped to $\left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right)$, which is required to be an element of $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$. The hyperbolicity of $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} \subset \mathbb{R}^2$ then implies that $a_{11} \geq 1$. Since $\left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right) \in \mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$, A must thus be of the form (3.7). With Definition 3.13 and the fact that $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ acts transitively on $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$, this shows that A can be written as

$$A = A_0 \cdot \widetilde{A}$$

where $A_0 \in G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ and $\widetilde{A} \in G^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ is contained in the stabilizer of the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$. Hence, we need to determine all $\widetilde{A} \in G^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$, such that $\widetilde{A} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. $\widetilde{A}$ must be of the form

$$\widetilde{A} = \begin{pmatrix} 1 & -\left.\frac{\partial_y h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}{\partial_x h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}\right|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \cdot r \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

for some $r \neq 0$. Then

$$h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\left(\widetilde{A}\cdot\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) = x^4 - r^2x^2y^2 + r^3\frac{2\sqrt{2}}{3\sqrt{3}}xy^3 - \frac{r^4}{12}y^4,$$

which shows that $r = 1$. Summarising, we have shown that

$$G^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}} \cong \mathbb{R}\times\mathbb{Z}_2,$$

where $\mathbb{R}$ acts as described in (7.58) and $\mathbb{Z}_2$ acts via $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \mapsto -\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$.

In order to complete the proof of Theorem 7.2, we still need to prove that the quartic CCGPSR curves a), b), c), and elements in the family of curves d) are pairwise inequivalent. We already have seen that this is true if one considers two elements in the one-parameter family d). Since the quartic CCGPSR curve a), that is $\mathcal{H}_{0,\frac{1}{4}}$, has a transitive action of the corresponding Lie group $G_0^{h_{0,\frac{1}{4}}}$, cf. Definition 3.13, it might only be equivalent to the quartic CCGPSR curve $\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$, that is b), which also has a transitive $G_0^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$-action. But with

$$\operatorname{dom}\left(\mathcal{H}_{0,\frac{1}{4}}\right) = \left(-\sqrt{2},\sqrt{2}\right)$$

and

$$\operatorname{dom}\left(\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\right) = \left(-\frac{\sqrt{2}}{\sqrt{3}},\sqrt{6}\right)$$

we find that

$$dh_{0,\frac{1}{4}}\Big|_{\left(\begin{smallmatrix}1\\-\sqrt{2}\end{smallmatrix}\right)} = dh_{0,\frac{1}{4}}\Big|_{\left(\begin{smallmatrix}1\\\sqrt{2}\end{smallmatrix}\right)} = 0$$

and

$$dh_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\Big|_{\left(\begin{smallmatrix}1\\-\frac{\sqrt{2}}{\sqrt{3}}\end{smallmatrix}\right)} = \frac{64}{27}dx + \frac{32\sqrt{2}}{9\sqrt{3}}dy \neq 0, \quad dh_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\Big|_{\left(\begin{smallmatrix}1\\\sqrt{6}\end{smallmatrix}\right)} = 0.$$

This means that $dh_{0,\frac{1}{4}}$ vanishes on $\partial\left(\mathbb{R}_{>0}\cdot\mathcal{H}_{0,\frac{1}{4}}\right)$, but $dh_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}$ vanishes only on one of the two connected components of $\partial\left(\mathbb{R}_{>0}\cdot\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\right)$. Hence, the quartic CCGPSR curves a) and b) can not be equivalent. Alternatively, we could have used that $G^{h_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}}$ has 2 connected components, while $G^{h_{0,\frac{1}{4}}}$ has 8 connected components. Now, in order to prove that there exist no quartic CCGPSR curve $\mathcal{H}_{0,K}$ in the one-parameter family d) that is equivalent to the quartic CCGPSR curve $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$, that is c), we will use a similar argument. We find that for $K < 0$, the equation

$$dh_{0,K}\left(\partial_y\right) = y\left(-2 + 4Ky^2\right)$$

has no other solutions than $y = 0$. For $0 < K < \frac{1}{4}$,

$$dh_{0,K}\left(\partial_y\right) = 0,\ y \neq 0 \quad \Leftrightarrow \quad y = \pm\frac{1}{\sqrt{2K}}.$$

But then

$$h_{0,K}\left(\left(\pm\frac{1}{\frac{1}{\sqrt{2K}}}\right)\right) = 1 - \frac{1}{4K} < 0 \quad \forall K \in \left(0, \frac{1}{4}\right).$$

Hence, for $0 < K < \frac{1}{4}$ the points $\pm\frac{1}{\sqrt{2K}}$ are not contained in $\partial\left(\mathrm{dom}\left(\mathcal{H}_{0,K}\right)\right)$. Summarising, we have shown that for all $K < \frac{1}{4}$, $dh_{0,K}|_{\partial\left(\mathbb{R}_{>0}\cdot\mathcal{H}_{0,K}\right)}$ is nowhere zero. But for $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$, we find $\mathrm{dom}\left(\mathcal{H}_{\frac{2}{3\sqrt{3}},0}\right) = \left(-\frac{\sqrt{3}}{2}, \sqrt{3}\right)$ and

$$dh_{\frac{2}{3\sqrt{3}},0}\Big|_{\left(\frac{1}{\sqrt{3}}\right)} \equiv 0.$$

Hence, for all $K < \frac{1}{4}$ the quartic CCGPSR curves $\mathcal{H}_{0,K}$ and $\mathcal{H}_{\frac{2}{3\sqrt{3}},0}$ cannot be equivalent.

This finishes the proof of Theorem 7.2. □

Remark 7.3. In [CNS, Thm. 2.9] it has been proven that every CCGPSR curve $\mathcal{H} \subset \{h = 1\}$ equipped with its Riemannian centro-affine fundamental form $g_{\mathcal{H}}$ is geodesically complete, independent of the homogeneity-degree $\tau \geq 3$ of h. This in particular implies that the curves a)–d) in Theorem 7.2 are geodesically complete.

Remark 7.4 (Comparison with CCPSR curves classification)**.** CCPSR curves have been classified in [CHM, Cor. 4]. One can also use the methods of Theorem 7.2 to find that classification. Roughly, it works as follows. We assume without loss of generality that $h = h_L := x^3 - xy^2 + Ly^3$, $L \in \mathbb{R}$. The first step is to find all L, such that the connected component $\mathcal{H}_L \subset \{h_L = 1\}$ that contains $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is a CCPSR curve. It turns out that $\mathcal{H}_L$ is a CCPSR curve if and only if $|L| \leq \frac{2}{3\sqrt{3}}$. Then, we would study an analogue to the vector field $\mathcal{V}$ (7.4) which was extensively used in the proof of Theorem 7.2, namely (recall (3.30))

$$\widetilde{\mathcal{V}} \in \Gamma(T\mathbb{R}), \quad \widetilde{\mathcal{V}} := \frac{\delta P_3(y)(\partial_z)}{y^3}\partial_L = \left(\frac{9}{2}L^2 - \frac{2}{3}\right)\partial_L. \tag{7.59}$$

Then we find that $\left\{\widetilde{\mathcal{V}} = 0\right\} = \{\pm\frac{2}{3\sqrt{3}}\}$ and that the maximally extended integral curve of $\widetilde{\mathcal{V}}|_{\mathbb{R}^2\setminus\left\{\widetilde{\mathcal{V}}=0\right\}}$ that contains $L = 0$ has the image $\left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$. Up to equivalence there are precisely two CCPSR curves, $\mathcal{H}_0$ and $\mathcal{H}_{\frac{2}{3\sqrt{3}}}$, the latter being a homogeneous space with respect to the action of $G_0^{h_{\frac{2}{3\sqrt{3}}}}$. We can now verify that $\mathcal{H}_0$ is exactly the curve $b)$ and $\mathcal{H}_{\frac{2}{3\sqrt{3}}}$ is equivalent to the curve $a)$ in [CHM, Cor. 4]. Note that there exists, up to equivalence, one more hyperbolic cubic homogeneous polynomial $h : \mathbb{R}^2 \to \mathbb{R}$ and a corresponding (maximal) non-closed PSR curve which is not equivalent to neither $\mathcal{H}_0$ nor $\mathcal{H}_{\frac{2}{3\sqrt{3}}}$. In [CHM, Thm. 7], it is given by $h = x(x^2 + y^2)$, and in our approach it is equivalent to h_L for all $|L| > \frac{2}{3\sqrt{3}}$. Note that h_L and h_{-L} are always equivalent, and that the maximal integral curve of $\widetilde{\mathcal{V}}|_{\mathbb{R}^2\setminus\left\{\widetilde{\mathcal{V}}=0\right\}}$ that contains any point $L > \frac{2}{3\sqrt{3}}$ has the image $\left\{L > \frac{2}{3\sqrt{3}}\right\} \subset \mathbb{R}$.

Remark 7.5 (Comparison of moduli spaces of quartic CCGPSR curves and of CCPSR curves)**.** It was shown in [CHM, Cor. 4] that the moduli space of CCPSR curves consists of two points and is thus compact. In Remark 7.4 we have described how to find the equivalence class of a CCPSR curve when the corresponding cubic polynomial is of the form (3.12) by parametrising the set of CCPSR curves over the compact interval $\left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$. A similar parametrisation of quartic CCGPSR curves over a compact set does not exist as we have seen in Theorem 7.2. Instead, we have identified a suitable non-compact subset of $\mathbb{R}^2$ over which the set of quartic CCGPSR curves can be parametrised. If we, however, restrict to quartic CCGPSR curves which are singular at infinity, we find that a corresponding compact interval continuously embedded in $\mathbb{R}^2$ is given by the set

$$\left\{ \left(\begin{smallmatrix} \pm \mathbf{u}(K) \\ K \end{smallmatrix} \right) \;\middle|\; K \in \left[-\frac{1}{12}, \frac{1}{4} \right] \right\},$$

see (7.18) and (7.21). Note that for one-dimensional CCGPSR manifolds, being singular at infinity as in Definition 3.16 is equivalent to having non-regular boundary behaviour as in Definition 5.1.

Remark 7.5 leads to the following question.

Open problem 7.6 (Possible compactness of non-regular quartic CCGPSR manifolds genrating set)**.** *Can we parametrise the set of quartic CCGPSR manifolds with non-regular boundary behaviour over a compact set in the sense as described in Proposition 5.8 for CCPSR manifolds*[22] *?*

If we find a positive answer to the open problem 7.6, then we could use an argument similar to the proof of Proposition 5.17 in order to solve the still open question whether quartic CCGPSR manifolds of dimension $n \geq 2$ are always complete or not.

At this point we will present an example for Proposition 3.33, that is for $\delta^2 P_k$ for quartic GPSR curves.

Example 7.7 ($\delta^2 P_3(y)$ and $\delta^2 P_4(y)$ for quartic curves)**.** *With Proposition 3.33, equations (3.56) and (3.57), one can show that for* $h_{L,K} = x^4 - x^2y^2 + Lxy^3 + Ky^4$ *(7.1) with* $P_3(y) = Ly^3$ *and* $P_4(y) = Ky^4$

$$\begin{aligned} \delta^2 P_3(y) &= \left(\frac{135}{4} L^3 + 54LK - \frac{11}{2} L \right) y^3 dz^2, \\ \delta^2 P_4(y) &= \left(54L^2K + \frac{9}{2} L^2 + 24K^2 - 4K - \frac{1}{2} \right) y^4 dz^2. \end{aligned}$$

With $L(T)$ *and* $K(T)$ *as in (7.8) and (7.9), respectively, one can similarly to the relations of* $\mathcal{V}$ *(7.4) and* $\left.\frac{\partial}{\partial T} \left(\begin{smallmatrix} L(T) \\ K(T) \end{smallmatrix} \right)\right|_{T=0}$ *as in (7.10) verify that*

$$\left. \frac{\partial^2}{\partial T^2} \begin{pmatrix} L(T) \\ K(T) \end{pmatrix} \right|_{T=0} = \begin{pmatrix} \frac{135}{4} L^3 + 54LK - \frac{11}{2} L \\ 54L^2K + \frac{9}{2} L^2 + 24K^2 - 4K - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \delta^2 P_3(y)/(y^3 dz^2) \\ \delta^2 P_4(y)/(y^4 dz^2) \end{pmatrix}$$

as expected.

Recall that for CCPSR manifolds $\mathcal{H} \subset \{h = 1\}$, h of the form (3.12), and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \in \mathcal{H}$, we have seen in Corollary 4.5 that all points $\overline{z} \in \partial \mathrm{dom}(\mathcal{H})$ fulfil the estimate $\frac{\sqrt{3}}{2} \leq \|\overline{z}\| \leq \sqrt{3}$. One application of Theorem 7.2 is an analogue of the upper bound for quartic CCGPSR manifolds of arbitrary dimension. We will also see that there exists no such lower positive bound for quartic CCGPSR manifolds that holds for any fixed dimension.

Proposition 7.8 ($\|z\| < \sqrt{6}$)**.** *Let* $\mathcal{H} \subset \{h = 1\}$ *be an* $n \geq 1$*-dimensional quartic CCGPSR manifold,* h *of the form (3.12), and* $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \in \mathcal{H}$*. Then*

$$\forall \overline{z} \in \partial \mathrm{dom}(\mathcal{H}) : \quad \|\overline{z}\| \leq \sqrt{6}, \tag{7.60}$$

[22] Recall that an n-dimensional CCPSR manifold $\mathcal{H} \subset \{h = 1\}$ has non-regular boundary behaviour if and only if it is equivalent to a CCPSR manifold corresponding to a polynomial in $\partial \mathcal{C}_n$.

where $\|\cdot\|$ denotes the norm induced by the Euclidean scalar product on $\mathbb{R}^n$ used in (3.12). On the other hand, there exist no general lower positive bound for the Euclidean norm of points in $\partial\mathrm{dom}(\mathcal{H})$ irrespective of the dimension of $\mathcal{H}$, i.e. for all $n \geq 1$ and all $\delta > 0$ we can find a quartic CCGPSR manifold $\mathcal{H}$ with $\dim(\mathcal{H}) = n$, such that there exists a point $\overline{z}_\delta \in \partial\mathrm{dom}(\mathcal{H})$ with $\|\overline{z}_\delta\| \leq \delta$.

Proof. First we show the existence of the upper positive bound. For an $n \geq 1$-dimensional quartic CCGPSR manifold $\mathcal{H}$ as used in this proposition, consider for $v \in \mathbb{R}^n \setminus \{0\}$, such that $\|v\| = 1$, the restriction $h|_{\mathrm{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\v\end{smallmatrix}\right)\right\}}$. Then with $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ denoting coordinates of $\mathbb{R}^2$, we define $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}$ as

$$h\left(\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)\right) = h\left(x\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + y\left(\begin{smallmatrix}0\\v\end{smallmatrix}\right)\right).$$

One can now easily show that $\widetilde{h}$ is of the form (3.12). Let $\widetilde{\mathcal{H}} \subset \widetilde{h}$ denote the quartic CCGPSR curve that contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{R}^2$. Note that $\widehat{\mathcal{H}}$ being a quartic CCGPSR curve follows from the fact that the map

$$\left(\widetilde{\mathcal{H}}, g_{\widetilde{\mathcal{H}}}\right) \to (\mathcal{H}, g_{\mathcal{H}}), \quad \begin{pmatrix}x\\y\end{pmatrix} \mapsto x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\v\end{pmatrix}$$

is, by construction, an isometric embedding. If we can now show that, independent of the chosen $v \in \mathbb{R}^n \setminus \{0\}$, for all $\overline{z} \in \partial\mathrm{dom}\left(\widetilde{\mathcal{H}}\right)$ the claimed estimate $\|\overline{z}\| \leq \sqrt{6}$ holds, we will have proven it for all dimensions. Since $\widehat{\mathcal{H}}$ is a quartic CCGPSR curve, we know by the proof of Theorem 7.2 that $\widetilde{\mathcal{H}} = \mathcal{H}_{L,K} \subset \{h_{L,K} = 1\}$ with $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ as in (7.1), with the following possible values for L and K:

$$\begin{aligned}
\begin{pmatrix}L\\K\end{pmatrix} &= \begin{pmatrix}0\\\frac{1}{4}\end{pmatrix},\\
\begin{pmatrix}L\\K\end{pmatrix} &= \begin{pmatrix}\frac{2\sqrt{2}}{3\sqrt{3}}\\-\frac{1}{12}\end{pmatrix},\\
\begin{pmatrix}L\\K\end{pmatrix} &\in \left\{K \in \left(-\frac{1}{12}, \frac{1}{4}\right),\ |L| \leq \mathbf{u}(K)\right\},\\
\begin{pmatrix}L\\K\end{pmatrix} &\in \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\},
\end{aligned}$$

$\mathbf{u}$ and $\mathbf{w}$ as defined in (7.18) and (7.28), respectively.

For $(L,K)^T = \left(0, \frac{1}{4}\right)^T$, $\mathrm{dom}\left(\mathcal{H}_{0,\frac{1}{4}}\right) = \left(-\sqrt{2}, \sqrt{2}\right)$, and for $(L,K)^T = \left(\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{1}{12}\right)^T$, $\mathrm{dom}\left(\mathcal{H}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}}\right) = \left(-\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{6}\right)$. Hence, in these two cases the estimate (7.60) holds.

Now we consider $(L,K)^T \in \left\{K \in \left(-\frac{1}{12}, \frac{1}{4}\right),\ |L| \leq \mathbf{u}(K)\right\}$. We want to determine

$$\sup_{(L,K)^T \in \left\{K \in \left(-\frac{1}{12}, \frac{1}{4}\right),\ |L| \leq \mathbf{u}(K)\right\}} \left(\max_{\overline{z} \in \partial\mathrm{dom}\left(\mathcal{H}_{L,K}\right)} \|\overline{z}\|\right).$$

In general, denote (whenever defined) $\mathrm{dom}\left(\mathcal{H}_{L,K}\right) = \left(\mathcal{N}_{L,K}, \mathcal{P}_{L,K}\right)$ and note that

$$\mathrm{dom}\left(\mathcal{H}_{-L,K}\right) = \left(-\mathcal{P}_{L,K}, \mathcal{N}_{L,K}\right).$$

Hence, we can without loss of generality assume that $L \geq 0$. Then we automatically have $\mathcal{P}_{L,K} \geq \|\mathcal{N}_{L,K}\|$. Furthermore, for $\widetilde{L} \geq L \geq 0$ such that $\mathcal{H}_{\widetilde{L},K}$ and $\mathcal{H}_{L,K}$ are both quartic CCGPSR curves it is easy to see that

$$\widetilde{L} \geq \widetilde{L} \quad \Rightarrow \quad \mathcal{P}_{\widetilde{L},K} \geq \mathcal{P}_{L,K}. \tag{7.61}$$

Hence,

$$\sup_{(L,K)^T\in\left\{K\in\left(-\frac{1}{12},\frac{1}{4}\right),\ |L|\leq\mathbf{u}(K)\right\}}\left(\max_{\overline{z}\in\partial\mathrm{dom}\left(\mathcal{H}_{L,K}\right)}\|\overline{z}\|\right)=\sup_{K\in\left(-\frac{1}{12},\frac{1}{4}\right)}\mathcal{P}_{\mathbf{u}(K),K}.$$

It turns out that we can find an explicit formula for $\mathcal{P}_{\mathbf{u}(K),K}$, $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$. Recall that

$$\mathbf{u}(K)=\frac{\sqrt{2}}{3\sqrt{3}}\sqrt{1-36K+\sqrt{(1+12K)^3}}.$$

We find that

$$\partial_K\mathbf{u}(K)=\frac{\sqrt{6}\left(\sqrt{1+12K}-2\right)}{\sqrt{1-36K+\sqrt{(1+12K)^3}}}$$

and claim that $\mathcal{P}_{\mathbf{u}(K),K}=-\partial_K\mathbf{u}(K)$ for all $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$. In order to show this, first note that $-\partial_K\mathbf{u}(K)>0$ for all $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$, and one can check that

$$h\left(1,-\partial_K\mathbf{u}(K)\right)_{\mathbf{u}(K),K)}=0\quad\forall K\in\left(-\frac{1}{12},\frac{1}{4}\right).\tag{7.62}$$

We still need to show that $-\partial_K\mathbf{u}(K)$ is not just some solution of $h\left(1,\overline{z}\right)_{\mathbf{u}(K),K)}=0$ for $\overline{z}$, but actually coincides with $\mathcal{P}_{\mathbf{u}(K),K}$. For $K=0$, $-\partial_K\mathbf{u}(0)=\sqrt{3}=\mathcal{P}_{\mathbf{u}(K),K}$. For $K\in\left(0,\frac{1}{4}\right)$, recall that the corresponding function (7.13)

$$f_{\mathbf{u}(K),K}(t)=h_{\mathbf{u}(K),K}\left(\left(\begin{smallmatrix}1\\t\end{smallmatrix}\right)\right)=1-t^2+\mathbf{u}(K)t^3+Kt^4$$

always has precisely one local maximum at $t=0$, and two distinct local minima (7.15), which are given by t_m (7.17) and t_M (7.23). For $(L,K)^T=(\mathbf{u}(K),K)^T$, $K\in\left(0,\frac{1}{4}\right)$, we have $t_M<0$. Also recall that for $(L,K)^T=(\mathbf{u}(K),K)^T$, $K\in\left(0,\frac{1}{4}\right)$, the point $t_m>0$ is a positive root of $f_{\mathbf{u}(K),K}(t)$, cf. (7.16) (7.17) (7.18). This shows that $f_{\mathbf{u}(K),K}(t)$ has a unique positive root t_m, which coincides by construction with $\mathcal{P}_{\mathbf{u}(K),K}$. One now verifies that, indeed, for $(L,K)^T=(\mathbf{u}(K),K)^T$, $K\in\left(0,\frac{1}{4}\right)$, we have that $t_m=-\partial_K\mathbf{u}(K)$. Now we want to determine

$$\sup_{K\in\left(0,\frac{1}{4}\right)}\mathcal{P}_{\mathbf{u}(K),K}.$$

For that we calculate

$$\partial_K(-\partial_K\mathbf{u})(K)=\frac{3\sqrt{6}\left((1+12K)^3+12(1+12K)^2+(-864K^2-240K-14)\sqrt{1+12K}\right)}{(1+12K)^2\sqrt{1-36K+\sqrt{(1+12K)^3}}^{\,3}}.$$

