Hyperbolic cubics and the geometry of the Kähler cone of smooth projective toric threefolds

David Lindemann

Aarhus University
Department of Mathematics

8. May 2023

1. **Introduction & motivation**

2. **Hyperbolic cubics & smooth projective toric 3-folds**

3. **Calculation & examples of the volume polynomial**

Main references:

"*Properties of the moduli set of complete connected projective special real manifolds*" (DL, Math. Z. 303(2) (2023)),

"*Torus Actions and Their Applications in Topology and Combinatorics*" (V.M. Buchstaber and T.E. Panov, American Mathematical Soc. (2002)),

"*Toric Varieties*" (D.A. Cox, J.B. Little, and H.K. Schenck, AMS Graduate Studies in Mathematics, Vol. 124 (2011)),

"*tba*" (DL and Andrew Swann, soon)

Hyperbolic polynomials

Definition

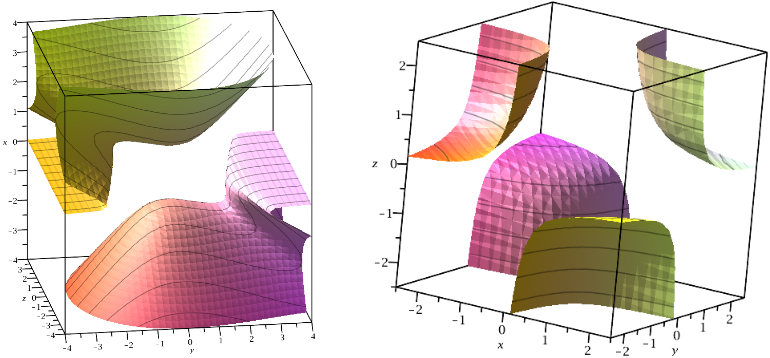
A homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is called **hyperbolic** if $\exists\, p \in \{h > 0\}$, such that $-\partial^2 h_p$ has **Minkowski signature**. Such a point p is called **hyperbolic point** of h.

- two hyperbolic polynomials $h, \widetilde{h}$ **equivalent** $:\Leftrightarrow \exists\, A \in \mathrm{GL}(n+1)$, such that $A^* \widetilde{h} = h$
- there is precisely **one** equivalence class of **quadratic** hyperbolic polynomials in each dimension
- there is **no general classification** for higher degree $\mathbf{deg}(\boldsymbol{h}) \geq \mathbf{3}$
- in the following: $\mathbf{hyp_1}(\boldsymbol{h}) := \{\textbf{hyperbolic points of } \boldsymbol{h}\} \cap \{\boldsymbol{h} = \mathbf{1}\}$

Definition

Open subsets of $\mathrm{hyp}_1(h)$ are called projective special real (**PSR**) manifolds for $\deg(h) = 3$, and generalised PSR (**GPSR**) manifolds for $\deg(h) \geq 4$.

Example: The **level set** $\{h_i = 1\}$, $i \in \{1,2\}$, for $h_1 = x^4 - x^2(y^2+z^2) - \frac{2\sqrt{2}}{3\sqrt{3}}xy^3$ and $h_2 = xyz$



- **note:** $\mathrm{hyp}_1(h_1) \subsetneq \{h_1 = 1\}$, $\mathrm{hyp}_1(h_2) = \{h_2 = 1\}$

Remark

$\mathrm{hyp}_1(h)$ admits a **natural Riemannian metric g** that is given by the restriction of

$$-\partial^2 h$$

to $T\,\mathrm{hyp}_1(h) \times T\,\mathrm{hyp}_1(h)$.

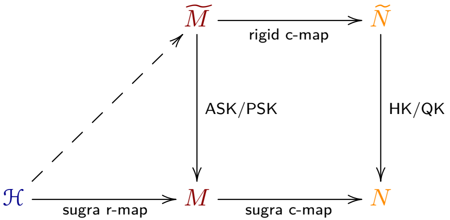
- g is the **centro-affine fundamental form** determined by the centro-affine Gauß equation

$$\mathrm{D}_X Y = \nabla^{\mathrm{ca}}_X Y + g(X,Y)\xi,$$

- $\mathrm{D} =$ **flat connection** on ambient $\mathbb{R}^{n+1}$
- $\nabla^{\mathrm{ca}} =$ induced **centro-affine connection** in $T\,\mathrm{hyp}_1(h)$
- $\xi =$ **position vector field** in $\mathbb{R}^{n+1}$