The denominator in the above formula is positive for all $K\in\left(-\frac{1}{12},\frac{1}{4}\right)$. Using a computer algebra system like MAPLE, we obtain

$$(1+12K)^3+12(1+12K)^2+\left(-864K^2-240K-14\right)\sqrt{1+12K}=0\quad\Leftrightarrow\quad K\in\left\{-\frac{1}{12},\frac{1}{4}\right\}.$$

At $K=0$ we find $-\partial_K^2\mathbf{u}(0)=-\frac{3\sqrt{3}}{2}$. This shows that

$$-\partial_K^2\mathbf{u}(K)<0\quad\forall K\in\left(-\frac{1}{12},\frac{1}{4}\right),\tag{7.63}$$

hence in particular for all $K \in \left(0, \frac{1}{4}\right)$. With $\mathbf{u}(0) = \frac{2}{3\sqrt{3}}$ this yields

$$\sup_{K\in\left(0,\frac{1}{4}\right)} \mathcal{P}_{\mathbf{u}(K),K} = \lim_{K\to 0,\, K>0} \mathcal{P}_{\mathbf{u}(K),K} = \mathcal{P}_{\frac{2}{3\sqrt{3}},0} = \sqrt{3}.$$

Next we need to consider $K \in \left(-\frac{1}{12}, 0\right)$ and show that for these K we also have $\mathcal{P}_{\mathbf{u}(K),K} = -\partial_K \mathbf{u}(K)$. With equation (7.62) and equations (7.16), (7.17), and (7.18), one only has to check that t_m and $-\partial_K \mathbf{u}(K)$ coincide, which turns out to be true. We can now use (7.63) again and obtain with $\mathbf{u}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}$

$$\sup_{K\in\left(-\frac{1}{12},0\right)} \mathcal{P}_{\mathbf{u}(K),K} = \lim_{K\to -\frac{1}{12},\, K>-\frac{1}{12}} \mathcal{P}_{\mathbf{u}(K),K} = \mathcal{P}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} = \sqrt{6}.$$

Next we need to consider points $(L,K)^T \in \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\}$ and determine

$$\sup_{(L,K)^T\in\left\{K<-\frac{1}{12},\ |L|<\mathbf{w}(K)\right\}} \left(\max_{\overline{z}\in\partial\mathrm{dom}\left(\mathcal{H}_{L,K}\right)} \|\overline{z}\| \right).$$

We want to use (7.61) again. However, contrary to points $(L,K)^T = (\mathbf{u}(K),K)^T$, maximally extended quartic GPSR curves $\mathcal{H}_{\mathbf{w}(K),K)}$ corresponding to points $(L,K)^T = (\mathbf{w}(K),K)^T$, $K < -\frac{1}{12}$, are <u>not</u> quartic CCGPSR curves, cf. (7.31) (7.32). But we can show that for fixed $K < -\frac{1}{12}$ and all $0 \leq L \leq \mathbf{w}(K)$, the corresponding function $f_{L,K}(t)$ (7.13) has precisely one positive and one negative real root. Solving $\dot{f}_{L,K}(t) = 0$ symbolically, we obtain (7.15). Hence, $\dot{f}_{L,K}(t) = 0$ has precisely one real root $t = 0$ for $K < -\frac{1}{12}$ and $0 \leq L \leq \mathbf{w}(K)$ if and only if $9L^2 + 32K < 0$. Thus if we can show that $9\mathbf{w}(K)^2 + 32K < 0$ for all $K < -\frac{1}{12}$, we will automatically have proven $9L^2 + 32K < 0$ for all $0 \leq L \leq \mathbf{w}(K)$ and all $K < -\frac{1}{12}$. We obtain

$$9\mathbf{w}(K)^2 + 32K = \frac{2}{3} + 8K,$$

which indeed is negative for all $K < -\frac{1}{12}$. This shows that $f_{L,K}(t)$ has exactly one local extremum at $t = 0$ for all $0 \leq L \leq \mathbf{w}(K)$ and all $K < -\frac{1}{12}$, and by the sign of the prefactor of the highest order monomial Kt^4 (in t) in $f_{L,K}(t)$ it follows that $f_{L,K}(t)$ (7.13) has precisely one positive and one negative real root for all $0 \leq L \leq \mathbf{w}(K)$ and all $K < -\frac{1}{12}$ as claimed. Now we use that the prefactors of the monomials in t of $f_{L,K}(t)$ depend smoothly on L, K, and can thus use (7.61) to get

$$\sup_{(L,K)^T\in\left\{K<-\frac{1}{12},\ |L|<\mathbf{w}(K)\right\}} \left(\max_{\overline{z}\in\partial\mathrm{dom}\left(\mathcal{H}_{L,K}\right)} \|\overline{z}\| \right) = \sup_{K<-\frac{1}{12}} \mathcal{P}_{\mathbf{w}(K),K},$$

where we denote by $\mathcal{P}_{\mathbf{w}(K),K}$ the (unique) positive real root of $f_{\mathbf{w}(K),K}(t)$. Since this is true for all $K < -\frac{1}{12}$ and the prefactors of the monomials in t of $f_{\mathbf{w}(K),K}(t)$ depend smoothly on K it follows that

$$\left(-\infty, -\frac{1}{12}\right) \ni K \mapsto \mathcal{P}_{\mathbf{w}(K),K} \in \mathbb{R}_{>0}$$

is smooth. Furthermore we have for all $K \in \left(-\infty, -\frac{1}{12}\right)$

$$\begin{aligned} 0 &\equiv \partial_K \left(f_{\mathbf{w}(K),K}\left(\mathcal{P}_{\mathbf{w}(K),K}\right)\right) \\ &= \left.\frac{\partial f_{\mathbf{w}(K),K}(t)}{\partial t}\right|_{\mathcal{P}_{\mathbf{w}(K),K}} \partial_K \mathcal{P}_{\mathbf{w}(K),K} + \mathcal{P}^3_{\mathbf{w}(K),K}\left(\partial_K \mathbf{w}(K) + \mathcal{P}_{\mathbf{w}(K),K}\right). \end{aligned} \tag{7.64}$$

Note that $f_{\mathbf{w}(K),K}(t)$ is strictly decreasing for $t>0$ (this follows from the uniqueness of its local maximum at $t=0$), which implies that

$$\left.\frac{\partial f_{\mathbf{w}(K),K}(t)}{\partial t}\right|_{\mathcal{P}_{\mathbf{w}(K),K}} < 0 \quad \forall K < -\frac{1}{12}.$$

We further find

$$\partial_K \mathbf{w}(K) + \mathcal{P}_{\mathbf{w}(K),K} = \frac{9(1+12K)^2}{(1-36K)^2} > 0 \quad \forall K < -\frac{1}{12}.$$

Hence, (7.64) implies with $\mathcal{P}_{\mathbf{w}(K),K} > 0$ for all $K < -\frac{1}{12}$ that

$$\partial_K \mathcal{P}_{\mathbf{w}(K),K} > 0 \quad \forall K < -\frac{1}{12}.$$

Together with $\mathbf{w}\left(-\frac{1}{12}\right) = \frac{2\sqrt{2}}{3\sqrt{3}}$, this shows

$$\sup_{K<-\frac{1}{12}} \mathcal{P}_{\mathbf{w}(K),K} = \lim_{K\to -\frac{1}{12},\, K<-\frac{1}{12}} \mathcal{P}_{\mathbf{w}(K),K} = \mathcal{P}_{\frac{2\sqrt{2}}{3\sqrt{3}},-\frac{1}{12}} = \sqrt{6}.$$

This finishes the proof of the estimate (7.60).

It remains to prove the second statement of this proposition. To do so, it suffices to construct for every $n\in\mathbb{N}$ a sequence of n-dimensional quartic CCGPSR manifolds $\mathcal{H}_i$, $i\in\mathbb{N}$, such that

$$\min_{\overline{z}\in\partial\mathrm{dom}(\mathcal{H}_i)} \|\overline{z}\| < c_i,$$

where $\{c_i, i\in\mathbb{N}\}\subset\mathbb{R}_{>0}$ is any strictly decreasing sequence of positive real numbers. As usual, $\mathcal{H}_i\subset\{h_i=1\}$ is assumed to contain the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and h_i is assumed to be of the form (3.12). Using the latter assumptions, we define for each $n\in\mathcal{N}$ a candidate for $h_i:\mathbb{R}^{n+1}\to\mathbb{R}$, where we let $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\begin{pmatrix}x\\y_1\\\vdots\\y_n\end{pmatrix}$ denote standard linear coordinates on $\mathbb{R}^{n+1}$ with standard Euclidean scalar product on $\mathbb{R}^n$ denoted by $\langle\cdot,\cdot\rangle$. Let

$$h_i := x^4 - x^2\langle y,y\rangle - i\langle y,y\rangle^2,$$

and $\mathcal{H}_i\subset\{h_i=1\}$ be the connected component of $\{h_i=1\}$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Then $\mathcal{H}_i$ is a quartic CCGPSR manifold for all $i\in\mathbb{N}$. This follows from the fact that the corresponding function β_i (3.22),

$$\beta_i:\mathbb{R}^n\to\mathbb{R},\quad \beta_i(z)=h_i\left(\left(\begin{smallmatrix}1\\z\end{smallmatrix}\right)\right)=1-\langle z,z\rangle - i\langle z,z\rangle^2,$$

is strictly concave and, hence, firstly the (unique) connected component of the set

$$\{1\}\times\{\beta_i(z)>0\} = (\mathbb{R}_{>0}\cdot\mathcal{H}_i)\cap\left\{\begin{pmatrix}1\\z\end{pmatrix}\ \middle|\ z\in\mathbb{R}^n\right\}$$

that contains the point$\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\subset\{h_i=1\}\subset\mathbb{R}^{n+1}$ is precompact, and secondly the right hand side of (3.33) coincides with the pullback of $-\partial^2 h_i$ and is positive definite on said connected component, showing that $\mathcal{H}_i$ is indeed a quartic CCGPSR manifold and that $\mathrm{dom}(\mathcal{H}_i)$ coincides precisely with that set projected to $\mathbb{R}^n$, cf. (3.13). We find that

$$\mathrm{dom}(\mathcal{H}_i)=\left\{\|z\|<\sqrt{\frac{\sqrt{1+4i}-1}{2i}}\right\}\subset\mathbb{R}^n.$$

Now $\sqrt{\frac{\sqrt{1+4i}-1}{2i}} \to 0$ as $i \to \infty$ shows that for all $\delta > 0$ there exists an $j \in \mathbb{N}$, such that $\sqrt{\frac{\sqrt{1+4j}-1}{2j}} < \delta$, and we have thus found with the corresponding $\mathcal{H}_j$ a suitable n-dimensional quartic CCGPSR manifold, such that for all $\overline{z} \in \partial\mathrm{dom}(\mathcal{H}_j)$, $\|\overline{z}\| \leq \delta$. This finishes the proof of Proposition 7.8. □

We will now use Proposition 7.8 to find a similar statement for quartic CCGPSR manifolds compared to Lemma 4.8 which describes a property of CCPSR manifolds.

Lemma 7.9 (Hyperbolicity condition for quartic CCGPSR manifolds with knowledge that $\|z\| < \sqrt{6}$). *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a quartic homogeneous polynomial of the form (3.12), $h = x^4 - x^2\langle y, y\rangle + xP_3(y) + P_4(y)$. Let $\mathcal{H} \subset \{h = 1\}$ be the connected component of $\{h = 1\} \subset \mathbb{R}^{n+1}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and assume that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is a hypersurface. Then $\mathcal{H}$ is a quartic CCGPSR manifold if and only if*

$$\|z\| < \sqrt{6} \tag{7.65}$$

and

$$\begin{aligned} &2\langle dz, dz\rangle - 6P_3(z, dz, dz) - 12P_4(z, z, dz, dz) \\ &+ \frac{1}{2(6 - \langle z, z\rangle)}\left(16\langle z, dz\rangle^2 - 24\langle z, dz\rangle P_3(z, z, dz) + 9P_3(z, z, dz)^2\right) > 0 \end{aligned} \tag{7.66}$$

for all $\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in (\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$.

Proof. The proof of this lemma is very similar to the proof of Lemma 4.8. The differences are mostly replacing "PSR" with "GPSR" and adding the label "quartic" when appropriate, and the calculation of $\det\left(-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)}\right)$ which additionally needs Proposition 7.8 (for PSR manifolds see formula (4.6) in Lemma 4.8).

Assume that $\mathcal{H}$ is a quartic CCGPSR manifold. Then Proposition 7.8 implies that $\|z\| < \sqrt{6}$ for all $z \in \mathrm{dom}(\mathcal{H})$ and we calculate

$$\begin{aligned} &\det\left(-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)}\right) \\ &= \det\left(\begin{array}{c|c} -12 + 2\langle z, z\rangle & 4z^T - 3P_3(z, z, \cdot) \\ \hline 4z - 3P_3(z, z, \cdot)^T & 2\mathbb{1} - 6P_3(z, \cdot, \cdot) - 12P_4(z, z, \cdot, \cdot) \end{array}\right) \\ &= (-12 + 2\langle z, z\rangle) \\ &\cdot \det\Bigg(2\mathbb{1} - 6P_3(z, \cdot, \cdot) - 12P_4(z, z, \cdot, \cdot) \\ &\quad - \frac{1}{-12 + 2\langle z, z\rangle}\left(4z - 3P_3(z, z, \cdot)^T\right) \otimes \left(4z^T - 3P_3(z, z, \cdot)\right)\Bigg). \end{aligned} \tag{7.67}$$

Furthermore, the sets $\mathrm{dom}(\mathcal{H})$ and $(\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$ coincide, cf. (3.13). Then (7.67) is equivalent to (7.66).

On the other hand, let $\mathcal{H}$ be the connected component of $\{h = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and assume that (7.65) and (7.66) hold. Then the sign of $\det\left(-\partial^2 h_{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right)}\right)$ is constantly -1 for all $\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in (\mathbb{R}_{>0} \cdot \mathcal{H}) \cap \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathbb{R}^n\}$, and since the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ is automatically a hyperbolic point of h since h is of the form (3.12), it follows that $\mathcal{H}$ consists only of hyperbolic points. Since $\mathcal{H}$ is by assumption a connected component of $\{h = 1\} \subset \mathbb{R}^{n+1}$, it is also closed. Hence, $\mathcal{H}$ is a quartic CCGPSR manifold as claimed. □

Next we will discuss some additional examples of quartic CCGPSR manifolds.

Example 7.10 ("Homogeneous hat"). *Consider the quartic homogeneous polynomial*

$$h : \mathbb{R}^{n+1} \to \mathbb{R}, \quad h = x^4 - x^2\langle y, y\rangle + \frac{1}{4}\langle y, y\rangle^2,$$

where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ *denote linear coordinates on* $\mathbb{R}^{n+1}$ *and* $\langle \cdot, \cdot \rangle$ *denotes the standard scalar product on* $\mathbb{R}^n$ *induced by* $y = (y_1, \ldots, y_n)^T$. *Then the connected component* $\mathcal{H} \subset \{h = 1\}$ *that contains the point* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^{n+1}$ *is a quartic CCGPSR manifold for all* $n \geq 1$. *Furthermore* G_0^h *(cf. Definition 3.13) acts transitively on* $\mathcal{H}$, *so that* $\mathcal{H}$ *is a homogeneous space with*

$$\mathcal{H} \cong \mathrm{SO}^+(n, 1)/\mathrm{SO}(n),$$

where $\mathrm{SO}^+(n, 1)/\mathrm{SO}(n)$ *is the oriented* n*-dimensional hyperbolic space. Here* $\mathrm{SO}^+(n, 1)$ *denotes the time-orientation preserving component of* $\mathrm{SO}(n, 1)$. *Furthermore, every point* $p \neq 0$ *in the boundary of* $U := \mathbb{R}_{>0} \cdot \mathcal{H}$ *does violate both conditions (i) and (ii) in Definition 5.1, where in (ii)*

$$\dim \ker \left(-\partial^2 h\Big|_{T(\partial U \setminus \{0\}) \times T(\partial U \setminus \{0\})} \right) = n,$$

so one might say that $\mathcal{H}$ *violates Def. 5.1 (i) and (ii) as much as possible.*

Note that for $n = 1$, $\mathcal{H}$ *and the quartic CCGPSR a) in Theorem 7.2 coincide (see also Figure 24), and one might think of* $\mathcal{H}$ *for* $n \geq 2$ *as the higher-dimensional analogues of the curve a).*

Proof. In order to show that all of the above claims are true, we check that for any $dB_0 \in \mathrm{Lin}\,(\mathbb{R}^n, \mathfrak{so}(n))$ (in (3.60)) both $\delta P_3(y)$ (3.31) and $\delta P_4(y)$ (3.32) identically vanish. Hence, Proposition 3.34 tells us that $\mathcal{H}$ is indeed a quartic CCGPSR manifold with transitively acting Lie group G_0^h. We still need to show that $\mathcal{H} \cong \mathrm{SO}^+(n, 1)/\mathrm{SO}(n)$. To do so we will first transform the linear coordinates $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ via $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} x \\ \sqrt{2}y \end{smallmatrix}\right)$ and obtain that h transforms to

$$\widetilde{h} = x^4 - 2x^2\langle y, y\rangle + \langle y, y\rangle^2 = \left(-x^2 + \langle y, y\rangle\right)^2.$$

Let $\widetilde{H} \subset \left\{\widetilde{h} = 1\right\}$ denote the quartic CCGPSR manifold that is the connected component of $\left\{\widetilde{h} = 1\right\} \subset \mathbb{R}^{n+1}$ containing $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and note that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are equivalent. We will now show that $\mathfrak{so}(n, 1) \subset T_{\mathbb{1}} G^{\widetilde{h}}$ and that the corresponding action of $\mathrm{SO}^+(n, 1)$ on $\widetilde{\mathcal{H}}$ is transitive. Let

$$a_i := \left(\begin{array}{c|c} & dy^T \\ \hline dy & \end{array}\right)(\partial_{y_i})$$

for $1 \leq i \leq n$ (note: in the untransformed coordinates, the a_i correspond to $d\mathcal{A}_0\left(\partial_{z_i}\right)$ (3.60), respectively). Then we check that

$$d\widetilde{h}_{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}\left(a_i \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \equiv 0,$$

and that $[a_i, a_j] = a_i a_j - a_j a_i = \delta_{i+1,j+1} - \delta_{j+1,i+1}$, where $\delta_{k,\ell}$ denotes the $(n+1)\times(n+1)$-matrix with only non-zero entry a 1 at the kth row, ℓth column. The set

$$\{a_i \mid 1 \leq i \leq n\}$$

is a generating set of the Lie algebra $\mathfrak{so}(n, 1)$, where $a \in \mathfrak{gl}(n + 1)$ is an element of $\mathfrak{so}(n, 1)$ if

$$a^T \begin{pmatrix} -1 & \\ & \mathbb{1} \end{pmatrix} + \begin{pmatrix} -1 & \\ & \mathbb{1} \end{pmatrix} a = 0. \tag{7.68}$$

This shows that $\widetilde{h}$ is $\mathfrak{so}(n,1)$-invariant and that we have an isometric action $\mu : \mathrm{SO}^+(n,1) \times \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$. In order to see that this action is transitive, observe that

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\mathcal{H}} \quad \Leftrightarrow \quad -x^2 + \langle y, y \rangle = -1,\ x > 0,$$

since by assumption $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \widetilde{\mathcal{H}}$. With our construction for $\mathfrak{so}(n,1)$ (7.68), $\mathrm{SO}(n,1)^+$ acts transitively on the set $\{-x^2+\langle y,y\rangle = -1,\ x>0\}$ and, hence, $\mathrm{SO}(n,1)^+$ also acts transitively on $\widetilde{\mathcal{H}}$. The isotropy group of any point in $\widetilde{\mathcal{H}}$, e.g. $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, is given by $\mathrm{SO}(n) \subset \mathrm{SO}^+(n,1)$, embedded via

$$\mathrm{SO}(n) \ni A \mapsto \begin{pmatrix} 1 & \\ & A \end{pmatrix} \in \mathrm{SO}^+(n,1).$$

Hence, $\widetilde{\mathcal{H}} \cong \mathrm{SO}^+(n,1)/\mathrm{SO}(n)$.

For the last claim, that is the violation of both Def. 5.1 (i) and (ii), observe that for $U = \mathbb{R}_{>0} \cdot \mathcal{H}$ we have

$$\partial U = \mathbb{R}_{>0} \cdot \left\{ \left(\begin{smallmatrix} 1 \\ v \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; \langle v, v \rangle = 2 \right\}.$$

Hence,

$$T_{\left(\begin{smallmatrix} 1 \\ v \end{smallmatrix}\right)}(\partial U \setminus \{0\}) = \mathbb{R} \cdot \left(\begin{smallmatrix} 1 \\ v \end{smallmatrix}\right) \oplus \left\{ \left(\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \;\middle|\; w \in \ker(\langle v, \cdot \rangle) \right\}$$

for all $\left(\begin{smallmatrix} 1 \\ v \end{smallmatrix}\right) \in \partial U \setminus \{0\}$. One can now easily check that $dh|_{\partial U} \equiv 0$, which shows that $\mathcal{H}$ violates Def. 5.1 (i), and that $-\partial^2 h|_{T\partial(\partial U\setminus\{0\}) \times T\partial(\partial U \setminus \{0\})} \equiv 0$, showing that $\mathcal{H}$ violates Def. 5.1 (ii) in the stated sense. □

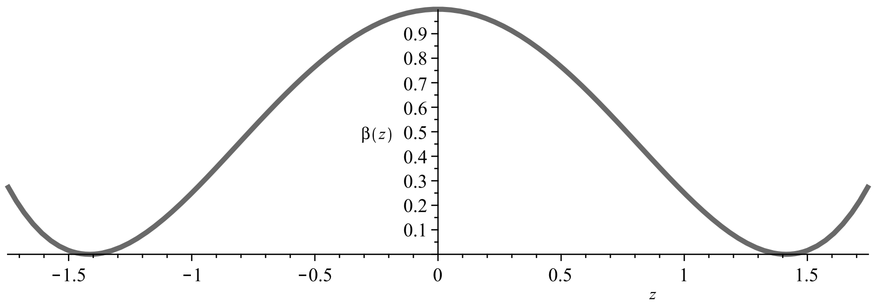


Figure 24: Plot of $\beta(z)$ as in (3.22) corresponding to $n = 1$, $h = x^4 - x^2y^2 + \frac{1}{4}y^4$. It resembles a hat.

Next we will present a family of inhomogeneous $n \geq 1$-dimensional quartic CCGPSR manifolds, which might be thought of as a higher-dimensional analogue of the family of quartic CCGPSR curves d) in Theorem 7.2 <u>with the additional restriction</u> $K \leq 0$.

Example 7.11 ($h = x^4 - x^2\langle y,y\rangle - (M(y,y))^2$-family)**.** *Let $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form. Then with $h = x^4 - x^2\langle y,y\rangle - (M(y,y))^2$, the connected component $\mathcal{H}$ of $\{h = 1\}$ that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \{h=1\} \subset \mathbb{R}^{n+1}$ is a quartic CCGPSR manifold. This can be seen by verifying that the corresponding function*

$$\beta : \mathbb{R}^n \to \mathbb{R}, \quad \beta(z) = 1 - \langle z, z \rangle - (M(z,z))^2,$$

as in (3.22) is globally strictly convex for any bilinear form $M \in \mathrm{Sym}\left((\mathbb{R}^n)^*\right)$ and with the formula (3.33) for the pullback of $g_{\mathcal{H}}$ to $\mathrm{dom}(\mathcal{H})$.

The quartic CCGPSR manifolds in Example 7.11 for $\dim(\mathcal{H}) = n \geq 2$ might be though of as a higher-dimensional analogue to the quartic CCGPSR curves described in Theorem 7.2 d) with corresponding $K \leq 0$. It is an interesting question if one can, similarly, generalise said curves with corresponding $K \in \left(0, \frac{1}{4}\right]$. Possible candidates would be the hypersurfaces corresponding to polynomials of the form $h = x^4 - x^2\langle y, y\rangle + (M(y,y))^2$. However, M must not have eigenvalues of absolute value bigger than $\frac{1}{2}$, which follows from the considerations in the proof of Theorem 7.2. At this point it is however unknown if that eigenvalue-condition for M automatically implies that the corresponding maximal quartic CCGPSR manifold $\mathcal{H} \subset \{h = 1\}$ is closed in $\mathbb{R}^{n+1}$ for $n \geq 2$ (note: examples of such hypersurfaces are studied in Example 7.10). We leave that question as a problem for future research.

Example 7.12 (Cubics h times x, same $\operatorname{dom}(\mathcal{H})$ and even easier metric)**.** *Another way to obtain quartic CCGPSR manifolds is as follows. Let* $\mathcal{H} \subset \{h = x^3 - x\langle y, y\rangle + P_3(y) = 1\} \subset \mathbb{R}^{n+1}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}$, *be a CCPSR manifold of dimension* $\dim(\mathcal{H}) = n$. *Then the connected component* $\widetilde{\mathcal{H}}$ *of the set* $\{x \cdot h = x^4 - x^2\langle y, y\rangle + xP_3(y) = 1\} \subset \mathbb{R}^{n+1}$ *that contains the point* $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \{x \cdot h = 1\} \subset \mathbb{R}^{n+1}$ *is a quartic CCGPSR manifold. In order to see that this is true, consider the functions* β *for* $\mathcal{H}$ *and* $\widetilde{\beta}$ *for* $\widetilde{\mathcal{H}}$*, both as in equation (3.22). We find that*

$$\beta(z) = 1 - \langle z, z\rangle + P_3(z) = \widetilde{\beta}(z)$$

for all $z \in \mathbb{R}^n$*. This in particular means that the two sets* $\operatorname{dom}(\mathcal{H}) \subset \mathbb{R}^n$ *and* $\operatorname{dom}\left(\widetilde{\mathcal{H}}\right)$ *coincide. Denote by* $\Phi_h : \operatorname{dom}(\mathcal{H}) \to \mathcal{H}$ *and* $\Phi_{x\cdot h} : \operatorname{dom}\left(\widetilde{\mathcal{H}}\right) \to \widetilde{\mathcal{H}}$ *the respective diffeomorphisms, cf. (3.14). Then we obtain using equation (3.33) that*

$$\begin{aligned}
\left(\Phi^*_{x\cdot h} g_{\widetilde{\mathcal{H}}}\right)_z &= -\frac{\partial^2\beta_z}{4\beta(z)} + \frac{3d\beta_z^2}{16\beta^2(z)} \\
&= \frac{3}{4}\left(-\frac{\partial^2\beta_z}{3\beta(z)} + \frac{1d\beta_z^2}{4\beta^2(z)}\right) \\
&\geq \frac{3}{4}\left(-\frac{\partial^2\beta_z}{3\beta(z)} + \frac{2d\beta_z^2}{9\beta^2(z)}\right) = \frac{3}{4}\left(\Phi^* g_{\mathcal{H}}\right)_z
\end{aligned}$$

for all $z \in \operatorname{dom}\left(\widetilde{\mathcal{H}}\right) = \operatorname{dom}(\mathcal{H})$*. This shows that* $\left(\widetilde{\mathcal{H}}, g_{\widetilde{\mathcal{H}}}\right)$ *is geodesically complete, and by using [CNS, Prop. 1.8] we deduce that* $\widetilde{\mathcal{H}}$ *is a quartic CCGPSR manifold as claimed.*

The construction described in Example 7.12 has, however, one important downside to it, which is that it does in general not preserve equivalence classes. This is meant in the sense that equivalent CCPSR manifolds need not be mapped to equivalent quartic CCGPSR manifolds in that way. See Section 9 for an example and a related discussion.

We will end this section with an open problem, which turned out to be more difficult than expected during the preparation of this thesis.

Open problem 7.13 (Existence of a quartic CCGPSR surface with $\|z\| = \sqrt{6}$)**.** *Does there exist a quartic CCGPSR surface* $\mathcal{H} \subset \{h = 1\}$*,* h *of the form (3.12),* $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^2$*, such that*

$$\sup_{z \in \operatorname{dom}(\mathcal{H})} \|z\| = \sqrt{6},$$

or, equivalently, such that there exists $\overline{z} \in \partial\operatorname{dom}(\mathcal{H})$ *with* $\|\overline{z}\| = \sqrt{6}$*?*

Note that for quartic CCGPSR curves, the curve $\mathcal{H}$ in Theorem 7.2 b) fulfils $\sup\limits_{z \in \operatorname{dom}(\mathcal{H})} \|z\| = \sqrt{6}$, but the existence of a quartic CCGPSR manifold $\mathcal{H}$ of dimension $\mathcal{H} \geq 2$ with that property is a priori not clear.

8 Scalar curvature of manifolds in the image of the generalised supergravity r-map

In this section we will study applications of our previous results to the theory of the generalised supergravity r-map. In particular we are interested in finding a formula for the scalar curvature and its first derivative of manifolds in the image of the r-map similar to (3.34) in Proposition 3.29, and (3.39), respectively (3.40), in Proposition 3.30.

Furthermore, we will provide an application of Theorem 5.6 for the theory of the supergravity q-map, that is, the composition of the supergravity r- and c- map. To do so we will first review basic definitions and results from that field. References for this subject are e.g. [GST], [FS], [DV], [CMMS1], [CMMS2], [CM], [C et al.], [CHM], [CDL], [CDMV], [D].

Definition 8.1 (Pseudo-Kähler manifold). *A pseudo-Kähler manifold is a triple (M, J, g), where M is a complex manifold with complex structure J and equipped with a pseudo-Hermitian metric g, such that*

$$\omega := g(\cdot, J\cdot) \tag{8.1}$$

is closed. ω is called the Kähler form of (M, J, g).

Now we will define the generalised supergravity r-map, cf. [CHM, Def. 2].

Definition 8.2 (Generalised supergravity r-map). *Let $U \subset \mathbb{R}^{n+1}$ be an open connected subset that is invariant under multiplication with positive numbers, that is for all $p \in U$ and all $r > 0$, $rp \in U$. For any[23] smooth homogeneous function $h : U \to \mathbb{R}_{>0}$ of homogeneity degree $\tau \in \mathbb{R} \setminus \{1, 0\}$, $\mathcal{H} = \{p \in U \mid h(p) = 1\}$ is a smooth hypersuface contained in $U \subset \mathbb{R}^{n+1}$ is a smooth hypersurface (which follows from the Euler identity for homogeneous functions). Further assume that*

$$g_{\mathcal{H}} := -\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}} > 0.$$

Then

$$g_U := -\frac{1}{\tau}\partial^2 (\ln h) \tag{8.2}$$

is a Riemmanian metric. Now consider the manifold

$$M := U \times \mathbb{R}^{n+1}. \tag{8.3}$$

Let $(x_1, \ldots, x_{n+1})^T$ denote the linear coordinates on U induced by the embedding $U \subset \mathbb{R}^{n+1}$ and let $(y_1, \ldots, y_{n+1})^T$ denote the standard linear coordinates on $\mathbb{R}^{n+1}$, so that M is equipped with the global coordinate system $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})^T$. Using the notation

$$g_U = \sum_{i,j=1}^{n+1} g_{ij} dx_i dx_j,$$

we equip M with the Riemannian metric

$$g_M := \frac{3}{4}\sum_{i,j=1}^{n+1} g_{ij} \left(dx_i dx_j + dy_i dy_j\right).$$

Then (M, g_M) is a Riemannian manifold and the correspondence

$$(\mathcal{H}, g_{\mathcal{H}}) \mapsto (M, g_M)$$

is called the generalised supergravity r-map. If $\mathcal{H}$ is a connected PSR manifold, then it is called the supergravity r-map. In the following, we will also denote $M = \mathrm{r}(\mathcal{H})$.

[23] No restriction on h to be a polynomial or a rational function.

Note that the generalised supergravity r-map can by applied in particular to CCPSR and CCGPSR manifolds. In order to describe manifolds in the image of the generalised supergravity r-map, we need the notion of projective special Kähler manifolds. Instead of rigorously introducing these this type of Kähler manifolds here, we refer the reader to [CHM, Def. 4] and the discussion leading to that definition, since a clean and complete introduction would go beyond the scope of this thesis. Having the definition of the latter type of Kähler manifolds in mind, we can now give to following characterisation of manifolds in the image of the generalised supergravity r-map.

Theorem 8.3 ([CHM] Thm. 4)**.** *The generalised supergravity r-map maps complete $n \geq 0$-dimensional Riemannian manifolds $(\mathcal{H}, g_{\mathcal{H}})$ to complete Kähler manifolds (M, g_M) of real dimension $2n+2$ with a free isometric action of the vector group $\mathbb{R}^{n+1}$. The supergravity r-map maps complete $n \geq 0$-dimensional PSR manifolds to complete projective special Kähler manifolds of real dimension $2n+2$.*

We are interested in the application of Theorem 8.3 to CCGPSR and, in particular, CCPSR manifolds. Parts of the following discussionhave been taken from the proof of [CHM, Thm. 4] in [CHM]. For an $n \geq 0$-dimensional CCGPSR manifold[24] $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, $M = \mathrm{r}(\mathcal{H})$ as in (8.3) fulfils

$$M = U \times \mathbb{R}^{n+1} = (\mathbb{R}_{>0} \cdot \mathcal{H}) \times \mathbb{R}^{n+1}$$

with the usual identification of the connected component U of $\{h > 0\} \subset \mathbb{R}^{n+1}$ containing $\mathcal{H}$ and $\mathbb{R}_{>0} \cdot \mathcal{H}$. Hence, $\mathcal{H}$ being connected implies that M is also connected. The Kähler (or projective special Kähler for CCPSR manifolds $\mathcal{H}$) manifold M is in particular a complex manifold. For chosen linear coordinates $(x_1, \ldots, x_{n+1})^T$ of the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H} \subset \mathbb{R}^{n+1}$ and induced coordinates of the open cone $U \subset \mathbb{R}^{n+1}$, $U \cong \mathbb{R}_{>0} \cdot \mathcal{H}$, together with the chosen linear coordinates $(y_1, \ldots, y_{n+1})^T$ of the $\mathbb{R}^{n+1}$-part in (8.3), the induced complex coordinates $(z_1, \ldots, z_{n+1})^T$ on M are given by

$$(z_1, \ldots, z_{n+1})^T = (y_1 + ix_1, \ldots, y_{n+1} + ix_{n+1})^T.$$

Hence, we can identify $M = U \times \mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1} + iU \subset \mathbb{C}^{n+1}$, where we think of the vector part $\mathbb{R}^{n+1}$ in (8.3) as the real part, and of the con part U in (8.3) as the imaginary part of M.

Whenever we are working with our usual standard form of the polynomial h as in Proposition 3.18, equation (3.12), we denote the linear coordinates of the ambient space $\mathbb{R}^{n+1}$ of the considered CCGPSR manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ by $(x, y_1, \ldots, y_n)^T$. For images of the generalised supergravity r-map $M = \mathrm{r}(\mathcal{H}) = U \times \mathbb{R}^{n+1}$, we will then denote the chosen linear coordinates of the vector part $\mathbb{R}^{n+1}$ in (8.3) by $(\widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T$ so that the induced complex coordinates of M are of the form $(\widetilde{x} + ix, \widetilde{y}_1 + iy_1, \ldots, \widetilde{y}_n + iy_n)^T$. When considering the induced real coordinates of $M = U \times \mathbb{R}^{n+1}$, we will use the ordering $(x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T$. In the following, we will frequently identify $\partial_i = \partial_{y_i}$ for all $1 \leq i \leq n$ whenever we use linear coordinates as in equation (3.12).

Remark 8.4 (Standard form analogue for $\mathrm{r}(\mathcal{H})$)**.** Let $\mathcal{H} \subset \{h = 1\}$ and $\widetilde{\mathcal{H}} \subset \left\{\widetilde{h} = 1\right\}$ be two equivalent n-dimensional CCGPSR manifolds related by $A \in \mathrm{GL}(n+1)$, that is

$$h \circ A = \widetilde{h}, \quad A\widetilde{\mathcal{H}} = \mathcal{H}.$$

[24]Recall that we view CCPSR manifolds as special cases of CCGPSR manifolds.

Then their respective images in the generalised supergravity r-map

$$M = \mathrm{r}(\mathcal{H}) = (\mathbb{R}_{>0} \cdot \mathcal{H}) \times \mathbb{R}^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \cong \mathbb{R}^{2n+2}$$

and

$$\widetilde{M} = \mathrm{r}\left(\widetilde{\mathcal{H}}\right) = \left(\mathbb{R}_{>0} \cdot \widetilde{\mathcal{H}}\right) \times \mathbb{R}^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \cong \mathbb{R}^{2n+2}$$

are isometric via the linear map

$$\begin{pmatrix} A & \\ & A \end{pmatrix} : \widetilde{M} \to M, \quad (rp, q) \mapsto (rAp, Aq),$$

where $r \in \mathbb{R}_{>0}$, $p \in \widetilde{\mathcal{H}}$, $Ap \in \mathcal{H}$, $q \in \mathbb{R}^{n+1}$ in the vector part of $\widetilde{M}$, and $Aq \in \mathbb{R}^{n+1}$ in the vector part of M. This means that we can make use of Proposition 3.18 in the setting of the generalised supergravity r-map. We are particularly interested in the scalar curvature S_M of manifolds $M = \mathrm{r}(\mathcal{H})$ in the image of the generalised supergravity r-map, $\mathcal{H} \subset \{h = 1\}$ being a CCGPSR manifold. If we want to know the value of S_M at some point $(rp, q) \in M = (\mathbb{R}_{>0} \cdot \mathcal{H}) \times \mathbb{R}^{n+1}$, $r \in \mathbb{R}_{>0}$, $p \in \mathcal{H}$, $q \in \mathbb{R}^{n+1}$, we use first use the fact that the vector group $\mathbb{R}^{n+1}$ acts via isometries on the vector part of M. This implies that $S_M(rp, q) = S_M(rp, 0)$ for all $q \in \mathbb{R}^{n+1}$ and all $rp \in \mathbb{R}_{>0} \cdot \mathcal{H}$. Then we determine $A(p) \in \mathrm{GL}(n+1)$ fulfilling Proposition 3.18 (i) and (ii). Together with the isometric action of $\mathbb{R}_{>0}$ on the $\mathbb{R}_{>0} \cdot \mathcal{H}$-part of M, this means in order to calculate S_M at any point, it suffices to find a formula for $S_M\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), 0\right)$ depending on the prefactors of the monomials in $P_3, \ldots, P_\tau$ assuming h is of the form (3.12) (see Lemma 8.6 for the result). Note the similarity to the process of determining $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$, cf. Proposition 3.29.

Lemma 8.5 (Homogeneity of $\mathrm{r}(\mathcal{H})$ for homogeneous $\mathcal{H}$)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a CCGPSR manifold and assume that the identity-component of the automorphism group of h, that is G_0^h (3.4), acts transitively on $\mathcal{H}$. Then the image of $\mathcal{H}$ in the generalised supergravity r-map,* $\mathrm{r}(\mathcal{H})$*, is a homogeneous space.*

Proof. [CHM, Prop. 1] implies that $\left(\mathrm{r}(\mathcal{H}), g_{\mathrm{r}(\mathcal{H})}\right)$ is isometric to

$$\left(\mathbb{R} \times \mathcal{H} \times \mathbb{R}^{n+1}, \frac{3}{4}\left(dr^2 + g_{\mathcal{H}} + \sum_{i,j=1}^{n+1} g_{ij} dy_i dy_j\right)\right),$$

where r denotes the standard linear coordinate on $\mathbb{R}$, $(x_1, \ldots, x_{n+1})^T$ denotes the chosen linear coordinates of the ambient space $\mathbb{R}^{n+1}$ of $\mathcal{H}$, $(y_1, \ldots, y_{n+1})^T$ denotes the chosen linear coordinates on $\mathbb{R}^{n+1}$, and $g_{ij} = g_U(\partial_{x_i}, \partial_{x_j})$ for $1 \leq i, j \leq n+1$ with (8.2) and the corresponding conventions. It is now easy to see that the product group

$$\mathbb{R} \times G_0^h \times \mathbb{R}^{n+1}$$

acts transitively on $\mathbb{R} \times \mathcal{H} \times \mathbb{R}^{n+1}$ via isometries. Here $\mathbb{R}$ and $\mathbb{R}^{n+1}$ act via translation on the $\mathbb{R}$- and $\mathbb{R}^{n+1}$-part of $\mathbb{R} \times \mathcal{H} \times \mathbb{R}^{n+1}$, respectively. Hence, $\mathrm{r}(\mathcal{H})$ is a homogeneous space. □

Having introduced all necessary concepts, we will now derive an analogue of Proposition 3.29 for manifolds in the image of the generalised supergravity r-map of the form $\mathrm{r}(\mathcal{H})$, where $\mathcal{H}$ is a connected GPSR manifold.

Lemma 8.6 (Scalar curvature of a manifold in image of the generalised supergravity r-map, [CDL] Corollary 3)**.** *Let* (M, g_M)*,* $M = \mathrm{r}(\mathcal{H})$*,* $\mathcal{H} \subset \{h = 1\}$*, be a Kähler manifold in the image of the generalised supergravity r-map and let* S_M *denote its scalar curvature. Let* $\tau \in \mathbb{R} \setminus \{0, 1\}$ *denote the homogeneity-degree of the corresponding function* $h : \mathbb{R}_{>0} \cdot \mathcal{H} \to \mathbb{R}$*. Then*

$$\begin{aligned} S_M = \frac{4\tau}{3} \Bigg(& -(n+1)^2 + \frac{\tau-2}{\tau-1}(n+1) \\ & + \frac{h}{\det\left(\partial^2 h\right)} \operatorname{tr}\left(\partial^2 h^{-1} \cdot \partial^2 \left(\det\left(\partial^2 h\right)\right)\right) \\ & - \frac{h}{\det\left(\partial^2 h\right)^2} \operatorname{tr}\left(\partial^2 h^{-1} \cdot \partial\left(\det\left(\partial^2 h\right)\right) \otimes \left\langle \partial\left(\det\left(\partial^2 h\right)\right), \cdot \right\rangle\right)\Bigg). \end{aligned} \tag{8.4}$$

In the above formula, all derivatives are taken with respect to the linear coordinates on the $\mathbb{R}_{>0} \cdot \mathcal{H}$*-part of* M *induced by the chosen linear coordinates of the ambient space* $\mathbb{R}^{n+1} \supset \mathcal{H}$*. The matrices inside* $\operatorname{tr}(\cdot)$ *are viewed as endomorphism fields of* $\mathbb{R}^{n+1}$*, so that their trace is just the sum of the respective diagonal entries.*

Proof. Up to the prefactor $\frac{4\tau}{3}$ and a different notation, this is precisely one of the statements in [CDL, Cor. 3]. The prefactor 2 comes from a slightly modified convention for the scalar curvature that was used in [CDL], and the prefactor $\frac{2\tau}{3}$ comes from a different metric of the base manifold $\mathcal{H}$ that was used in [CDL, Ch. 5], namely $-\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ instead of $-\frac{1}{\tau}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$, and a different metric used on M, namely $\sum\limits_{i,j=1}^{n+1} g_{ij}\left(dx_i dx_j + dy_i dy_j\right)$ instead of $g_M = \frac{3}{4}\sum\limits_{i,j=1}^{n+1} g_{ij}\left(dx_i dx_j + dy_i dy_j\right)$. □

Lemma 8.7 (Properties of the scalar curvature of manifolds in the image of the generalised supergravity r-map)**.** *Let* (M, g_M) *be a Kähler manifold in the image of the supergravity r-map, such that* $M = \mathrm{r}(\mathcal{H})$ *for a connected GPSR manifold* $\mathcal{H}$*. Then the scalar curvature* S_M *of* (M, g_M) *is invariant under translations in the vector-part* $\mathbb{R}^{n+1}$ *of* $M = \mathbb{R}_{>0} \cdot \mathcal{H} \times \mathbb{R}^{n+1}$*. Furthermore, for any chosen linear coordinates* $(x_1, \ldots, x_{n+1})^T$ *of the cone* $\mathbb{R}_{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ *which are induced by a choice of the linear coordinates of the ambient space* $\mathbb{R}^{n+1} \supset \mathcal{H}$*, the scalar curvature* S_M *is a homogeneous rational function of degree zero in the variables* $x_1, \ldots, x_{n+1}$*.*

Proof. This follows from Lemma 8.6, which follows from [CDL, Thm. 3] or, equivalently, from [CDL, Cor. 3]. All one needs to check is that entry-wise, for any chosen linear coordinates $(x_1, \ldots, x_{n+1})^T$ of the ambient space $\mathbb{R}^{n+1} \supset \mathcal{H}$, the inverse of the symmetric matrix $\partial^2 h$ (which is by assumption of Lorentz type at any point $p \in \mathbb{R}_{>0} \cdot \mathcal{H}$) is a rational function in the variables $x_1, \ldots, x_{n+1}$, which follows from the general formula for inverse matrices

$$\left(\partial^2 h\right)^{-1} = \frac{1}{\det\left(\partial^2 h\right)} \operatorname{adj}\left(\partial^2 h\right),$$

where $\operatorname{adj}\left(\partial^2 h\right)$ denotes the adjunct matrix of $\partial^2 h$. □

Proposition 8.8 (Scalar curvature of manifolds in the image of the r-map)**.** *Let* $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ *be an* $n \geq 1$*-dimensional connected GPSR manifold and* h *of homogeneity-degree* $\tau \geq 3$*. Assume that* h *is of the form (3.12), that is* $h = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum\limits_{k=3}^{\tau} x^{\tau-k} P_k(y)$*, and that* $\mathcal{H}$ *contains the point* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$*, where* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \begin{pmatrix} x \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$ *denote the standard linear* *coordinates on $\mathbb{R}^{n+1}$. Let $M = \mathrm{r}(\mathcal{H}) \cong (\mathbb{R}_{>0}\cdot\mathcal{H}) \times \mathbb{R}^{n+1}$ be the Kähler manifold obtained by applying the generalised supergravity r-map to $\mathcal{H}$ and let $(x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T$ denote the induces (real) coordinates on M, where $(x, y_1, \ldots, y_n)^T$ denotes the coordinates on the $\mathbb{R}_{>0}\cdot\mathcal{H}$-part and $(\widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T$ denote the coordinates on the $\mathbb{R}^{n+1}$-part of M. Let S_M denote the scalar curvature of M. Then at the point*

$$\mathbf{p} := (x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1, 0, \ldots, 0, 0, 0, \ldots, 0)^T \in M,$$

S_M takes the value

$$S_M(\mathbf{p}) = \frac{4\tau}{3}\left(-n^2 - 2n - \frac{2}{\tau} + \frac{9}{2}\left(\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k)^2\right) + 6\sum_{i,j} P_4(\partial_i, \partial_i, \partial_j, \partial_j)\right). \tag{8.5}$$

For $\tau = 3$, the P_4-part in (8.5) is omitted.