Explicit constructions of special Kähler and quaternionic Kähler manifolds:

- **supergravity r-map** constructs from given **PSR manifold** $\mathcal{H}$ a **projective special Kähler** (**PSK**) **manifold** $M \cong \mathbb{R}^{n+1} + i\,\mathbb{R}_{>0} \cdot \mathcal{H}$ [DV'92, CHM'12]
- **supergravity c-map** constructs from given **PSK manifold** M a (non-compact) **quaternionic Kähler manifold** $N \cong M \times \mathbb{R}^{2n+5} \times \mathbb{R}_{>0}$ [FS'90]
- above constructions **preserve geodesic completeness**



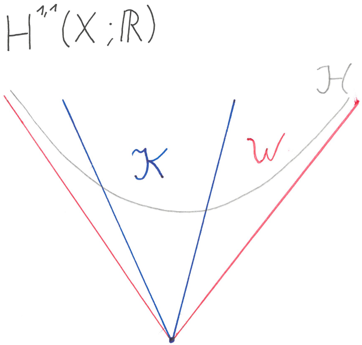
Motivation 2: Kähler geometry

Geometry of Kähler cones [DP'04, W'04, TW'11]**:**

- for X a compact Kähler τ-fold, the homogeneous polynomial

$$h : H^{1,1}(X;\mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_X \omega^\tau,$$

  is **hyperbolic** since every point in the **Kähler cone** $\mathcal{K} \subset H^{1,1}(X;\mathbb{R})$ is hyperbolic by the **Hodge-Riemann bilinear relations**
- $\mathcal{H} := \{h = 1\} \cap \mathcal{K}$ is a **(G)PSR manifold** for $\tau \geq 3$
- in general, $\mathcal{H}$ is **not** a **connected component** of $\mathrm{hyp}_1(h)$



Motivation 3: Combinatorial geometry

Lorentzian polynomials [BH]**:**

- a degree $\tau \geq 2$ homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is **strictly Lorentzian** if
  - **(i)** $-\partial_{i_1} \dots \partial_{i_{n-1}} h$ has **Minkowski signature** $\forall\ i_1, \dots, i_{n-1} \in \{1, \dots, n-1\}$
  - **(ii)** h has only **positive coefficients**
- **Lorentzian polynomials** := limits of Lorentzian polynomials in vector space $\mathrm{Sym}^{\tau}(\mathbb{R}^{n+1})^*$
- Lorentzian polynomials have **applications** in **matroid theory** and in the **geometry of Kähler cones** [BH]

Remark [BH, Thm, 2.16]

Strictly Lorentzian polynomials are **hyperbolic**, i.e. **every point** in $\mathbb{R}^{n+1}_{>0}$ is hyperbolic.

Question 1: Which **hyperbolic/strictly Lorentzian** polynomial can be **realised** as the volume polynomial of some compact Kähler manifold?

Question 2: What does the **geometry** of the volume polynomial, i.e. of the Riemannian manifold $\mathrm{hyp}_1(h)$, tell us about the **underlying** Kähler manifold?

$\rightsquigarrow$ We take the following (hopefully **realistic**) **approach**:

- **Restriction 1:** **cubic** hyperbolic polynomials, respectively compact Kähler **3-folds**
- **Restriction 2:** **smooth projective toric 3-folds** for the considered Kähler manifolds

Why these restrictions?

Cubic hyperbolic polynomials

- **global geometry** a connected component $\mathcal{H}$ of $\mathrm{hyp}_1(h)$ is **complete** w.r.t centro-affine metric g **iff** $\mathcal{H} \subset \mathbb{R}^{n+1}$ is **closed**
- have some **classification results** for corresponding PSR manifolds:
  - (i) **curves** [CHM'12], 3 equivalence classes (2 closed, 1 homogeneous space)
  - (ii) **surfaces** [CDL'14], 7 equivalence classes (5 + 1 one-parameter family closed, 2 homogeneous spaces)
  - (iii) **reducible** h [CDJL'17]
  - (iv) **homogeneous** PSR manifolds [DV'92]

While not completely understood in **general dimension**, the **moduli space** of hyperbolic cubics cubics has the following characterisation:

Theorem [L'19]

Let $y := (y_1, \ldots, y_n)^T$, and let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a hyperbolic cubic. Then

(i) $h \cong x^3 - x(y_1^2 + \ldots + y_n^2) + P_3(y)$, where $P_3 : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 3

(ii) $\mathrm{hyp}_1(h)$ contains a **complete** connected component **iff** $\exists$ choice for P_3, such that $\|P_3\| := \max\limits_{|y|=1} P_3(y) \leq \frac{2}{3\sqrt{3}}$.