Proof. In order to obtain the above formula for S_M we will use (8.4). Recall that for $h = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum\limits_{k=3}^{\tau} x^{\tau-k}P_k(y)$ we have

$$\begin{aligned}\partial^2 h = \Bigg(&\tau(\tau-1)x^{\tau-2} - (\tau-2)(\tau-3)x^{\tau-4}\langle y, y\rangle \\ &+ \sum_{k=3}^{\tau-2}(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y)\Bigg)dx^2 \\ &+ 2\left(-2(\tau-2)x^{\tau-3}\langle y, dy\rangle + \sum_{k=3}^{\tau-1} k(\tau-k)x^{\tau-k-1}P_k(y, \ldots, y, dy)\right)dx \\ &- 2x^{\tau-2}\langle dy, dy\rangle + \sum_{k=3}^{\tau} k(k-1)x^{\tau-k}P_k(y, \ldots, y, dy, dy).\end{aligned}$$

Thus, written as a symmetric matrix, we get at $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$

$$\partial^2 h|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \left(\begin{array}{c|c} \tau(\tau-1) & \\ \hline & -2\mathbb{1} \end{array}\right), \quad \det\left(\partial^2 h\right)\Big|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = (-1)^n 2^n \tau(\tau-1)$$

and

$$\partial^2 h^{-1}|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \left(\begin{array}{c|c} \frac{1}{\tau(\tau-1)} & \\ \hline & -\frac{1}{2}\mathbb{1} \end{array}\right).$$

For $\mu \in \{x, y_1, \ldots, y_n\}$, recall that

$$\partial_\mu \det\left(\partial^2 h\right) = \det\left(\partial^2 h\right)\operatorname{tr}\left(\partial^2 h^{-1}\cdot\partial_\mu\partial^2 h\right). \tag{8.6}$$

We obtain

$$\partial_x\partial^2 h = \left(\begin{array}{c|c} \begin{array}{c}\tau(\tau-1)(\tau-2)x^{\tau-3} - (\tau-2)(\tau-3)(\tau-4)x^{\tau-5}\langle y,y\rangle \\ +\sum\limits_{k=3}^{\tau-3}(\tau-k)(\tau-k-1)(\tau-k-2)x^{\tau-k-3}P_k(y)\end{array} & \begin{array}{c}-2(\tau-2)(\tau-3)x^{\tau-4}\langle y,\cdot\rangle \\ +\sum\limits_{k=3}^{\tau-2}k(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y,\ldots,y,\cdot)\end{array} \\ \hline \begin{array}{c}-2(\tau-2)(\tau-3)x^{\tau-4}y \\ +\sum\limits_{k=3}^{\tau-2}k(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y,\ldots,y,\cdot)^T\end{array} & \begin{array}{c}-2(\tau-2)x^{\tau-1}\mathbb{1} \\ +\sum\limits_{k=3}^{\tau-1}k(k-1)(\tau-k)x^{\tau-k-1}P_k(y,\ldots,y,\cdot,\cdot)\end{array}\end{array}\right)$$

and, hence,

$$\partial_x\partial^2 h|_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \left(\begin{array}{c|c} \tau(\tau-1)(\tau-2) & \\ \hline & -2(\tau-2)\mathbb{1} \end{array}\right).$$

This implies with (8.6)

$$\partial_x \det\left(\partial^2 h\right)\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = (-1)^n 2^n \tau(\tau-1)(n+1)(\tau-2).$$

As before, e.g. similar to Proposition 3.29, abbreviate $\partial_{y_i} = \partial_i$ for $1 \leq i \leq n$. We then have

$$\partial_i \partial^2 h = \left(\begin{array}{c|c} \begin{array}{c} -2(\tau-2)(\tau-3)x^{\tau-4}y_i \\ +\sum\limits_{k=3}^{\tau-2} k(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y,\ldots,y,\partial_i) \end{array} & \begin{array}{c} -2(\tau-2)x^{\tau-3}\langle\partial_i,\cdot\rangle \\ +\sum\limits_{k=3}^{\tau-1} k(k-1)(\tau-k)x^{\tau-k-1}P_k(y,\ldots,y,\partial_i,\cdot) \end{array} \\ \hline \begin{array}{c} -2(\tau-2)x^{\tau-3}\partial_i \\ +\sum\limits_{k=3}^{\tau-1} k(k-1)(\tau-k)x^{\tau-k-1}P_k(y,\ldots,y,\partial_i,\cdot)^T \end{array} & \sum\limits_{k=3}^{\tau} k(k-1)(k-2)x^{\tau-k}P_k(y,\ldots,y,\partial_i,\cdot,\cdot) \end{array}\right)$$

and

$$\partial_i \partial^2 h|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \left(\begin{array}{c|c} 0 & -2(\tau-2)\langle\partial_i,\cdot\rangle \\ \hline -2(\tau-2)\partial_i & 6P_3(\partial_i,\cdot,\cdot) \end{array}\right).$$

Hence, (8.6) yields

$$\partial_i \det\left(\partial^2 h\right)\Big|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = (-1)^n 2^n \tau(\tau-1)\left(-3\sum_{j=1}^{n} P_3(\partial_i,\partial_j,\partial_j)\right)$$

for all $1 \leq i \leq n$. Observe that $\partial^2 h^{-1}|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$ is diagonal, which in particular means that for the $\frac{h}{\det(\partial^2 h)}\mathrm{tr}\left(\partial^2 h^{-1}\cdot\partial^2\left(\det\left(\partial^2 h\right)\right)\right)$-part of (8.4) we only need to calculate

$$\begin{aligned} \partial_\mu^2 \det\left(\partial^2 h\right) &= \det\left(\partial^2 h\right)\left(\mathrm{tr}\left(\partial^2 h^{-1}\cdot\partial_\mu\partial^2 h\right)\right)^2 \\ &\quad + \det\left(\partial^2 h\right)\mathrm{tr}\left(-\partial^2 h^{-1}\cdot\partial_\mu\partial^2 h\cdot\partial^2 h^{-1}\cdot\partial_\mu\partial^2 h + \partial^2 h^{-1}\partial_\mu\partial_\mu\partial^2 h\right) \end{aligned} \tag{8.7}$$

at $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for all $\mu \in \{x, y_1, \ldots, y_n\}$. We find

$$\partial_x^2 \partial^2 h =$$

$$\left(\begin{array}{c|c} \begin{array}{c} \tau(\tau-1)(\tau-2)(\tau-3)x^{\tau-4} \\ -(\tau-2)(\tau-3)(\tau-4(\tau-5)x^{\tau-6}\langle y,y\rangle \\ +\sum\limits_{k=3}^{\tau-4} k(\tau-k)(\tau-k-1)(\tau-k-2)(\tau-k-3)x^{\tau-k-4}P_k(y) \end{array} & \begin{array}{c} -2(\tau-2)(\tau-3)(\tau-4)x^{\tau-5}\langle y,\cdot\rangle \\ +\sum\limits_{k=3}^{\tau-3} k(\tau-k)(\tau-k-1)(\tau-k-2)x^{\tau-k-3}P_k(y,\ldots,y,dy) \end{array} \\ \hline \begin{array}{c} -2(\tau-2)(\tau-3)(\tau-4)x^{\tau-5}y \\ +\sum\limits_{k=3}^{\tau-3} k(\tau-k)(\tau-k-1)(\tau-k-2)x^{\tau-k-3}P_k(y,\ldots,y,dy)^T \end{array} & \begin{array}{c} -2(\tau-2)(\tau-3)x^{\tau-4}\mathbb{1} \\ \sum\limits_{k=3}^{\tau-2} k(k-1)(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y,\ldots,y,\cdot,\cdot) \end{array} \end{array}\right)$$

and

$$\partial_i^2 \partial^2 h =$$

$$\left(\begin{array}{c|c} \begin{array}{c} -2(\tau-2)(\tau-3)x^{\tau-4} \\ +\sum\limits_{k=3}^{\tau-2} k(k-1)(\tau-k)(\tau-k-1)x^{\tau-k-2}P_k(y,\ldots,y,\partial_i,\partial_i) \end{array} & \sum\limits_{k=3}^{\tau-1} k(k-1)(k-2)(\tau-k)x^{\tau-k-1}P_k(y,\ldots,y,\partial_i,\partial_i,\cdot) \\ \hline \sum\limits_{k=3}^{\tau-1} k(k-1)(k-2)(\tau-k)x^{\tau-k-1}P_k(y,\ldots,y,\partial_i,\partial_i,\cdot)^T & \sum\limits_{k=4}^{\tau} k(k-1)(k-2)(k-3)x^{\tau-k}P_k(y,\ldots,y,\partial_i,\partial_i,\cdot,\cdot) \end{array}\right)$$

for all $1 \leq i \leq n$. Hence,

$$\partial_x^2 \partial^2 h|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \left(\begin{array}{c|c} \tau(\tau-1)(\tau-2)(\tau-3) & \\ \hline & -2(\tau-2)(\tau-3)\mathbb{1} \end{array}\right)$$

and

$$\partial_i^2 \partial^2 h|_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} = \left(\begin{array}{c|c} -2(\tau-2)(\tau-3) & 6(\tau-3)P_3(\partial_i,\partial_i,\cdot) \\ \hline 6(\tau-3)P_3(\partial_i,\partial_i,\cdot)^T & 24P_4(\partial_i,\partial_i,\cdot,\cdot) \end{array}\right)$$

for all $1 \leq i \leq n$. Observe that for $\tau = 3$, the P_4-term in $\partial_i^2\partial^2 h|_{\binom{1}{0}}$ is omitted, and furthermore $\partial_x^2\partial^2 h|_{\binom{1}{0}} \equiv \partial_i^2\partial^2 h|_{\binom{1}{0}} \equiv 0$ for $\tau = 3$ as expected. Thus we find with (8.7)

$$\partial_x^2 \det\left(\partial^2 h\right)\Big|_{\binom{1}{0}} = (-1)^n 2^n \tau(\tau-1)(n+1)(\tau-2)(n(\tau-2)+(\tau-3))$$

and

$$\begin{aligned}\partial_i^2 \det\left(\partial^2 h\right)\Big|_{\binom{1}{0}} = (-1)^n 2^n \tau(\tau-1)\Bigg(&\frac{4(\tau-2)^2 - 2(\tau-2)(\tau-3)}{\tau(\tau-1)}\\ &+\left(9\sum_{j,k} P_3(\partial_i,\partial_j,\partial_j)P_3(\partial_i,\partial_k,\partial_k)\right)\\ &+\left(-9\sum_{j,k} P_3(\partial_i,\partial_j,\partial_k)^2\right)\\ &+\left(-12\sum_{j} P_4(\partial_i,\partial_i,\partial_j,\partial_j)\right)\Bigg).\end{aligned}$$

We can now use our calculations to determine the

$$-\frac{h}{\det\left(\partial^2 h\right)^2}\operatorname{tr}\left(\partial^2 h^{-1}\cdot\partial\left(\det\left(\partial^2 h\right)\right)\otimes\left\langle\partial\left(\det\left(\partial^2 h\right)\right),\cdot\right\rangle\right)\text{-part}$$

of S_M (8.4):

$$\begin{aligned}&-\frac{h}{\det\left(\partial^2 h\right)^2}\operatorname{tr}\left(\partial^2 h^{-1}\cdot\partial\left(\det\left(\partial^2 h\right)\right)\otimes\left\langle\partial\left(\det\left(\partial^2 h\right)\right),\cdot\right\rangle\right)\Bigg|_{\binom{1}{0}}\\ &= -\frac{h}{\det\left(\partial^2 h\right)^2}\partial\left(\det\left(\partial^2 h\right)\right)^T\cdot\partial^2 h^{-1}\cdot\partial\left(\det\left(\partial^2 h\right)\right)\Bigg|_{\binom{1}{0}}\\ &= -\frac{(n+1)^2(\tau-2)^2}{\tau(\tau-1)} + \frac{9}{2}\sum_{i,j,k} P_3(\partial_i,\partial_j,\partial_j)P_3(\partial_i,\partial_k,\partial_k).\end{aligned}$$

For the $\frac{h}{\det(\partial^2 h)}\operatorname{tr}\left(\partial^2 h^{-1}\cdot\partial^2\left(\det\left(\partial^2 h\right)\right)\right)$-part of S_M (8.4) we find

$$\begin{aligned}&\frac{h}{\det\left(\partial^2 h\right)}\operatorname{tr}\left(\partial^2 h^{-1}\cdot\partial^2\left(\det\left(\partial^2 h\right)\right)\right)\Bigg|_{\binom{1}{0}}\\ &= \frac{n(n-1)(\tau-2)^2 + (2n+1)(\tau-2)(\tau-3)}{\tau(\tau-1)}\\ &+\frac{9}{2}\left(\sum_{i,j,k}\left(-P_3(\partial_i,\partial_j,\partial_j)P_3(\partial_i,\partial_k,\partial_k) + P_3(\partial_i,\partial_j,\partial_k)^2\right)\right)\\ &+6\sum_{i,j} P_4(\partial_i,\partial_i,\partial_j,\partial_j).\end{aligned}$$

Summarising, we obtain

$$S_M(\mathbf{p}) = \frac{4\tau}{3}\left(-n^2 - 2n - \frac{2}{\tau} + \frac{9}{2}\left(\sum_{i,j,k} P_3(\partial_i,\partial_j,\partial_k)^2\right) + 6\sum_{i,j} P_4(\partial_i,\partial_i,\partial_j,\partial_j)\right)$$

which is precisely the formula (8.5). □

Note that, in comparison with $S_{\mathcal{H}}$ at $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ in Proposition 3.29, the scalar curvature $S_M = S_{\mathrm{r}(\mathcal{H})}$ at $(x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1, 0, \ldots, 0, 0, 0, \ldots, 0)^T$ (for connected GPSR manifolds $\mathcal{H}$ with the corresponding assumptions) does depend on P_4 for $\tau \geq 4$ and not just on P_3.

We can now for fixed $n \in \mathbb{N}$, similar to Theorem 4.13, show that the scalar curvature of manifolds in the image of the supergravity r-map $M = \mathrm{r}(\mathcal{H})$ is globally bounded by constants depending only on n whenever $\mathcal{H}$ is a n-dimensional CCPSR manifold.

Proposition 8.9 ((Non-sharp) $S_{\mathrm{r}(\mathcal{H})}$ bounds for n-dimensional CCGPSR manifolds $\mathcal{H}$)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be an $n \geq 1$-dimensional CCPSR manifold and $M = \mathrm{r}(\mathcal{H})$ be the corresponding projective special Kähler manifold after applying the supergravity r-map to $\mathcal{H}$. Then the scalar curvature of (M, g_M) is globally bounded by*

$$-\frac{25}{6}n^3 - \frac{86}{9}n^2 - \frac{28}{3}n - \frac{4}{3} \leq S_M \leq \frac{25}{6}n^3 + \frac{14}{9}n^2 - \frac{28}{3}n - \frac{4}{3}, \tag{8.8}$$

independent of the considered n-dimensional CCPSR manifold $\mathcal{H}$.

Proof. We can without loss of generality assume that h is of the form (3.12) and that $\mathcal{H} \subset \{h = 1\}$ coincided with the connected component that contains the point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then (with $\tau = 3$) S_M at $\mathbf{p}^T = (x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1, 0, \ldots, 0, 0, 0, \ldots, 0)^T$ is of the form

$$S_M(\mathbf{p}) = 4\left(-n^2 - 2n - \frac{2}{3} + \frac{9}{2}\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k)^2\right).$$

Using (4.16) and the estimate (4.12), we obtain

$$4\left(-\frac{25}{24}n^3 - n^2 - 2n - \frac{2}{3}\right) \leq S_M(\mathbf{p}) \leq 4\left(\frac{25}{24}n^3 - n^2 - 2n - \frac{2}{3}\right).$$

Now we use Lemma 8.7 and Remark 8.4 and conclude that

$$4\left(-\frac{25}{24}n^3 - n^2 - 2n - \frac{2}{3}\right) \leq S_M(p) \leq 4\left(\frac{25}{24}n^3 - n^2 - 2n - \frac{2}{3}\right) \tag{8.9}$$

for all $p \in M = \mathrm{r}(\mathcal{H})$. In particular, (8.9) is depends only on the dimension $n = \dim(\mathcal{H})$ of $\mathcal{H}$, not on the choice of the particular CCPSR manifold $\mathcal{H}$. □

One consequence of Proposition (8.9) is the following.

Corollary 8.10 (Negativity of $S_{\mathrm{r}(\mathcal{H})}$ for $\dim(\mathcal{H}) = 1$)**.** *Let $\mathcal{H}$ be CCPSR curve. Then its corresponding image in the supergravity r-map, $M = \mathrm{r}(\mathcal{H})$, has negative scalar curvature.*

Proof. For $n = \dim(\mathcal{H}) = 1$, the upper bound in (8.8) reads

$$\frac{25}{24} - 1 - 2 - \frac{2}{3} = -\frac{63}{24} < 0.$$

□

We can, however, improve Corollary 8.10 and find a sharp estimate for S_M, $M = \mathrm{r}(\mathcal{H})$, for $\dim(\mathcal{H}) = 1$ independent of the considered CCPSR curve $\mathcal{H}$.

Lemma 8.11 (Global sharp estimate for $S_{\mathrm{r}(\mathcal{H})}$, $\dim(\mathcal{H})=1$)**.** *Let $\mathcal{H}$ be CCPSR curve. Then the scalar curvature S_M of its image in the supergravity r-map $M=\mathrm{r}(\mathcal{H})$ is globally bounded by*

$$-\frac{44}{3}\leq S_M\leq -12. \tag{8.10}$$

This estimate is sharp in the sense that for all $s\in\left[-\frac{44}{3},-12\right]$ there exists a CCPSR curve $\mathcal{H}_s$ and a point $p_s\in\mathcal{H}_s$, such that for $\mathbf{p}_s:=\left(\begin{smallmatrix}p_s\\0\end{smallmatrix}\right)\in M_s=\mathrm{r}\left(\mathcal{H}_s\right)$, we have $S_{M_s}\left(\mathbf{p}_s\right)=s$.

Proof. We can without loss of generality assume that $h_L=x^3-xy^2+Ly^3$ and that $\mathcal{H}$ is the connected component of the level set $\subset\{h_L=1\}\subset\mathbb{R}^2$ that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, cf. Proposition 3.18. Theorem 5.6 implies that $\mathcal{H}$ is a CCPSR curve if and only if $|L|\leq\frac{2}{3\sqrt{3}}$. Hence, we can use equation (8.5) and find for $\mathbf{p}^T=(1,0,0,0)^T\in M=\mathrm{r}(\mathcal{H})$ that $S_M(\mathbf{p})=4\left(-\frac{11}{3}+\frac{9}{2}L^2\right)$ and, hence,

$$\min_{|L|\leq\frac{2}{3\sqrt{3}}}S_M(\mathbf{p})=-\frac{44}{3}\leq S_M(\mathbf{p})\leq\max_{|L|\leq\frac{2}{3\sqrt{3}}}S_M(\mathbf{p})=-12.$$

With Remark 8.4 we conclude that (8.10) holds true globally. To prove that the estimate in this lemma is sharp in the stated sence, we choose for $s=-\frac{44}{3}$ the CCPSR curve $\mathcal{H}_{-\frac{44}{3}}$ associated to h_0, and for $s=-12$ the CCPSR curve $\mathcal{H}_{-12}$ associated to $h_{\frac{2}{3\sqrt{3}}}$ and find that at $\mathbf{p}^T=(1,0,0,0)^T$ (which is, by construction, contained in both CCPSR curves $\mathcal{H}_{-\frac{44}{3}}$ and $\mathcal{H}_{-12}$)

$$S_{\mathrm{r}\left(\mathcal{H}_{-\frac{44}{3}}\right)}(\mathbf{p})=-\frac{44}{3}$$

and

$$S_{\mathrm{r}(\mathcal{H}_{-12})}(\mathbf{p})=-12.$$

Since $S_{\mathrm{r}(\mathcal{H})}(\mathbf{p})=$, $\mathcal{H}\subset\{h_L=1\}$, depends continuously on $L\in\left[-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}}\right]$, we conclude that the estimate (8.10) is indeed sharp in the stated sense. This finishes the proof. □

What one might ask for next is an analogue of Proposition 5.12 for the scalar curvature of manifolds $M=\mathrm{r}(\mathcal{H})$ in the image of the r-map for CCPSR surfaces $\mathcal{H}$, that is for $\dim(\mathcal{H})=2$. We will formulate this as an open problem, since it turns out that this is the setting of Proposition 5.12) with $P_3\left(\left(\begin{smallmatrix}y\\z\end{smallmatrix}\right)\right)=r\left(\frac{2}{3\sqrt{3}}y^3+kyz^2+\ell z^3\right)$ and $\mathbf{p}^t=(1,0,0,0,0,0)^T\in M=\mathrm{r}(\mathcal{H})$,

$$S_M(\mathbf{p})=-\frac{104}{3}+r^2\left(\frac{8}{3}+6k^2+18\ell^2\right),$$

which, in comparison with $S_{\mathcal{H}}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)=-2+r\left(\frac{3}{4}k^2-\frac{\sqrt{3}}{2}k\right)$, contains a non-trivial ℓ-term. This unfortunately prevents an "easy" analogue for global S_M-bounds when one tries to use the proof of Proposition 5.12.

Open problem 8.12 (Sharp $S_{\mathrm{r}(\mathcal{H})}$-bounds for $\dim(\mathcal{H})=2$)**.** *For CCPSR surfaces $\mathcal{H}$ and their corresponding special real Kähler manifolds in the image of the supergravity r-map $M=\mathrm{r}(\mathcal{H})$, find sharp global bounds for the scalar curvature S_M analogous to Proposition 5.12.*

Remark 8.13. In [CDL, Prop. 9], the image of the scalar curvature $S_{\mathrm{r}(\mathcal{H})}$ for $\mathcal{H}$ as in Theorem 2.45 a)–d) has been precisely determined. For e) it was shown that $S_{\mathrm{r}(\mathcal{H})}$ is not constant. In order to solve Open problem 8.12, it would thus be sufficient to consider only the cases e) and the one-parameter family f) in Theorem 2.45, calculate the image of the corresponding scalar curvature, and then compare the results with [CDL, Prop. 9].

Next we will derive an analogue of Proposition 3.30 for the scalar curvature manifolds in the image of the generalised supergravity r-map. Recall equation (3.24) in Proposition 3.26 and Definition 3.27.

Proposition 8.14 (First derivative of $S_{\mathrm{r}(\mathcal{H})}$)**.** *Let $\mathcal{H} \subset \{h = 1\}$ be a connected GPSR manifold with h of the form (3.12) and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$. Then the first derivative of the scalar curvature S_M of the Kähler manifold $M = \mathrm{r}(\mathcal{H})$ obtained via the generalised supergravity r-map at the point $\mathbf{p} = (x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1, 0, \ldots, 0, 0, 0, \ldots, 0)^T \in M$ fulfils*

$$
\begin{aligned}
dS_M\big|_{\mathbf{p}} = \frac{4\tau}{3}\Bigg(&-6\left(\sum_i P_3(\partial_i, \partial_i, dy)\right) \\
&+ \frac{81}{2}\left(\sum_{i,j,k,\ell} P_3(\partial_i, \partial_j, \partial_k) P_3(\partial_i, \partial_k, \partial_\ell) P_3(\partial_j, \partial_\ell, dy)\right) \\
&+ 36\left(\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k) P_4(\partial_i, \partial_j, \partial_k, dy)\right) \\
&+ 36\left(\sum_{i,j,k} P_3(\partial_j, \partial_k, dy) P_4(\partial_i, \partial_i, \partial_j, \partial_k)\right) \\
&+ 30\left(\sum_{i,j} P_5(\partial_i, \partial_i, \partial_j, \partial_j, dy)\right)\Bigg).
\end{aligned} \tag{8.11}
$$

For $\tau = 3$, the P_4- and P_5-parts in (8.11) are to be left out. For $\tau = 4$, one omits the P_5-part in (8.11).

Proof. Equation (8.5) and Definition 3.27 imply that

$$
dS_M\big|_{\mathbf{p}} = \frac{4\tau}{3}\left(9\left(\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k)\delta P_3(\partial_i, \partial_j, \partial_k)\right) + 6\left(\sum_{i,j} \delta P_4(\partial_i, \partial_i, \partial_j, \partial_j)\right)\right). \tag{8.12}
$$

From equation (3.24) in Proposition 3.26 obtain that

$$
\delta P_3(y) = -\frac{2(\tau-2)}{\tau}\langle y, y\rangle\langle y, dy\rangle + 3P_3\left(y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dy)^T\right) + 4P_4(y, y, y, dy)
$$

and

$$
\delta P_4(y) = \frac{2(\tau-3)}{\tau}P_3(y)\langle y, dy\rangle + 4P_4\left(y, y, y, dB_0 y + \frac{3}{2}P_3(y, \cdot, dy)^T\right) + 5P_5(y, y, y, y, dy),
$$

where we recall that $dB_0 \in \mathrm{Lin}(\mathbb{R}^n; \mathfrak{so}(n))$, cf. (3.25), and omit the identification $T_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}\mathcal{H} \cong T_0\mathrm{dom}\,(\mathcal{H})$ so that we can use dy instead of dz. Up to a slightly different notation and different names for the indices, we have seen in (3.44) that

$$
\begin{aligned}
\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k)\delta P_3(\partial_i, \partial_j, \partial_k) = &-\frac{2(\tau-2)}{\tau}\left(\sum_i P_3(\partial_i, \partial_i, dy)\right) \\
&+ \frac{9}{2}\left(\sum_{i,j,k,\ell} P_3(\partial_i, \partial_j, \partial_k) P_3(\partial_i, \partial_k, \partial_\ell) P_3(\partial_j, \partial_\ell, dy)\right) \\
&+ 4\sum_{i,j,k} P_3(\partial_i, \partial_j, \partial_k) P_4(\partial_i, \partial_j, \partial_k, dy).
\end{aligned}
$$

Using $\partial^2 \left(\delta P_4\right)_y (v,w) = 12\delta P_4(y,y,v,w)$, we get

$$\begin{aligned}\delta P_4(\partial_i,\partial_i,\partial_j,\partial_j) &= \frac{\tau-3}{\tau}\left(P_3(\partial_i,\partial_j,\partial_j)dy_i + P_3(\partial_i,\partial_i,\partial_j)dy_j\right)\\ &\quad + 2P_4\left(\partial_i,\partial_j,\partial_j,dB_0\partial_i + \frac{3}{2}P_3(\partial_i,\cdot,dy)^T\right)\\ &\quad + 2P_4\left(\partial_i,\partial_i,\partial_j,dB_0\partial_j + \frac{3}{2}P_3(\partial_j,\cdot,dy)^T\right)\\ &\quad + 5P_5(\partial_i,\partial_i,\partial_j,\partial_j,dy)\end{aligned}$$

and, hence,

$$\begin{aligned}\sum_{i,j}\delta P_4(\partial_i,\partial_i,\partial_j,\partial_j) &= \frac{2(\tau-3)}{\tau}\left(\sum_i P_3(\partial_i,\partial_i,dy)\right)\\ &\quad + 6\left(\sum_{i,j,k} P_3(\partial_j,\partial_k,dy)P_4(\partial_i,\partial_i,\partial_j,\partial_k)\right)\\ &\quad + 5\sum_{ij} P_5(\partial_i,\partial_i,\partial_j,\partial_j,dy).\end{aligned}$$

Summarising we obtain with (8.12) the formula (8.11) as claimed. One now verifies that for $\tau = 3$, the prefactor of the $\sum\limits_i P_3(\partial_i,\partial_i,dy)$-part in $dS_M|_{\mathbf{p}}$ (8.11) is -24 which is the correct values and, hence, the formula for $dS_M|_{\mathbf{p}}$ is indeed consistent for all $\tau \geq 3$ when leaving out the P_4- and P_5-part if appropriate. □

As an application of Proposition 8.14 we will present an r-map analogue of Proposition 6.9. In order to omit confusion with the letter "M" used in the definition of the multi-parameter families in Theorem 6.1 we will not use the notation $M = \mathrm{r}(\mathcal{H})$ in the following proposition and instead simply use $\mathrm{r}(\mathcal{H})$ for a manifold in the image of the supergravity r-map.

Proposition 8.15 (Inhomogeneity of elements of $\mathrm{r}\left(\mathcal{F}\cup\mathcal{G}\right)$)**.** *Let $h \in \mathcal{F}\cup\mathcal{G}$ and $\mathcal{H}(h)$ be the corresponding CCPSR manifold as in 6.3, respectively 6.4, and let $\mathrm{r}(\mathcal{H})$ be their respective projective special Kähler manifold in the image of the supergravity r-map. Then $\mathrm{r}(\mathcal{H})$ is inhomogeneous.*

Proof. We will proceed very similar to the proof of Proposition (6.9) and we will use the same terminology. For connected PSR manifolds $\mathcal{H}$ (that is for $\tau = 3$), $dS|_{\mathrm{r}(\mathcal{H})}$ at the point $\mathbf{p} = (x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1,0,\ldots,0,0,0,\ldots,0)^T \in \mathrm{r}(\mathcal{H})$ is of the form

$$\begin{aligned}dS_{\mathrm{r}(\mathcal{H})}\Big|_{\mathbf{p}} &= 36\sum_{i,j,k} P_3(\partial_i,\partial_j,\partial_k)\delta P_3(\partial_i,\partial_j,\partial_k)\\ &= -24\left(\sum_i P_3(\partial_i,\partial_i,dy)\right) + 162\sum_{i,j,k,\ell} P_3(\partial_i,\partial_j,\partial_k)P_3(\partial_i,\partial_k,\partial_\ell)P_3(\partial_j,\partial_\ell,dy),\end{aligned}$$

cf. (8.11) and (8.12). Recall that for $h \in \mathcal{F}\cup\mathcal{G}$, the corresponding CCPSR manifold $\mathcal{H}(h)$ as in 6.3, respectively 6.4, is equivalent to the connected component $\widetilde{\mathcal{H}} \subset \left\{\widetilde{h} = 1\right\}$, $\widetilde{h} = x^3 - x\langle y,y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 + \frac{\sqrt{2}}{\sqrt{3}}\sum\limits_{i=1}^{n-1}\frac{\mu_i - \sqrt{2}\eta_i}{\sqrt{2}\mu_i+\eta_i}y_i^2\right)$, that contains the point $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{R}^{n+1}$. Here, $\mu_1,\ldots,\mu_{n-1},\eta_1,\ldots,\eta_{n-1} \geq 0$, and furthermore $\sum\limits_{i=1}^{n-1}\mu_i dy_i^2 > 0$ for $h \in \mathcal{G}$ and $\sum\limits_{i=1}^{n-1}\eta_i dy_i^2 > 0$ for

$h \in \mathcal{F}$, cf. Proposition 6.6. With $\sigma_k = \frac{\mu_k - \sqrt{2}\eta_k}{\sqrt{2}\mu_k + \eta_k}$ (6.50) and $P_3(y) = y_n \left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \sigma_i y_i^2 \right)$ (6.51), we have $\widetilde{h} = x^3 - x\langle y, y\rangle + P_3(y)$ and we recall that

$$P_3(\partial_i, \partial_j, \partial_k) = \begin{cases} \frac{\sqrt{2}}{3\sqrt{3}} \sigma_i \delta_i^j \delta_k^n, & 1 \leq i \leq n-1,\ 1 \leq j \leq n-1,\ 1 \leq k \leq n, \\ 0, & 1 \leq i \leq n-1,\ j = k = n, \\ \frac{2}{3\sqrt{3}}, & i = j = k = n. \end{cases},$$

see (6.52). We now calculate and find

$$dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_i) = 0 \quad \forall 1 \leq i \leq n-1$$

and

$$dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_n) = \frac{8}{\sqrt{3}} \left(-\sqrt{2} \left(\sum_{i=1}^{n-1} \sigma_i \right) + \left(\sum_{i=1}^{n-1} \sigma_i^2 \right) + \sqrt{2} \left(\sum_{i=1}^{n-1} \sigma_i^3 \right) \right).$$

Hence,

$$\frac{\partial}{\partial \sigma_k} \left(dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_n) \right) = \frac{8}{\sqrt{3}} \left(-\sqrt{2} + 2\sigma_k + 3\sqrt{2}\sigma_k^2 \right).$$

Note that

$$\frac{\partial}{\partial \sigma_k} \left(dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_n) \right)\Bigg|_{\sigma_k = -\sqrt{2}} = 8\sqrt{6} > 0 \tag{8.13}$$

and

$$\frac{\partial}{\partial \sigma_k} \left(dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_n) \right)\Bigg|_{\sigma_k = \frac{1}{\sqrt{2}}} = 4\sqrt{6} > 0. \tag{8.14}$$

For $h \in \mathcal{F}$, we have $\eta_1, \ldots, \eta_{n-1} > 0$, $\mu_1, \ldots, \mu_{n-1} \geq 0$, and there exists at least one $k \in \{1, \ldots, n-1\}$, such that $\mu_k > 0$. We have seen in (6.57) that

$$\widetilde{\mathcal{H}} \subset \left\{ \widetilde{h} = x^3 - x\langle y, y\rangle + y_n \left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \frac{\mu_i - \sqrt{2}}{\sqrt{2}\mu_i + 1} y_i^2 \right) = 1 \right\}$$

is equivalent to

$$\widetilde{\mathcal{H}}_r \subset \left\{ \widetilde{h}_r := x^3 - x\langle y, y\rangle + y_n \left(\frac{2}{3\sqrt{3}} y_n^2 + \frac{\sqrt{2}}{\sqrt{3}} \sum_{i=1}^{n-1} \frac{r\mu_i - \sqrt{2}}{r\sqrt{2}\mu_i + 1} y_i^2 \right) = 1 \right\}$$

for all $r > 0$. The polynomial $\widetilde{h}_r$ corresponds to the choices $\sigma_k = \widetilde{\sigma}_k(r) = \frac{r\mu_k - \sqrt{2}}{r\sqrt{2}\mu_k + 1}$ (6.58) for all $1 \leq k \leq n-1$ with $\frac{\partial}{\partial r}(\widetilde{\sigma}_k(r))\Big|_{r=0} = 3\mu_k \geq 0$ (6.59) for all $1 \leq k \leq n-1$. Since

$$dS_{\mathrm{r}(\widetilde{\mathcal{H}}_r)}\Big|_{\mathbf{p}}(\partial_n) = \frac{8}{\sqrt{3}} \left(-\sqrt{2} \left(\sum_{i=1}^{n-1} \widetilde{\sigma}_i(r) \right) + \left(\sum_{i=1}^{n-1} \widetilde{\sigma}_i^2(r) \right) + \sqrt{2} \left(\sum_{i=1}^{n-1} \widetilde{\sigma}_i^3(r) \right) \right)$$

is analytic near $r = 0$ (since η_k is positive for all $1 \leq k \leq n-1$), we can use (8.13) and obtain

$$\frac{\partial}{\partial r} \left(dS_{\mathrm{r}(\widetilde{\mathcal{H}}_r)}\Big|_{\mathbf{p}}(\partial_n) \right)\Bigg|_{r=0} = 8\sqrt{6} \sum_{i=1}^{n-1} 3\mu_i > 0,$$

by the existence of at least one positive μ_k, $1 \leq k \leq n-1$. Since all $\widetilde{\mathcal{H}}_r$, $r > 0$, are equivalent to $\widetilde{\mathcal{H}}$, this shows that $dS_{\mathrm{r}(\widetilde{\mathcal{H}})}\Big|_{\mathbf{p}}(\partial_n)$ does not identically vanish on $\mathrm{r}\left(\widetilde{\mathcal{H}}\right)$ and, hence, that

$S_{\mathrm{r}(\widetilde{\mathcal{H}})}$ is not constant. By construction, $\widetilde{\mathcal{H}}$ is equivalent to $\mathcal{H}$ and, hence, this proves that $\mathrm{r}(\mathcal{H}(h))$ (which is by definition of the supergravity r-map isometric to $\mathrm{r}\left(\widetilde{\mathcal{H}}\right)$, see Remark 8.4) is not a homogeneous space for all $h \in \mathcal{F}$.

For $h \in \mathcal{G}$, the proof of inhomogeneity of $\mathrm{r}(\mathcal{H}(h))$ has the same steps. In that case we use $\sigma_k = \overline{\sigma}_k(r) = \frac{1-r\sqrt{2}\eta_k}{\sqrt{2}+r\eta_k}$ (6.64) for $1 \leq k \leq n-1$ with $\frac{\partial}{\partial r}\left(\overline{\sigma}_k(r)\right)\Big|_{r=0} = -\frac{3}{2}\eta_k$ 6.65 for $1 \leq k \leq n-1$. Furthermore, by assumption of $h \in \mathcal{G}$ there exists at least one $k \in \{1,\ldots,n-1\}$, such that $\eta_k > 0$. This, together with (8.14) and the notation analogous to $\widetilde{\mathcal{H}}_r$,

$$\overline{\mathcal{H}}_r \subset \left\{\overline{h}_r = x^3 - x\langle y,y\rangle + y_n\left(\frac{2}{3\sqrt{3}}y_n^2 + \frac{\sqrt{2}}{\sqrt{3}}\sum_{i=1}^{n-1}\frac{1-r\sqrt{2}\eta_i}{\sqrt{2}+r\eta_i}y_i^2\right) = 1\right\}$$

implies

$$\frac{\partial}{\partial r}\left(dS_{\mathrm{r}(\overline{\mathcal{H}}_r)}\Big|_{\mathbf{p}}(\partial_n)\right)\Bigg|_{r=0} = -2\sqrt{6}\sum_{i=1}^{n-1}3\eta_i < 0,$$

showing that $dS_{\mathrm{r}(\overline{\mathcal{H}}_r)}$ is not constant which, as before, shows that $\mathrm{r}\left(\widetilde{\mathcal{H}}\right)$ is not a homogeneous space. Hence, $\mathrm{r}\left(\mathcal{H}(h)\right)$ is not a homogeneous space for all $h \in \mathcal{G}$. □

We will now use Proposition 8.8 to determine the scalar curvature of the r-map-images of the homogeneous CCPSR manifolds $\mathcal{H}_{1,n}$ (6.44) and $\mathcal{H}_{2,n}$ (6.45).

Lemma 8.16 (Scalar curvature of $\mathrm{r}(\mathcal{H}_{1,n})$ and $\mathrm{r}(\mathcal{H}_{2,n})$)**.** *Let $\mathcal{H}_{1,n}$ and $\mathcal{H}_{2,n}$ be the $n \geq 3$-dimensional CCPSR manifolds as in Proposition 6.9, equation (6.44) and (6.45), respectively. Then the scalar curvature of their respective image under the supergravity r-map is constant and given by*

$$S_{\mathrm{r}(\mathcal{H}_{1,n})} \equiv -4n^2 - 6n - 2$$

and

$$S_{\mathrm{r}(\mathcal{H}_{2,n})} \equiv -4n^2 - 8.$$

Proof. The CCPSR manifolds $\mathcal{H}_{1,n}$ and $\mathcal{H}_{2,n}$ are homogeneous spaces [DV, C]. Hence, Lemma 8.5 implies that $\mathrm{r}(\mathcal{H}_{1,n})$ and $\mathrm{r}(\mathcal{H}_{2,n})$ are also homogeneous spaces and have thus constant scalar curvature. With the convention (6.50) from Proposition 6.9 we have

$$\text{for } \mathcal{H}_{1,n}: \quad \sigma_k = \frac{1}{\sqrt{2}} \quad \forall 1 \leq k \leq n-1$$

and

$$\text{for } \mathcal{H}_{2,n}: \quad \sigma_k = -\sqrt{2} \quad \forall 1 \leq k \leq n-1.$$

We obtain with (8.5) for the CCPSR manifolds $\widetilde{\mathcal{H}}$ corresponding to general values of σ_k as in Proposition 6.9, (6.52), at the point

$$\mathbf{p} = (x, y_1, \ldots, y_n, \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_n)^T = (1, 0, \ldots, 0, 0, 0, \ldots, 0)^T \in \mathrm{r}\left(\widetilde{\mathcal{H}}\right)$$

for the value of $S_{\mathrm{r}(\mathcal{H})}$

$$S_{\mathrm{r}(\mathcal{H})}(\mathbf{p}) = 4\left(-n^2 - 2n + \sum_{i=1}^{n-1}\sigma_i^2\right).$$

We can now check that indeed $S_{\mathrm{r}(\mathcal{H}_{1,n})} \equiv -4n^2 - 6n - 2$ and $S_{\mathrm{r}(\mathcal{H}_{2,n})} \equiv -4n^2 - 8$ as claimed. □

We have seen in Proposition 8.15 and Lemma 8.16 that the results of Proposition 3.18 have applications to the geometry of the (generalised) supergravity r-map, in particular in the sense that one does not need to calculate S_M (8.4) in full generality for some given h just to prove inhomogeneity or check the value of the scalar curvature of manifolds that are in the image of the supergravity r-map and homogeneous. However, before working with this "machinery" of Proposition 8.15, we have calculated the scalar curvature of the r-map image of CCPSR manifolds $\mathcal{H}(h)$ as in (6.3) for all $h \in \mathcal{F}$ (6.1) and for h corresponding to $\mathcal{H}_{2,n}$ using the conventions that were used in Theorem 6.1. Similar calculations should be possible for $h \in \mathcal{G}$ (6.2). These calculations could be of interest in theoretical physics as indicated in [MMT] since the manifolds $\mathcal{H}(h)$ can be interpreted as a deformation of $\mathcal{H}_{2,n}$ which has a reducible prepotential h, and hence we will present them here. Recall that CCPSR manifolds of dimension $n \geq 3$ with reducible prepotential have been classified in [CDJL], see Theorem 2.46.