- can **roughly** split up study of the **moduli space** of hyperbolic cubics h in **standard form** $x^3 - x\langle y, y\rangle + P_3(y)$ into whether $\|P_3\| \leq \frac{2}{3\sqrt{3}}$, or $\|P_3\| > \frac{2}{3\sqrt{3}}$
- if h is in **standard form**, P_3-term gives **information** about the connected component of $\mathrm{hyp}_1(h)$ that contains $(x, y) = (1, 0)$
- the **standard form** with $\|P_3\| \leq \frac{2}{3\sqrt{3}}$ allows us to describe the **asymptotic geometry** of complete connected components of $\mathrm{hyp}_1(h)$, these are **again** complete PSR manifolds [L'20] and describe the **boundary points** of $\mathrm{GL}(n+1)$-orbits in the **moduli space**.

Smooth projective toric 3-folds

- for our purpose, need the **fan picture** to describe torics
- toric 3-folds X are described by their **moment polytope** M in $\mathbb{R}^3$
- alternatively, describe X by the **fan** $\boldsymbol{\Sigma}$ with cones spanned by the faces/edges/vertices of the **dual polytope** N

Remark [BP]

A toric 3-fold X_Σ corresponding to a finite fan $\boldsymbol{\Sigma}$ in $\mathbb{R}^3$ is **smooth** & **projective** if $\boldsymbol{\Sigma}$ is

(i) **complete**, i.e. the union of the cones in Σ is $\mathbb{R}^3$,

(ii) **simplicial**, i.e. the generators $\eta_1, \ldots, \eta_m$ of the rays in Σ are contained in an **integer lattice**,such that for each 3-d. cone $\mathrm{C}(\eta_i, \eta_j, \eta_k)$ in Σ, we have

$$|\det(\eta_i|\eta_j|\eta_k)| = 1.$$

Calculating the volume polynomial

Question: How do we calculate the **volume polynomial** h of X_Σ from the combinatorial data in $\boldsymbol{\Sigma}$?

Theorem [BP, CLS]

Let X_Σ be a smooth projective toric 3-fold with fan Σ. Let $\eta_1, \ldots, \eta_m$ denote the generators of the **rays** in Σ, and assign a **formal variable** v_i to each η_i. Then there is a **ring isomorphism**

$$H^*(X_\Sigma, \mathbb{Z}) \cong \mathbb{Z}[v_1, \ldots, v_m]/(I_\Sigma + J_\Sigma)$$

where

- the v_i on the right hand side are of **degree two**
- I_Σ is the **Stanley-Reisner ring** (or: face ring) of Σ, i.e.

$$I_\Sigma := (v_{i_1} \ldots v_{i_n} \mid i_j \neq i_k,\ \mathrm{C}(\eta_{i_1}, \ldots, \eta_{i_n}) \notin \Sigma),$$

- J_Σ is the ideal generated by solutions of

$$(\eta_1 | \ldots | \eta_m) \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = 0$$

note:

- $m \geq 4$, otherwise **completeness** cannot be satisfied
- $H^*(X_\Sigma, \mathbb{R}) \cong H^*(X_\Sigma, \mathbb{Z}) \otimes \mathbb{R}$
- $H^2(X_\Sigma, \mathbb{Z}) \cong H^{1,1}(X_\Sigma, \mathbb{Z})$

$\rightsquigarrow$ in the following, will assume **wlog** that $(\eta_{m-2}|\eta_{m-1}|\eta_m) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, can always be obtained via acting with $\mathrm{SL}(2, \mathbb{Z})$

Calculating h_Σ

With our assumptions, $\{[v_1], \ldots, [v_{m-3}]\}$ is a **basis** of $H^{1,1}(X_\Sigma, \mathbb{R})$. Since $H^3(X_\Sigma, \mathbb{R})$ is **1-dimensional**, we have

$$h_\Sigma = h_\Sigma(x_1, \ldots, x_{m-3}) = \left(\sum_{i=1}^{m-3} x_i [v_i] \right)^3 .$$