Lemma 8.17 ($S_{\mathrm{r}(\mathcal{H}(h))}$, $h \in \mathcal{F}$, alternative form)**.** *Let*

$$h \in \mathcal{F} = \left\{ h = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} b_i z_i^2 \;\middle|\; 1 = b_1 \geq \ldots \geq b_{n-1} \geq 0 \right\}$$

as in Theorem 6.1 equation (6.1) and let $\mathcal{H}(h)$ be the corresponding CCPSR manifold of dimension $n \geq 3$, cf. Theorem 6.1 equation (6.3). Let[25] $g = -\frac{1}{2}\partial^2 h$ *and let c_{ij}, $1 \leq i, j \leq n+1$, denote the entries of the cofactor matrix of g, where the index n corresponds to the coordinate w and the index $n+1$ corresponds to the coordinate x. Then the scalar curvature of the supergravity r-map image of $\mathcal{H}(h)$, that is* $\mathrm{r}(\mathcal{H}(h))$, *is given by*

$$\begin{aligned} S_{\mathrm{r}(\mathcal{H}(h))} &= -4n^2 - 6n - 2 \\ &\quad + \frac{2h}{(\det g)^3} \Bigg(\Bigg(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} 3c_{ij}\left(b_i b_j (2c_{ni}c_{nj} + c_{nn}c_{ij}) + b_i(4c_{n+1i}c_{nj} + 2c_{n+1n}c_{ij})\right. \\ &\quad + \left.(2c_{n+1i}c_{n+1j} + c_{n+1n+1}c_{ij})\right) \Bigg) \\ &\quad + \left(\sum_{i=1}^{n-1} 3c_{ni}((b_i(-4c_{nn}c_{n+1i} - 2c_{n+1n}c_{ni}) + (-4c_{n+1n}c_{n+1i} - 2c_{n+1n+1}c_{ni})) \right) \\ &\quad + 3c_{nn}\left(2{c_{n+1n}}^2 + c_{n+1n+1}c_{nn}\right) \Bigg). \end{aligned} \tag{8.15}$$

The values of c_{ij} are given by

$$\begin{aligned} c_{n+1i} &= \left(\sum_{j=1}^{n-1} (b_i - b_j) b_j z_j^2 z_i \prod_{\substack{k\neq i \\ k\neq j}} (-x - b_k w) \right) + (x - b_i w) z_i \prod_{k \neq i} (-x - b_k w), \\ c_{n+1n} &= \left(\sum_{i=1}^{n-1} b_i z_i^2 \prod_{k\neq i} (-x - b_k w) \right) - w \prod_{k=1}^{n-1} (-x - b_k w), \\ c_{n+1n+1} &= \left(\sum_{i=1}^{n-1} (-b_i^2 z_i^2) \prod_{k\neq i} (-x - b_k w) \right) + x \prod_{k=1}^{n-1} (-x - b_k w), \end{aligned}$$

[25] Note that the prefactor $-\frac{1}{2}$ was chosen so that the calculations contain less symbols.

$$c_{ni} = \left(\sum_{j=1}^{n-1} (b_j - b_i) z_j^2 z_i \prod_{\substack{k\neq i\\ k\neq j}} (-x - b_k w) \right) - w z_i \prod_{k\neq i} (-x - b_k w),$$
$$c_{nn} = \sum_{i=1}^{n-1} (-z_i^2) \prod_{k\neq i} (-x - b_k w),$$

(the above formulas for c_{n+1i} and c_{ni} hold for all $1 \leq i \leq n-1$, respectively) and

$$\begin{aligned} c_{ij} = {} & \delta^i_j \frac{\det\left(-\frac{1}{2}\partial^2 h\right)}{-x - b_j w} \\ & + \left(\sum_{\mu=1}^{n-1} \frac{(b_\mu b_i + b_\mu b_j - b_\mu^2 - b_i b_j) z_\mu^2 z_i z_j}{(-x - b_\mu w)(-x - b_i w)(-x - b_j w)} + \frac{(x - b_i w - b_j w) z_i z_j}{(-x - b_i w)(-x - b_j w)} \right) \prod_{k=1}^{n-1} (-x - b_k w) \end{aligned}$$

for all $1 \leq i, j \leq n-1$.

Proof. Recall formula (8.4) for $S_{\mathrm{r}(\mathcal{H}(h))}$, which reads with $\tau = \deg(h) = 3$

$$\begin{aligned} S_{\mathrm{r}(\mathcal{H}(h))} = 4 \Bigg(& -(n+1)^2 + \frac{2}{3}(n+1) \\ & + \frac{h}{\det(\partial^2 h)} \operatorname{tr}\left(\partial^2 h^{-1} \cdot \partial^2 \left(\det\left(\partial^2 h\right) \right) \right) \\ & - \frac{h}{\det(\partial^2 h)^2} \operatorname{tr}\left(\partial^2 h^{-1} \cdot \partial \left(\det\left(\partial^2 h\right) \right) \otimes \left\langle \partial \left(\det\left(\partial^2 h\right) \right), \cdot \right\rangle \right) \Bigg). \end{aligned}$$

With the terminology $g = -\frac{1}{2}\partial^2 h$, we can rewrite $S_{\mathrm{r}(\mathcal{H}(h))}$ and obtain

$$\begin{aligned} \frac{1}{4} S_{\mathrm{r}(\mathcal{H}(h))} = & - n^2 - \frac{3}{2} n - \frac{1}{2} \\ & - \frac{h}{2\det g} \operatorname{tr}\left(g^{-1} \partial^2 \det g \right) \\ & + \frac{h}{2(\det g)^2} \operatorname{tr}\left(g^{-1} \partial \det g \otimes (\partial \det g)^T \right), \end{aligned}$$

where $g = -\frac{1}{2}\partial^2 h$ and $\partial \det g \otimes (\partial \det g)^T$ denotes the symmetric $(n+1)\times(n+1)$-matrix

$$\left(\frac{\partial \det g}{\partial s} \frac{\partial \det g}{\partial t} \right)_{st}, \quad s, t \in \{z_1, \ldots, z_{n-1}, w, x\}.$$

In order to simplify the above expression for $h : \mathbb{R}^{n+1} \to \mathbb{R}$ of the form

$$h = x(-w^2 + \langle z, z \rangle) + w \sum_{i=1}^{n-1} b_i z_i^2, \quad b_i \geq 0 \ \forall i \in \{1, \ldots, n-1\}$$

with

$$g = -\frac{1}{2}\partial^2 h = \begin{pmatrix} -x - b_1 w & & & -b_1 z_1 & -z_1 \\ & \ddots & & \vdots & \vdots \\ & & -x - b_{n-1} w & -b_{n-1} z_{n-1} & -z_{n-1} \\ -b_1 z_1 & \ldots & -b_{n-1} z_{n-1} & x & w \\ -z_1 & \ldots & -z_{n-1} & w & 0 \end{pmatrix},$$

we will use that $\frac{\partial g}{\partial s}$ is a sparse matrix for all $s \in \{z_1, \dots, z_{n-1}, w, x\}$. Observe that

$$\partial \det g = \det g \operatorname{tr}\left(g^{-1}\partial g\right)$$

and

$$\begin{aligned}
\partial^2 \det g &= \det g \left(\operatorname{tr}\left(g^{-1}\partial g\right)\right)^2 + \det g \operatorname{tr}\left(\partial(g^{-1})\partial g + g^{-1}\partial^2 g\right) \\
&= \det g \left(\left(\operatorname{tr}\left(g^{-1}\partial g\right)\right)^2 - \operatorname{tr}\left(g^{-1}\partial g g^{-1}\partial g\right)\right),
\end{aligned}$$

where we used that $\partial^2 g$ has only zero entries. Hence,

$$\begin{aligned}
\frac{1}{4} S_{\mathrm{r}(\mathcal{H}(h))} &= -n^2 - \frac{3}{2}n - \frac{1}{2} \\
&\quad - \frac{h}{2} \operatorname{tr}\left(g^{-1}\left(\operatorname{tr}\left(g^{-1}\partial g\right)\right)^2 - g^{-1}\operatorname{tr}\left(g^{-1}\partial g g^{-1}\partial g\right)\right) \\
&\quad + \frac{h}{2} \operatorname{tr}\left(g^{-1}\left(\operatorname{tr}\left(g^{-1}\partial g\right)\right)^2\right) \\
&= -n^2 - \frac{3}{2}n - \frac{1}{2} \\
&\quad + \frac{h}{2} \operatorname{tr}\left(g^{-1}\operatorname{tr}\left(g^{-1}\partial g g^{-1}\partial g\right)\right).
\end{aligned}$$

Note that

$$\operatorname{tr}\left(g^{-1}\partial g g^{-1}\partial g\right) = \left(\operatorname{tr}\left(g^{-1}\frac{\partial g}{\partial s} g^{-1}\frac{\partial g}{\partial t}\right)\right)_{st}, \quad s,t \in \{z_1, \dots, z_{n-1}, w, x\},$$

is a symmetric $(n+1)\times(n+1)$-matrix. This follows from the fact that for any two square matrices A and B one has $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. We further obtain

$$\frac{\partial g}{\partial z_\eta} = \begin{pmatrix}
 & & & & & & & 0 & 0 \\
 & & & & & & & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & -b_\eta & -1 \\
 & & & & & & & 0 & 0 \\
 & & & & & & & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
0 & \dots & 0 & -b_\eta & 0 & \dots & 0 & & \\
0 & \dots & 0 & -1 & 0 & \dots & 0 & &
\end{pmatrix}, \quad \frac{\partial g}{\partial w} = \begin{pmatrix}
-b_1 & & & & \\
 & \ddots & & & \\
 & & -b_{n-1} & & \\
 & & & 0 & 1 \\
 & & & 1 & 0
\end{pmatrix}, \quad \frac{\partial g}{\partial x} = \begin{pmatrix}
-1 & & & & \\
 & \ddots & & & \\
 & & -1 & & \\
 & & & 1 & 0 \\
 & & & 0 & 0
\end{pmatrix},$$

where the only non-zero entries in $\frac{\partial g}{\partial z_\eta}$ are contained in the ηth row and ηth column. In order to calculate g^{-1} recall that for any invertible matrix $F = (F_{ij}) \in \mathrm{Mat}(m \times m, \mathbb{R})$, its inverse F^{-1} is given by

$$F^{-1} = \frac{1}{\det F}\begin{pmatrix}
\operatorname{cof}(F,1,1) & \dots & \operatorname{cof}(F,1,m) \\
\vdots & \ddots & \vdots \\
\operatorname{cof}(F,m,1) & \dots & \operatorname{cof}(F,m,m)
\end{pmatrix}^T.$$

Here, $\operatorname{cof}(F,i,j)$ denotes the (i,j)-cofactor of F:

$$\operatorname{cof}(F,i,j) = (-1)^{i+j}\det[F]_{ij},$$

where $[F]_{ij}$ denotes the $(m-1)\times(m-1)$-matrix obtained by deleting the i-th row and the j-th column of F. In our case $\operatorname{cof}(g,i,j) = \operatorname{cof}(g,j,i)$ since g is symmetric. We define

$(\mathrm{cof}(g))_{ij} = c_{ij} := (\mathrm{cof}(g,i,j))$, $1 \leq i,j \leq n+1$, to minimise the necessary symbols in the following calculations. With this notation we have $g = \frac{\mathrm{cof}(g)}{\det g}$ and

$$\begin{aligned}\frac{1}{4}S_{\mathrm{r}(\mathcal{H}(h))} &= -n^2 - \frac{3}{2}n - \frac{1}{2}\\ &\quad + \frac{h}{2(\det g)^3}\,\mathrm{tr}\left(\mathrm{cof}(g)\,\mathrm{tr}\left(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g\right)\right).\end{aligned}$$

We can now calculate $\frac{1}{4}S_{\mathrm{r}(\mathcal{H}(h))}$ in terms of the cofactors of g. We obtain

$$\mathrm{cof}(g)\frac{\partial g}{\partial z_\eta} = \begin{pmatrix} 0 & \dots & 0 & -b_\eta c_{n1} - c_{n+1\,1} & 0 & \dots & 0 & -b_\eta c_{\eta 1} & -c_{\eta 1}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & \dots & 0 & -b_\eta c_{nn+1} - c_{n+1n+1} & 0 & \dots & 0 & -b_\eta c_{\eta n+1} & -c_{\eta n+1}\end{pmatrix},$$

where the first non-trivial column is the η-th column,

$$\mathrm{cof}(g)\frac{\partial g}{\partial w} = \begin{pmatrix} -b_1 c_{11} & \dots & -b_{n-1}c_{1n-1} & c_{1n+1} & c_{1n}\\ \vdots & & \vdots & \vdots & \vdots\\ -b_1 c_{n+1\,1} & \dots & -b_{n-1}c_{n+1n-1} & c_{n+1n+1} & c_{n+1n}\end{pmatrix},$$

and

$$\mathrm{cof}(g)\frac{\partial g}{\partial x} = \begin{pmatrix} -c_{11} & \dots & -c_{1n-1} & c_{1n} & 0\\ \vdots & & \vdots & \vdots & \vdots\\ -c_{n+1\,1} & \dots & -c_{n+1n-1} & c_{n+1n} & 0\end{pmatrix}.$$

The calculation of $\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g)$ requires only the diagonal values of the matrix

$$\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g,$$

which are given by

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_j}\right)_{\nu\nu} =$$

$$\text{for } \nu = j: \begin{cases} & b_i b_j (c_{ni}c_{nj} + c_{nn}c_{ij})\\ + & b_i(c_{n+1i}c_{nj} + c_{n+1n}c_{ij}) + b_j(c_{ni}c_{n+1j} + c_{n+1n}c_{ij})\\ + & (c_{n+1i}c_{n+1j} + c_{n+1n+1}c_{ij}),\end{cases}$$

$$\text{for } \nu = n: \begin{cases} & b_i b_j (c_{ni}c_{nj} + c_{nn}c_{ij})\\ + & b_j(c_{n+1n}c_{ij} + c_{ni}c_{n+1j}),\end{cases}$$

$$\text{for } \nu = n+1: \begin{cases} & b_i(c_{n+1i}c_{nj} + c_{n+1n}c_{ij})\\ + & (c_{n+1i}c_{n+1j} + c_{n+1n+1}c_{ij}),\end{cases}$$

$$\text{for } \nu \notin \{j, n, n+1\}: \ 0.$$

Thus we obtain

$$\begin{aligned}(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{ij} &= \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_j}\right)\\ &= \begin{cases} & b_i b_j(2c_{ni}c_{nj} + 2c_{nn}c_{ij})\\ + & b_i(2c_{n+1i}c_{nj} + 2c_{n+1n}c_{ij})\\ + & b_j(2c_{ni}c_{n+1j} + 2c_{n+1n}c_{ij})\\ + & (2c_{n+1i}c_{n+1j} + 2c_{n+1n+1}c_{ij}).\end{cases}\end{aligned}$$

Similar calculations show

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\right)_{\nu\nu} =$$

$$\text{for } \nu = i: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1}(b_i b_\mu(c_{n\mu}c_{i\mu}) + b_\mu(c_{n+1\mu}c_{i\mu}))\right) \\ + & b_i(-c_{nn}c_{n+1i} - c_{n+1n}c_{ni}) \\ + & (-c_{n+1n+1}c_{ni} - c_{n+1n}c_{n+1i}), \end{cases}$$

$$\text{for } \nu = n: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_i b_\mu(c_{n\mu}c_{i\mu})\right) \\ + & b_i(-c_{n+1n}c_{ni} - c_{nn}c_{n+1i}), \end{cases}$$

$$\text{for } \nu = n+1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\mu(c_{n+1\mu}c_{i\mu})\right) \\ + & (-c_{n+1n}c_{n+1i} - c_{n+1n+1}c_{ni}), \end{cases}$$

$$\text{for } \nu \notin \{i, n, n+1\}: \ 0,$$

and

$$\begin{aligned}(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{in} &= \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\right) \\ &= \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1}(b_i b_\mu(2c_{n\mu}c_{i\mu}) + b_\mu(2c_{n+1\mu}c_{i\mu}))\right) \\ + & b_i(-2c_{n+1n}c_{ni} - 2c_{nn}c_{n+1i}) \\ + & (-2c_{n+1n}c_{n+1i} - 2c_{n+1n+1}c_{ni}). \end{cases}\end{aligned}$$

We continue and obtain

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial x}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\right)_{\nu\nu} =$$

$$\text{for } \nu = i: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1}(b_i(c_{n\mu}c_{i\mu}) + (c_{n+1\mu}c_{i\mu}))\right) \\ + & b_i(-c_{nn}c_{ni}) \\ + & (-c_{n+1n}c_{ni}), \end{cases}$$

$$\text{for } \nu = n: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_i(c_{n\mu}c_{i\mu})\right) \\ + & b_i(-c_{nn}c_{ni}), \end{cases}$$

$$\text{for } \nu = n+1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} c_{n+1\mu}c_{i\mu}\right) \\ + & (-c_{n+1n}c_{ni}), \end{cases}$$

$$\text{for } \nu \notin \{i, n, n+1\}: \ 0,$$

and

$$(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{in+1} = \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial x}\,\mathrm{cof}(g)\frac{\partial g}{\partial z_i}\right)$$

$$= \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1}(b_i(2c_{n\mu}c_{i\mu}) + (2c_{n+1\mu}c_{i\mu}))\right) \\ + & b_i(-2c_{nn}c_{ni}) \\ + & (-2c_{n+1n}c_{ni}). \end{cases}$$

We further calculate

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial w}\right)_{\nu\nu} =$$

$$\text{for } 1 \leq \nu \leq n-1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\nu b_\mu ({c_{\nu\mu}}^2)\right) \\ + & b_\nu(-2c_{n+1\nu}c_{n\nu}), \end{cases}$$

$$\text{for } \nu = n: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\mu(-c_{n+1\mu}c_{n\mu})\right) \\ + & (c_{n+1n+1}c_{nn} + {c_{n+1n}}^2), \end{cases}$$

$$\text{for } \nu = n+1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\mu(-c_{n+1\mu}c_{n\mu})\right) \\ + & (c_{n+1n+1}c_{nn} + {c_{n+1n}}^2), \end{cases}$$

and

$$(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{nn} = \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial w}\right)$$

$$= \begin{cases} & \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} b_\nu b_\mu ({c_{\nu\mu}}^2)\right) \\ + & \left(\sum\limits_{\mu=1}^{n-1} b_\mu(-4c_{n+1\mu}c_{n\mu})\right) \\ + & (2c_{n+1n+1}c_{nn} + 2{c_{n+1n}}^2). \end{cases}$$

Note that in the last equation we once relabelled ν as μ. This swapping of indices will be of importance and used frequently from here on.

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial x}\right)_{\nu\nu} =$$

$$\text{for } 1 \leq \nu \leq n-1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\mu ({c_{\nu\mu}}^2)\right) \\ + & (-2c_{n+1\nu}c_{n\nu}), \end{cases}$$

$$\text{for } \nu = n: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} b_\mu(-{c_{n\mu}}^2)\right) \\ + & (2c_{n+1n}c_{nn}), \end{cases}$$

$$\text{for } \nu = n+1: \; 0,$$

and

$$(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{nn+1} = \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial w}\,\mathrm{cof}(g)\frac{\partial g}{\partial x}\right) = \begin{cases} & \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} b_\mu({c_{\nu\mu}}^2)\right) \\ + & \left(\sum\limits_{\mu=1}^{n-1}(b_\mu(-{c_{n\mu}}^2) + (-2c_{n+1\mu}c_{n\mu}))\right) \\ + & (2c_{n+1n}c_{nn}). \end{cases}$$

Lastly, we compute

$$\left(\mathrm{cof}(g)\frac{\partial g}{\partial x}\,\mathrm{cof}(g)\frac{\partial g}{\partial x}\right)_{\nu\nu} =$$

$$\text{for } 1 \le \nu \le n-1: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1} {c_{\nu\mu}}^2\right) \\ + & (-{c_{n\nu}}^2), \end{cases}$$

$$\text{for } \nu = n: \begin{cases} & \left(\sum\limits_{\mu=1}^{n-1}(-{c_{n\mu}}^2)\right) \\ + & ({c_{nn}}^2), \end{cases}$$

$$\text{for } \nu = n+1: \ 0,$$

and

$$(\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g))_{n+1n+1} = \mathrm{tr}\left(\mathrm{cof}(g)\frac{\partial g}{\partial x}\,\mathrm{cof}(g)\frac{\partial g}{\partial x}\right) = \begin{cases} & \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} {c_{\nu\mu}}^2\right) \\ + & \left(\sum\limits_{\mu=1}^{n-1}(-2{c_{n\mu}}^2)\right) \\ + & ({c_{nn}}^2). \end{cases}.$$

Summarising, we have shown that

$$\mathrm{tr}(\mathrm{cof}(g)\partial g\,\mathrm{cof}(g)\partial g) =$$

$$\left(\begin{array}{c|c|c}
\begin{matrix} b_ib_j(2c_{ni}c_{nj}+2c_{nn}c_{ij}) \\ +b_i(2c_{n+1i}c_{nj}+2c_{n+1n}c_{ij}) \\ +b_j(2c_{ni}c_{n+1j}+2c_{n+1n}c_{ij}) \\ +(2c_{n+1i}c_{n+1j}+2c_{n+1n+1}c_{ij}) \end{matrix} & * & * \\
\hline
\begin{matrix} \left(\sum\limits_{\mu=1}^{n-1}(b_jb_\mu(2c_{n\mu}c_{j\mu})+b_\mu(2c_{n+1\mu}c_{j\mu}))\right) \\ b_j(-2c_{n+1n}c_{nj}-2c_{nn}c_{n+1j}) \\ +(-2c_{n+1n}c_{n+1j}-2c_{n+1n+1}c_{nj}) \end{matrix} & \begin{matrix} \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} b_\nu b_\mu({c_{\nu\mu}}^2)\right) \\ +\left(\sum\limits_{\mu=1}^{n-1} b_\mu(-4c_{n+1\mu}c_{n\mu})\right) \\ +(2c_{n+1n+1}c_{nn}+2{c_{n+1n}}^2) \end{matrix} & * \\
\hline
\begin{matrix} \left(\sum\limits_{\mu=1}^{n-1}(b_j(2c_{n\mu}c_{j\mu})+(2c_{n+1\mu}c_{j\mu}))\right) \\ +b_j(-2c_{nn}c_{nj}) \\ +(-2c_{n+1n}c_{nj}) \end{matrix} & \begin{matrix} \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} b_\mu({c_{\nu\mu}}^2)\right) \\ +\left(\sum\limits_{\mu=1}^{n-1}(b_\mu(-{c_{n\mu}}^2)+(-2c_{n+1\mu}c_{n\mu}))\right) \\ +(2c_{n+1n}c_{nn}) \end{matrix} & \begin{matrix} \left(\sum\limits_{\nu=1}^{n-1}\sum\limits_{\mu=1}^{n-1} {c_{\nu\mu}}^2\right) \\ +\left(\sum\limits_{\mu=1}^{n-1}(-2{c_{n\mu}}^2)\right) \\ +({c_{nn}}^2) \end{matrix}
\end{array}\right).$$

In the matrix above, i denotes the number of the row and j denotes the columns number and $*$ is meant to be replaced accordingly to the matrix' symmetry. We use this result and obtain

$$
\begin{aligned}
\frac{1}{4}S_{\mathrm{r}(\mathcal{H}(h))} &= -n^2 - \frac{3}{2}n - \frac{1}{2} \\
&\quad + \frac{h}{2(\det g)^3}\operatorname{tr}\left(\operatorname{cof}(g)\operatorname{tr}\left(\operatorname{cof}(g)\partial g\operatorname{cof}(g)\partial g\right)\right) \\
&= -n^2 - \frac{3}{2}n - \frac{1}{2} \\
&\quad + \frac{h}{2(\det g)^3}\Bigg(\Bigg(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} 3c_{ij}\Big(b_ib_j(2c_{ni}c_{nj} + c_{nn}c_{ij}) + b_i(4c_{n+1i}c_{nj} + 2c_{n+1n}c_{ij}) \\
&\qquad + (2c_{n+1i}c_{n+1j} + c_{n+1n+1}c_{ij})\Big)\Bigg) \\
&\quad + \Bigg(\sum_{i=1}^{n-1} 3c_{ni}\Big(b_i(-4c_{nn}c_{n+1i} - 2c_{n+1n}c_{ni}) + (-4c_{n+1n}c_{n+1i} - 2c_{n+1n+1}c_{ni})\Big)\Bigg) \\
&\quad + 3c_{nn}\Big(2{c_{n+1n}}^2 + c_{n+1n+1}c_{nn}\Big)\Bigg).
\end{aligned}
$$

It remains to calculate $\det g$ and the cofactor matrix $\operatorname{cof}(g)$. Using the Laplace expansion for $\det g$, we obtain

$$
\begin{aligned}
\det g &= -w\det\overbrace{\begin{pmatrix} -x-b_1w & & & -z_1 \\ & \ddots & & \vdots \\ & & -x-b_{n-1} & -z_{n-1} \\ -b_1z_1 & \dots & -b_{n-1}z_{n-1} & w \end{pmatrix}}^{\bar{N}:=} \\
&\quad + \sum_{i=1}^{n-1}(-1)^{(n+1)+i}(-z_i)\det\overbrace{\begin{pmatrix}
-x-b_1w & & & & & & -b_1z_1 & -z_1 \\
& \ddots & & & & & \vdots & \vdots \\
& & -x-b_{i-1}w & 0 & & & -b_{i-1}z_{i-1} & -z_{i-1} \\
& & 0 & 0 & & & -b_iz_i & -z_i \\
& & 0 & -x-b_{i+1}w & & & -b_{i+1}z_{i+1} & -z_{i+1} \\
& & & & \ddots & & \vdots & \vdots \\
& & & & & -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1} \\
-b_1z_1 & \dots & -b_{i-1}z_{i-1} & -b_{i+1}z_{i+1} & \dots & -b_{n-1}z_{n-1} & x & w
\end{pmatrix}}^{N_i:=}.
\end{aligned}
$$

We will use the Laplace expansion again to calculate $\det N_i$ for $1 \leq i \leq n-1$, and $\det \bar{N}$.

$$
\begin{aligned}
\det N_i &= \Big(\sum_{j=1}^{i-1}(-1)^{n+j}(-b_jz_j) \\
&\cdot\det\begin{pmatrix}
-x-b_1w & & & & & & & & & -b_1z_1 & -z_1 \\
& \ddots & & & & & & & & \vdots & \vdots \\
& & -x-b_{j-1}w & 0 & & & & & & -b_{j-1}z_{j-1} & -z_{j-1} \\
& & 0 & 0 & & & & & & -b_jz_j & -z_j \\
& & 0 & -x-b_{j+1}w & & & & & & -b_{j+1}z_{j+1} & -z_{j+1} \\
& & & & \ddots & & & & & \vdots & \vdots \\
& & & & & -x-b_{i-1}w & 0 & & & -b_{i-1}z_{i-1} & -z_{i-1} \\
& & & & & 0 & 0 & & & -b_iz_i & -z_i \\
& & & & & 0 & -x-b_{i+1}w & & & -b_{i+1}z_{i+1} & -z_{i+1} \\
& & & & & & & \ddots & & \vdots & \vdots \\
& & & & & & & & -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1}
\end{pmatrix}\Big) \\
&+ \Big(\sum_{j=i+1}^{n-1}(-1)^{n+j-1}(-b_jz_j)
\end{aligned}
$$

$$
\cdot \det \begin{pmatrix}
-x-b_1w &&&&&&&&& -b_1z_1 & -z_1 \\
& \ddots &&&&&&&& \vdots & \vdots \\
&& -x-b_{i-1}w & 0 &&&&&& -b_{i-1}z_{i-1} & -z_{i-1} \\
&& 0 & 0 &&&&&& -b_iz_i & -z_i \\
&& 0 & -x-b_{i+1}w &&&&&& -b_{i+1}z_{i+1} & -z_{i+1} \\
&&&& \ddots &&&&& \vdots & \vdots \\
&&&&& -x-b_{j-1}w & 0 &&& -b_{j-1}z_{j-1} & -z_{j-1} \\
&&&&& 0 & 0 &&& -b_jz_j & -z_j \\
&&&&& 0 & -x-b_{j+1}w &&& -b_{j+1}z_{j+1} & -z_{j+1} \\
&&&&&&& \ddots && \vdots & \vdots \\
&&&&&&&& -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1}
\end{pmatrix} \Bigg)
$$

$$
- x \det \begin{pmatrix}
-x-b_1w &&&&&& -z_1 \\
& \ddots &&&&& \vdots \\
&& -x-b_{i-1}w & 0 &&& -z_{i-1} \\
&& 0 & 0 &&& -z_i \\
&& 0 & -x-b_{i+1}w &&& -z_{i+1} \\
&&&& \ddots && \vdots \\
&&&&& -x-b_{n-1}w & -z_{n-1}
\end{pmatrix}
$$

$$
+ w \det \begin{pmatrix}
-x-b_1w &&&&&& -b_1z_1 \\
& \ddots &&&&& \vdots \\
&& -x-b_{i-1}w & 0 &&& -b_{i-1}z_{i-1} \\
&& 0 & 0 &&& -b_iz_i \\
&& 0 & -x-b_{i+1}w &&& -b_{i+1}z_{i+1} \\
&&&& \ddots && \vdots \\
&&&&& -x-b_{n-1}w & -b_{n-1}z_{n-1}
\end{pmatrix}.
$$

Observe that

$$
\det \begin{pmatrix}
-x-b_1w &&&&&&&&& -b_1z_1 & -z_1 \\
& \ddots &&&&&&&& \vdots & \vdots \\
&& -x-b_{j-1}w & 0 &&&&&& -b_{j-1}z_{j-1} & -z_{j-1} \\
&& 0 & 0 &&&&&& -b_jz_j & -z_j \\
&& 0 & -x-b_{j+1}w &&&&&& -b_{j+1}z_{j+1} & -z_{j+1} \\
&&&& \ddots &&&&& \vdots & \vdots \\
&&&&& -x-b_{i-1}w & 0 &&& -b_{i-1}z_{i-1} & -z_{i-1} \\
&&&&& 0 & 0 &&& -b_iz_i & -z_i \\
&&&&& 0 & -x-b_{i+1}w &&& -b_{i+1}z_{i+1} & -z_{i+1} \\
&&&&&&& \ddots && \vdots & \vdots \\
&&&&&&&& -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1}
\end{pmatrix}
$$

$$
= (-1)^{i+j} \det \begin{pmatrix}
-x-b_1w \\
& \ddots \\
&& -x-b_{j-1}w \\
&&& -z_j &&&& -b_jz_j \\
&&&& -x-b_{j+1}w \\
&&&&& \ddots \\
&&&&&& -x-b_{i-1}w \\
&&& -z_i &&&& -b_iz_i \\
&&&&&&&& -x-b_{i+1}w \\
&&&&&&&&& \ddots \\
&&&&&&&&&& -x-b_{n-1}w
\end{pmatrix}
$$

$$
= (-1)^{i+j}(b_i - b_j)z_iz_j \prod_{\substack{k\neq i\\ k\neq j}} (-x - b_kw).
$$

Using this, one finds

$$
\det N_i = (-1)^{n+i} \left(\left(\sum_{j=1}^{n-1} (b_j - b_i) b_j z_j^2 z_i \prod_{\substack{k\neq i\\ k\neq j}} (-x - b_kw) \right) + (-x + b_iw) z_i \prod_{k\neq i} (-x - b_kw) \right).
$$

For the matrix $\bar{N}$ we obtain

$$\det \bar{N} = \left(w \prod_{k=1}^{n-1} (-x - b_k w) \right)$$
$$+ \sum_{i=1}^{n-1} (-1)^{n+i} (-b_i z_i) \det \begin{pmatrix} -x-b_1 w & & & & & & -z_1 \\ & \ddots & & & & & \vdots \\ & & -x-b_{i-1}w & 0 & & & -z_{i-1} \\ & & 0 & 0 & & & -z_i \\ & & 0 & -x-b_{i+1}w & & & -z_{i+1} \\ & & & & \ddots & & \vdots \\ & & & & & -x-b_{n-1}w & -z_{n-1} \end{pmatrix}$$
$$= \left(\sum_{i=1}^{n-1} (-b_i z_i^2) \prod_{k \neq i} (-x - b_k w) \right) + w \prod_{k=1}^{n-1} (-x - b_k w).$$

Hence,

$$\begin{aligned} \det g &= \left(\sum_{i=1}^{n-1} (-1)^{n+i} z_i \det N_i \right) - w \det M \\ &= \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (b_j - b_i) b_j z_j^2 z_i^2 \prod_{\substack{k \neq i \\ k \neq j}} (-x - b_k w) \right) \\ &+ \left(\sum_{i=1}^{n-1} (-x + b_i w) z_i^2 \prod_{k \neq i} (-x - b_k w) \right) \\ &+ \left(\sum_{i=1}^{n-1} w b_i z_i^2 \prod_{k \neq i} (-x - b_k w) \right) - w^2 \prod_{k=1}^{n-1} (-x - b_k w) \\ &= \left(\sum_{i>j} (b_i - b_j)^2 z_i^2 z_j^2 \prod_{\substack{k \neq i \\ k \neq j}} (-x - b_k w) \right) \\ &+ \left(\sum_{i=1}^{n-1} (-x + 2 b_i w) z_i^2 \prod_{k \neq i} (-x - b_k w) \right) - w^2 \prod_{k=1}^{n-1} (-x - b_k w) \\ &= \left(\prod_{k=1}^{n-1} (-x - b_k w) \right) \left(\left(\sum_{i>j} \frac{(b_i - b_j)^2 z_i^2 z_j^2}{(-x - b_i w)(-x - b_j w)} \right) + \sum_{i=1}^{n-1} \frac{(-x + 2 b_i w) z_i^2}{-x - b_i w} - w^2 \right). \end{aligned}$$

For the cofactors of g we obtain

$$\begin{aligned} c_{n+1 i} &= (-1)^{(n+1)+i} \det N_i \\ &= \left(\sum_{j=1}^{n-1} (b_i - b_j) b_j z_j^2 z_i \prod_{\substack{k \neq i \\ k \neq j}} (-x - b_k w) \right) + (x - b_i w) z_i \prod_{k \neq i} (-x - b_k w), \\ c_{n+1 n} &= - \det \widetilde{N} \\ &= \left(\sum_{i=1}^{n-1} b_i z_i^2 \prod_{k \neq i} (-x - b_k w) \right) - w \prod_{k=1}^{n-1} (-x - b_k w), \\ c_{n+1 n+1} &= \det \begin{pmatrix} -x-b_1 w & & & -b_1 z_1 \\ & \ddots & & \vdots \\ & & -x-b_{n-1}w & -b_{n-1} z_{n-1} \\ -b_1 z_1 & \dots & -b_{n-1} z_{n-1} & x \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \left(x \prod_{k=1}^{n-1} (-x - b_k w) \right) \\
&\quad + \sum_{i=1}^{n-1} (-1)^{n+i} (-b_i z_i) \det \begin{pmatrix}
-x-b_1 w & & & & & -b_1 z_1 \\
& \ddots & & & & \vdots \\
& & -x-b_{i-1}w & 0 & & -b_{i-1}z_{i-1} \\
& & 0 & 0 & & -b_i z_i \\
& & 0 & -x-b_{i+1}w & & -b_{i+1}z_{i+1} \\
& & & & \ddots & \vdots \\
& & & & -x-b_{n-1}w & -b_{n-1}z_{n-1}
\end{pmatrix} \\
&= \left(\sum_{i=1}^{n-1} (-b_i^2 z_i^2) \prod_{k \neq i} (-x - b_k w) \right) + x \prod_{k=1}^{n-1} (-x - b_k w).
\end{aligned}
$$

The (n,i)-cofactors require similar calculations as above. For $1 \leq i \leq n-1$ we recall the calculation of $\det N_i$ and obtain

$$
\begin{aligned}
c_{ni} &= (-1)^{n+i} \det \begin{pmatrix}
-x-b_1 w & & & & & & -b_1 z_1 & -z_1 \\
& \ddots & & & & & \vdots & \vdots \\
& & -x-b_{i-1}w & 0 & & & -b_{i-1}z_{i-1} & -z_{i-1} \\
& & 0 & 0 & & & -b_i z_i & -z_i \\
& & 0 & -x-b_{i+1}w & & & -b_{i+1}z_{i+1} & -z_{i+1} \\
& & & & \ddots & & \vdots & \vdots \\
& & & & & -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1} \\
-z_1 & \dots & -z_{i-1} & -z_{i+1} & \dots & -z_{n-1} & w & 0
\end{pmatrix} \\
&= \left(\sum_{j=1}^{n-1} (b_j - b_i) z_j^2 z_i \prod_{\substack{k \neq i \\ k \neq j}} (-x - b_k w) \right) - w z_i \prod_{k \neq i} (-x - b_k w).
\end{aligned}
$$

For $i = n$,

$$
\begin{aligned}
c_{nn} &= \det \begin{pmatrix}
-x-b_1 w & & & -z_1 \\
& \ddots & & \vdots \\
& & -x-b_{n-1}w & -z_{n-1} \\
-z_1 & \dots & -z_{n-1} & 0
\end{pmatrix} \\
&= \sum_{i=1}^{n-1} (-1)^{n+i} (-z_i) \det \begin{pmatrix}
-x-b_1 w & & & & & -z_1 \\
& \ddots & & & & \vdots \\
& & -x-b_{i-1}w & 0 & & -z_{i-1} \\
& & 0 & 0 & & -z_i \\
& & 0 & -x-b_{i+1}w & & -z_{i+1} \\
& & & & \ddots & \vdots \\
& & & & -x-b_{n-1}w & -z_{n-1}
\end{pmatrix} \\
&= \sum_{i=1}^{n-1} (-z_i^2) \prod_{k \neq i} (-x - b_k w).
\end{aligned}
$$

It remains to calculate c_{ij} for $1 \leq i, j \leq n-1$. Do do so, observe that

$$
\begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix} = \begin{pmatrix}
-x-b_1 w & & & -b_1 z_1 & -z_1 \\
& \ddots & & \vdots & \vdots \\
& & -x-b_{n-1}w & -b_{n-1}z_{n-1} & -z_{n-1} \\
-b_1 z_1 & \dots & -b_{n-1}z_{n-1} & x & w \\
-z_1 & \dots & -z_{n-1} & w & 0
\end{pmatrix} \cdot \frac{1}{\det g} \begin{pmatrix}
c_{11} & \dots & c_{1n+1} \\
\vdots & \ddots & \vdots \\
c_{n+1\,1} & \dots & c_{n+1n+1}
\end{pmatrix},
$$

and, hence,

$$
\begin{aligned}
c_{ij} &= \frac{1}{-x - b_j w} \left(\delta^i_j \det \left(-\frac{1}{2} \partial^2 h \right) + b_j z_j \mathrm{cof} \left(-\frac{1}{2} \partial^2 h, n, i \right) + z_j \mathrm{cof} \left(-\frac{1}{2} \partial^2 h, n+1, i \right) \right) \\
&= \delta^i_j \frac{\det \left(-\frac{1}{2} \partial^2 h \right)}{-x - b_j w}
\end{aligned}
$$

$$
+\left(\left(\sum_{\mu=1}^{n-1}\frac{(b_\mu b_i+b_\mu b_j-b_\mu^2-b_ib_j)z_\mu^2 z_iz_j}{(-x-b_\mu w)(-x-b_iw)(-x-b_jw)}\right)+\frac{(x-b_iw-b_jw)z_iz_j}{(-x-b_iw)(-x-b_jw)}\right)
\cdot\prod_{k=1}^{n-1}(-x-b_kw)
$$

for all $1\leq i,j\leq n-1$. □

Remark 8.18 (Limit case $b_1=\ldots=b_{n-1}=0$)**.** In the case $b_1=\ldots=b_{n-1}=0$, that is $h=x(-w^2+\langle z,z\rangle)$, one can check that the steps used to acquire the formula (8.15) for $S_{\mathrm{r}(\mathcal{H}(h))}$ are still valid. Hence, we can use (8.15) to calculate the scalar curvature of $\mathcal{H}_{2,n}$ (Proposition 6.9 equation (6.45)), which is equivalent to $\mathcal{H}(h)$ corresponding to $b_1=\ldots=b_{n-1}=0$. In that case we obtain for the determinant of $g=-\frac{1}{2}\partial^2 h$ and the cofactors c_{ij} of g

$$
\begin{aligned}
\det g &= (-1)^{n-1}x^{n-2}h,\\
c_{n+1i} &= (-1)^n x^{n-1}z_i,\\
c_{n+1n} &= (-1)^n x^{n-1}w,\\
c_{n+1n+1} &= (-1)^{n+1}x^n,\\
c_{ni} &= (-1)^{n-1}x^{n-2}wz_i,\\
c_{nn} &= (-1)^{n-1}x^{n-2}\langle z,z\rangle,\\
c_{ij} &= \delta_i^j(-1)^n x^{n-3}h+(-1)^{n-1}x^{n-2}z_iz_j
\end{aligned}
$$

for all $1\leq i,j\leq n-1$. One can now verify that $S_{\mathrm{r}(\mathcal{H}(h))}$ calculated via (8.15) is constant with value $S_{\mathrm{r}(\mathcal{H}(h))}\equiv -4n^2-8$, which coincides with the result for $S_{\mathrm{r}(\mathcal{H}_{2,n})}$ obtained in Lemma 8.16 as expected.

Another important construction originating in the physics literature [FS] is the supergravity c-map. Since we did not work directly with this construction, we refer the reader for an introduction to Section 3 of [CHM]. From [CHM, Thm. 5] we obtain the following.

Lemma 8.19 (Properties of manifolds in the supergravity q-map)**.** *The composition of the supergravity r-map and c-map maps CCPSR manifolds $\mathcal{H}$ of dimension $\dim(\mathcal{H})=n$ to complete quaternionic Kähler manifolds of real dimension $4n+8$ that have negative scalar curvature.*

Lastly, we will briefly discuss applications and open questions in physics related to our research.

The composition of the supergravity r- and c-maps is called the supegravity q-map. Note also that until now, there is no known generalisation of the supergravity c-map to CCGPSR manifolds with corresponding homogeneity-degree $\tau\geq 4$.

The reasons for mentioning this field of research are the following. Mathematically, we can use Theorem 5.6 and obtain a method of deforming the Kähler manifolds in the image of the supgravity r- and q-map. Furthermore, we now know that these manifolds can be parametrised over a compact convex set as described in Theorem 5.6. In [D], the curvature properties of manifolds in the image of the supergravity c-map (and, in particular, q-map) have been studies. Thus, for future research, it is an interesting question how to use the information we obtained for CCPSR manifolds in Theorem 5.6 in combination with the results of [D], for example to find curvature bounds of manifolds in the image of the supergravity q-map or to study the following question using our results and the results specifically from [D, Ch. 7]. One task is the following.

Open problem 8.20 ($\|R\|^2$ and $d\|R\|^2$ in standard form for QK manifolds in image of q-map)**.** *Find a closed formula for the squared pointwise norm of the curvature tensor R of manifolds in the image of the supergravity q-map using the notation from (3.12). Furthermore, in low dimensions try to solve $d\|R\|^2 = 0$ using computer algebra software to obtain all candidates for homogeneous spaces in the image of the supergravity q-map.*

From a physics standpoint, Theorem 5.6 allows one to deform supergravity theories with scalar-fields defined for all time corresponding to CCPSR manifolds and their images in the supergravity r- and c-map. There has been no research in this direction from our part so far, but trying to interpret the physical implications might be an interesting task for future studies.

9 Outlook

In the last part of this thesis we will discuss some open questions, possible ways to solve them, and general ideas that came up during the its preparation.

One of the driving forces for our studies in this thesis has been the open question of completeness for quartic CCGPSR manifolds, cf. Open problem 7.1. Different ideas and tries to solve this question ultimately let to a better understanding of GPSR manifolds and in particular CCPSR manifolds. If one tries to obtain similar results for quartic CCGPSR manifolds as for CCPSR manifolds as in sections 4 and 5, it will quickly be obvious that quartic CCGPSR manifolds are a lot more complicated to work with. As an example, the proof of Theorem 7.2 where we were able to classify one-dimensional quartic CCGPSR curves needs far more technicalities than an analogous proof for the classification of CCPSR curves, cf. Remark 7.4. However, we expect that the following open questions might be solvable:

- Determine if a statement as in Theorem 5.3 also holds for quartic CCGPSR manifolds, i.e. check if Def. 5.1 (i) always implies Def. 5.1 (ii) for quartic CCGPSR manifolds.

- Classify quartic CCGPSR surfaces up to equivalence.

In the proof of Theorem 5.3 we have used the known classification of CCPSR surfaces [CDL], but one might want to try to solve an analogous result for quartic CCGPSR manifolds without classifying quartic CCGPSR surfaces. The latter is probably even more complicated than the classification of quartic CCGPSR curves in Theorem 7.2, but might not be impossible to manage. If one manages to solve one of the above open questions, one might use them to solve another interesting question.

- Can the set of quartic CCGPSR manifolds with non-regular boundary behaviour (Definition 5.1) be parametrised over a compact subset of $\mathrm{Sym}^3\left(\mathbb{R}^n\right)^* \oplus \mathrm{Sym}^4\left(\mathbb{R}^n\right)^*$ similar to the statement of Proposition 5.8 (cf. Open problem 7.6)?

If the answer to the above question is positive, then one could prove completeness for all quartic CCGPSR manifolds using a method as for the proof of Proposition 5.17. Note this is in particular motivated by the following consequence of Theorem 7.2. A quartic CCGPSR curve $\mathcal{H}_{L,K} \subset \{h_{L,K} = x^4 - x^2y^2 + Lxy^3 + Ky^4 = 1\} \subset \mathbb{R}^2$, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}_{L,K}$, is singular at infinity in the sense of Definition 3.16 if and only if $L = \pm\mathbf{u}(K)$, $K \in \left[-\frac{1}{12}, \frac{1}{4}\right]$, cf. equation (7.18), which shows that the set of singular-at-infinity quartic CCGPSR curves can be parametrised over a compact subset of $\mathbb{R}^2$, namely

$$\left\{ \begin{pmatrix} L \\ K \end{pmatrix} \;\middle|\; L = \pm\mathbf{u}(K),\ K \in \left[-\frac{1}{12}, \frac{1}{4}\right] \right\}.$$

However, the one-parameter family d) of quartic CCGPSR curves in Theorem 7.2 also shows that the set of all quartic CCGPSR curves can not be parametrised over a compact set in that way.

We want to stress that there might exist incomplete quartic CCGPSR manifolds of dimension $n \geq 2$, so one could also try to solve the following problems.

- Find an incomplete quartic CCGPSR manifold. Or, more generally:

- Find an incomplete CCGPSR manifold.

Another open question is the classification of homogeneous quartic CCGPSR manifolds $\mathcal{H} \subset \{h = 1\}$ of dimension $\dim(\mathcal{H}) \geq 2$ with transitive action of the identity component G_0^h of the corresponding automorphism group G^h. One ansatz would be to generalise the proof in [DV] where homogeneous CCPSR manifolds with with transitive action of the identity component of the corresponding automorphism group have been classified. At least for low dimensions, one could use Proposition 3.34 and use a computer algebra system to answer that question. The involved equations (3.31) and (3.32) are quartic equations in the prefactors of the monomials in P_3 and P_4, so they should be solvable with any computer algebra system for low dimensions. One would however still have to check (most likely by hand) which of the obtained solutions are equivalent as quartic CCGPSR manifolds.

Apart from the open questions in the setting of known results, there is another idea for a way to compare CCGPSR manifolds of different homogeneity-degree which came up while working on a way of an alternative proof of the classification of CCPSR curves similar to the proof of Theorem 7.2, cf. Remark 7.4. Recall Example 7.12. There we described how to obtain an n-dimensional quartic CCGPSR manifold $\mathcal{H}^4$ from a given n-dimensional CCPSR manifold $\mathcal{H}^3 \subset \{h = 1\}$ with our usual assumptions h of the form (3.12) and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}^3$, where $\mathcal{H}^4$ was defined to be the connected component of $\{xh = 1\}$ that contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \{xh = 1\}$. Furthermore we have shown that quartic CCGPSR manifolds obtained in this way are complete. This construction has however the flaw that it does not respect equivalence classes, that is equivalent CCPSR manifolds might yield inequivalent quartic CCGPSR manifolds. To see this, consider CCPSR and quartic CCGPSR curves. We can without loss of generality assume that a CCPSR curve $\mathcal{H}^3 = \mathcal{H}^3_L$ is the connected component of the level set $\{h_L = x^3 - xy^2 + Ly^3 = 1\}$ that contains the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \{h_L = 1\} \subset \mathbb{R}^2$ with $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$. Then

$$xh_L = x^4 - x^2y^2 + Lxy^3 =: h_{L,0}, \tag{9.1}$$

where we chose the notation $h_{L,0}$ in accordance with the proof of Theorem 7.2 and we denote the corresponding quartic CCGPSR curve by $\mathcal{H}^4_{L,K}$. We know that two CCPSR curves $\mathcal{H}^3_L$ and $\mathcal{H}^3_{L'}$ are equivalent if either $L, L' \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ or $L, L' \in \left\{-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right\}$. But the corresponding quartic CCGPSR curves $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ are equivalent if and only if $|L| = |L'|$. To see that this is true and in particular that $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ are inequivalent if $|L| \neq |L'|$, recall that for $|L| = |L'| = \frac{2}{3\sqrt{3}}$, $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ are both equivalent to the quartic CCGPSR curve c) in Theorem 7.2. Suppose that there exist $L, L' \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ with $|L| \neq |L'|$, such that $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ are equivalent. We have seen in the proof of Theorem 7.2 that the considered vector field $\mathcal{V} \in \Gamma(T\mathbb{R}^2)$ (7.4) is transversal to the set

$$\left\{ \begin{pmatrix} L \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; L \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right) \right\}$$

at all points. This and the smoothness of $\mathcal{V}$ already show that not every quartic CCGPSR curves $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ with $L, L' \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ can be equivalent. We have also shown that every maximal integral curve of $\mathcal{V}|_{\mathbb{R}^2 \setminus \{\mathcal{V}=0\}}$ starting at a point in the set

$$\left\{ \begin{pmatrix} L \\ K \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -\frac{1}{12} < K < \frac{1}{4},\ |L| < \mathbf{u}(K) \right\},$$

with $\mathbf{u}(K)$ as in (7.18), meets the set

$$\left\{ \begin{pmatrix} 0 \\ K \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -\frac{1}{12} < K < \frac{1}{4} \right\}$$

in precisely one point. But we know that (cf. the one-parameter family of curves d) in Theorem 7.2) two quartic CCGPSR surfaces $\mathcal{H}^4_{0,K} \subset \{h_{0,K} = x^4 - x^2y^2 + Ky^4 = 1\}$ and $\mathcal{H}^4_{0,K} \subset \{h_{0,K'} = x^4 - x^2y^2 + K'y^4 = 1\}$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}^4_{0,K}$ and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}^4_{0,K'}$, are equivalent if and only if $K = K'$. Together with the fact that $\mathcal{V}$ vanishes at no point in the set $\left\{\left(\begin{smallmatrix}L\\K\end{smallmatrix}\right) \in \mathbb{R}^2 \;\middle|\; -\frac{1}{12} < K < \frac{1}{4},\, |L| < \mathbf{u}(K)\right\} \supset \left\{\left(\begin{smallmatrix}L\\0\end{smallmatrix}\right) \in \mathbb{R}^2 \;\middle|\; |L| < \mathbf{u}(0) = \frac{2}{3\sqrt{3}}\right\}$, we deduce with the symmetry $dL\left(\mathcal{V}_{-L,K}\right) = dL\left(\mathcal{V}_{L,K}\right)$ and $dK\left(\mathcal{V}_{-L,K}\right) = -dK\left(\mathcal{V}_{L,K}\right)$ that $\mathcal{H}^4_{L,0}$ and $\mathcal{H}^4_{L',0}$ with $L, L' \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ are equivalent if and only if $|L| = |L'|$ as claimed.

In comparison, consider the following assignment. To a CCPSR curve $\mathcal{H}^3_L$, $|L| \leq \frac{2}{3\sqrt{3}}$, as before we assign the quartic CCGPSR curve

$$\mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}} \subset \left\{h_{\sqrt{2}L,-\frac{1}{12}} = x^4 - x^2y^2 + \sqrt{2}Lxy^3 - \frac{1}{12}y^4 = 1\right\}$$

with $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}$. The vector field $\mathcal{V}$ is tangent to the set

$$\left\{\begin{pmatrix}\sqrt{2}L\\ -\frac{1}{12}\end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |L| < \frac{2}{3\sqrt{3}}\right\}$$

and vanishes at the points $\begin{pmatrix}\pm\frac{2\sqrt{2}}{3\sqrt{3}}\\ -\frac{1}{12}\end{pmatrix} \in \mathbb{R}^2$ which correspond to $L = \pm\frac{2}{3\sqrt{3}}$, respectively. Hence, two such quartic CCGPSR curves $\mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}$ and $\mathcal{H}^4_{\sqrt{2}L',-\frac{1}{12}}$, $L, L' \in \left[\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$ are equivalent if either $|L| = |L'| = \frac{2}{3\sqrt{3}}$ (cf. Thm. 7.2 b)) or if $L, L' \in \left(\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ (cf. Thm. 7.2 c)). These are precisely the conditions for the corresponding CCPSR curves $\mathcal{H}^3_L$ and $\mathcal{H}^3_{L'}$ to be equivalent. Thus, the correspondence

$$\mathcal{H}^3_L \to \mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}, \tag{9.2}$$

or, when considering the corresponding polynomials,

$$h_L \to h_{\sqrt{2}L,-\frac{1}{12}},$$

has the advantage over the previous construction (9.1), that is $\mathcal{H}^3_L \to \mathcal{H}^4_{L,0}$, that it respects equivalence classes in the sense that for all $L, L' \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$, the CCPSR curves $\mathcal{H}^3_L$ and $\mathcal{H}^3_{L'}$ are equivalent if and only if the quartic CCGPSR curves $\mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}$ and $\mathcal{H}^4_{\sqrt{2}L',-\frac{1}{12}}$ are equivalent. Furthermore, note that $\mathcal{H}^3_L$ is singular at infinity if and only if $\mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}$ is singular at infinity in the sense of Definition 3.16.

Recall the definition of $\mathcal{C}_n$ in Proposition 5.8 and note that the assignment $\mathcal{H}^3_L \to \mathcal{H}^4_{\sqrt{2}L,-\frac{1}{12}}$ when considered on the level of $\mathcal{C}_1 = \left\{x^3 - xy^2 + P_3(y) \;\middle|\; \max\limits_{\|z\|=1} P_3(z) \leq \frac{2}{3\sqrt{3}}\right\} \cong \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$ is given by the affine linear map

$$\Xi : L \mapsto \begin{pmatrix}\sqrt{2}L\\ -\frac{1}{12}\end{pmatrix}. \tag{9.3}$$

There are, however, other possible ways to assign to each CCPSR curve a quartic CCGPSR curve, such that equivalence classes and the property of being either singular at infinity or not singular at infinity are conserved. For example, consider with $\mathbf{u}$ and $\mathbf{w}$ as in (7.18) and (7.28), respectively, for a chosen point

$$\begin{pmatrix}L_0\\K_0\end{pmatrix} \in \left\{-\frac{1}{12} < K < \frac{1}{4},\ |L| < \mathbf{u}(K)\right\} \cup \left\{K < -\frac{1}{12},\ |L| < \mathbf{w}(K)\right\}$$

the maximal integral curve $\gamma_{L_0,K_0} : I \to \mathbb{R}^2$, $0 \in I$, of $\mathcal{V}|_{\mathbb{R}^2\setminus\{\mathcal{V}=0\}}$ that fulfils the initial condition $\gamma_{L_0,K_0}(0) = \left(\begin{smallmatrix} L_0 \\ K_0 \end{smallmatrix}\right)$. At this point, we will assume that the following statement is true in general. Independent of the initial values L_0 and K_0 as above with $I = (I_-, I_+)$ (note: $I_- \in \mathbb{R}_{<0} \cup \{-\infty\}$ and $I_+ \in \mathbb{R}_{>0} \cup \{\infty\}$), γ_{L_0,K_0} fulfils

$$\lim_{t\to I_+,\, t<I_+} \gamma_{L_0,K_0}(t) = \begin{pmatrix} -\frac{2\sqrt{2}}{3\sqrt{3}} \\ -\frac{1}{12} \end{pmatrix}$$

and

$$\lim_{t\to I_-,\, t>I_-} \gamma_{L_0,K_0}(t) = \begin{pmatrix} \frac{2\sqrt{2}}{3\sqrt{3}} \\ -\frac{1}{12} \end{pmatrix}.$$

We expect that this holds from checking it specific values with MAPLE. The proof of the latter statement is most likely obtainable with some modifications to the techniques used in the proof of Theorem 7.2. We can now choose any smooth diffeomorphism

$$F : \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right] \to \overline{\gamma_{L_0,K_0}(I)}, \quad F(L) = \begin{pmatrix} F_1(L) \\ F_2(L) \end{pmatrix}$$

and obtain in comparison with (9.3) another way to construct a quartic CCGPSR curve from a CCPSR curve via

$$\mathcal{H}^3_L \to \mathcal{H}^4_{F_1(L),F_2(L)}. \tag{9.4}$$

The above construction (9.4) respects equivalence classes and the (non-)singular-at-infinity property for all $L \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$. For future research of this topic, there are two main goals:

- Find a construction associating to each n-dimensional CCPSR manifold $\mathcal{H}^3$ an n-dimensional quartic CCGPSR manifold $\mathcal{H}^4$,

  $$\mathcal{H}^3 \to \mathcal{H}^4,$$

  such that equivalence classes and the (non-)singular-at-infinity property are preserved.

- More generally, find for all $\tau \geq 3$ a way to map n-dimensional CCGPSR manifolds of homogeneity-degree τ, $\mathcal{H}^\tau$, to n-dimensional CCGPSR manifolds of homogeneity-degree $\tau + 1$, such that equivalence classes and the (non-)singular-at-infinity property are preserved. This would yield a sequence of constructions

  $$\mathcal{H}^3 \to \mathcal{H}^4 \to \mathcal{H}^5 \to \mathcal{H}^6 \to \ldots.$$

To obtain such results, extensive study of the corresponding δP_k's as in Definition 3.27 will probably be necessary. For example for the construction (9.2), one can check that with the affine linear map Ξ (9.3) (cf. Figure 25) and the vector fields $\mathcal{V}$ (7.4) and $\widetilde{\mathcal{V}}$ (7.59) (recall the correspondence of $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ with $\delta P_3(y)$ as in (7.59), respectively $\delta P_3(y)$ and $\delta P_4(y)$ as in (7.2) and (7.3)) we obtain

$$\left(\Xi_* \widetilde{\mathcal{V}}\right)_{\left(\begin{smallmatrix} L \\ -\frac{1}{12} \end{smallmatrix}\right)} = \frac{1}{\sqrt{2}} \left(\frac{9}{2}L^2 - \frac{4}{3}\right) \partial_L = \frac{1}{\sqrt{2}} \mathcal{V}_{\left(\begin{smallmatrix} L \\ -\frac{1}{12} \end{smallmatrix}\right)}.$$

Apart from purely mathematical open questions, one open task is how to interpret and use our results in the theory of supergravity. In particular, we have shown in Proposition 5.8 that we can "parametrise" theories obtained from complete PSR manifolds over a certain

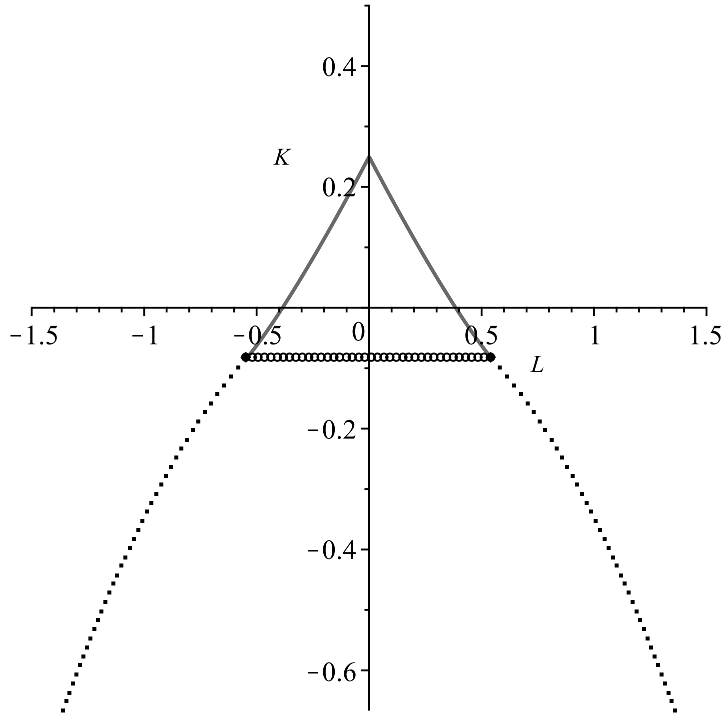


Figure 25: The graph of Ξ marked with a line consisting of small circles.

convex compact set. This might allow to find a physically meaningful way to construct a measure in the space of theories obtained this way. Furthermore, the results for the curvature of CCPSR manifolds, that is Theorem 4.13 and Proposition 5.12, might be interesting for physicists to consider in their studies.

Our more general results for CCGPSR manifolds, in particular the classification of quartic CCGPSR curves in Theorem 7.2, might be useful in scattering theory, cf. the discussion below Theorem 1.18 in [CNS] and also [Me, Ch. 8].

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den **Unterschrift**

Changes in comparison with the submitted version:

page	change
116	in eqn. (6.49) $\mathbb{R} \rightsquigarrow \mathbb{R}_{>0}$, in next eqn. added a missing ") "
92	Prop. 5.8: "Characterisation moduli ..." $\rightsquigarrow$ "Characterisation of the moduli..."
7	"... real curves found in [CDL, Thm. 1] ..." $\rightsquigarrow$ "... real surfaces found in [CDL, Thm. 1] ..."
8	In the description of Thm. 6.1, moved "pairwise inequivalent" to correct position
73	4.19 $\rightsquigarrow$ (4.19) in first line
162	"... independent of the dimension of $\mathcal{H}$, ..." $\rightsquigarrow$ "... irrespective of the dimension of $\mathcal{H}$, ..."
200	in the caption of Figure 25: "image" $\rightsquigarrow$ "graph"
61	after equation at the bottom, added missing end of sentence: "is singular at infinity."
98	$-\frac{\partial^2\beta_{P_3}}{\beta_{P_3}} + \frac{2}{3}\frac{d\beta_{P_3}^2}{\beta_{P_3}^2} \rightsquigarrow -\frac{\partial^2\beta_{P_3}}{3\beta_{P_3}} + \frac{2d\beta_{P_3}^2}{9\beta_{P_3}^2}$, pre-factor $\frac{1}{3}$ was missing
98	"(were we identify ..." $\rightsquigarrow$ "(where we identify ..."
27	$(h \circ A(p))(x, y) \rightsquigarrow (h \circ A(p))\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)$
32	$\begin{pmatrix} 1 \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ y \end{pmatrix} \in \mathbb{R}^{n+1}$ in eqn. (3.13)
35	added missing y^3 in formula of $\widetilde{h}$
63	"... $\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$, $\|\check{z}\| = \frac{\sqrt{3}}{2}$ ..." $\rightsquigarrow$ "... $\check{z} \in \overline{\mathrm{dom}(\mathcal{H})}$ with $\|\check{z}\| = \frac{\sqrt{3}}{2}$ ..."
81	"... is can ..." $\rightsquigarrow$ "... can ..."
80	$\{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \mid z \in \mathrm{dom}(\mathcal{H})\} \rightsquigarrow \{\left(\begin{smallmatrix} 1 \\ z \end{smallmatrix}\right) \in \mathbb{R}^{n+1} \mid z \in \mathrm{dom}(\mathcal{H})\}$
98	"... idea of ..." $\rightsquigarrow$ "... ideas behind ..."
161	footnote: CCGPSR $\rightsquigarrow$ CCPSR
2	$\Gamma(M) \rightsquigarrow \Gamma(TM)$
58	$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H} \rightsquigarrow \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathcal{H}$ in the statement of Lem. 4.2
178	$h_L = x^2 - y^2 + Ly^3 \rightsquigarrow h_L = x^3 - xy^2 + Ly^3$ in the first line of the proof of Lem. 8.11
130	$1/4 \rightsquigarrow \frac{1}{4}$ in eqn. (7.21)
40	removed unnecessary brackets in eqn. (3.34)
160	"Comparison with CCPSR curves classification, method of similar proof for them" $\rightsquigarrow$ "Comparison with CCPSR curves classification" (in Rem. 7.4)
92	"the set of homogeneous cubic polynomials" $\rightsquigarrow$ "the set of hyperbolic homogeneous cubic polynomials" (in Prop. 5.8)
92	"affine $\frac{n^3+3n^2+2n}{6}$-dimensional hyperplane" $\rightsquigarrow$ "affine $\frac{n^3+3n^2+2n}{6}$-dimensional affine subspace" (in Prop. 5.8)
92	"continuous hypersurface $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$" $\rightsquigarrow$ "continuous submanifold of $\mathrm{Sym}^3\left(\mathbb{R}^{n+1}\right)^*$" (in Prop. 5.8)
92	$\mathbb{R}^{\frac{n^3+3n^2+2n}{6}} \rightsquigarrow \mathbb{R}^{\frac{n^3+6n^2+11n+6}{6}}$ (in footnote)
52	$(P_3(y, y, \cdot)dB_0^T\cdot)^T) \rightsquigarrow \left(P_3(y, y, \cdot)dB_0^T\cdot\right)^T$ (line 5)
52	added comma after "equivalently" in line 4
53	added comma after "(see equations (3.53) and (3.54))"
23	"of degree $\tau \geq 30$" $\rightsquigarrow$ "of degree $\tau \geq 3$" (in Lem. 3.4)
39	"in Proposition (3.18)" $\rightsquigarrow$ "in Proposition 3.18"
54	"One for the first variation of the P_i's" $\rightsquigarrow$ "One application of the first variation of the P_k's"