Example 1

$\mathbb{C}P^3$ is **smooth projective toric**, and one fan $\mathbf{\Sigma}$ is given by

$$\begin{aligned}\mathbf{\Sigma} = &\{\mathrm{C}(e_1, e_2, e_3), \mathrm{C}(e_1, e_2, \eta), \mathrm{C}(e_1, \eta, e_3), \mathrm{C}(\eta, e_2, e_3)\}\\ &\cup \{\mathrm{C}(e_1, e_2), \mathrm{C}(e_1, e_3), \mathrm{C}(e_1, \eta), \mathrm{C}(e_2, e_3), \mathrm{C}(e_2, \eta), \mathrm{C}(e_3, \eta)\}\\ &\cup \{\mathrm{C}(e_1), \mathrm{C}(e_2), \mathrm{C}(e_3), \mathrm{C}(\eta)\},\end{aligned}$$

$\eta = -e_1 - e_2 - e_3$. Then

$$h_\Sigma = x_1^3[v_1^3].$$

$\rightsquigarrow$ as expected, but **boring** (for *our* purposes)

Example 2

$(\mathbb{C}P^1)^3$ is **smooth projective toric**, and one fan $\boldsymbol{\Sigma}$ is determined by its **3-dimensional cones**

$$\text{3-d. cones of } \boldsymbol{\Sigma} = \mathrm{C}(\pm e_1, \pm e_2, \pm e_3).$$

The volume polynomial is given by

$$\begin{aligned} h_\Sigma &= (x_1[v_1] + x_2[v_2] + x_3[v_3])^3 \\ &= 3x_1x_2x_3[v_1v_2v_3]. \end{aligned}$$

$\rightsquigarrow$ to **actually find** the above polynomial, make use of

$$\begin{aligned} I_\Sigma &= (v_1v_4, v_2v_5, v_3v_6), \\ J_\Sigma &= (v_4 - v_1, v_5 - v_2, v_6 - v_3), \end{aligned}$$

$\Rightarrow$ $\boldsymbol{[v_i^2] = [0]}$ for all $1 \leq i \leq 3$

$\rightsquigarrow$ $\mathrm{hyp}_1(h_\Sigma)$ is a **homogeneous surface**, which is **flat** w.r.t. centro-affine fundamental form

Blowup construction on the level of fans

- in order to make use of the **toric minimal model programme** (**tmmp**), we need to understand blowups at a **point** and along **curves**, and "flips" on the level of **fans**
- need to **make sure** to stay in class of **smooth projective toric 3-folds**

Blowup in a point

Blowing up X_Σ in a **point** correspond to

- choosing a 3-d. cone $\mathrm{C}(\eta_i, \eta_j, \eta_k)$ in $\boldsymbol{\Sigma}$
- constructing a **new ray** $\eta_{m+1} = \eta_i + \eta_j + \eta_k$
- building a **new fan** $\boldsymbol{\Sigma}'$ via

$$\text{3-d. cones of } \boldsymbol{\Sigma}' = \text{3-d. cones of } \boldsymbol{\Sigma} \setminus \{\mathrm{C}(\eta_i, \eta_j, \eta_k)\}$$
$$\cup \{\mathrm{C}(\eta_i, \eta_j, \eta_{m+1}), \mathrm{C}(\eta_i, \eta_{m+1}, \eta_k), \mathrm{C}(\eta_{m+1}, \eta_j, \eta_k)\}$$

- this **completely** determines $\boldsymbol{\Sigma}'$
- $\boldsymbol{\Sigma}'$ is **complete & simplicial**, hence $X_{\Sigma'}$ is a **smooth projective toric 3-fold**

$\rightsquigarrow$ the above process is a **certain type** of **star subdivision**

Proposition (DL, AS)

Let $\boldsymbol{\Sigma}$ be a complete simplicial fan. Suppose $\boldsymbol{\Sigma}'$ is obtained via a one-point blowup (in the tmmp). Then

(i) $h_{\Sigma'} \cong h_{\Sigma} + x_{m+1}^3$, (which is ***nice***)

(ii) **every** connected component of $\mathrm{hyp}_1(h_{\Sigma'})$ is **not closed** in $\mathbb{R}^{m-2}$

Proof sketch:

- (i) follows from a **calculation** and uses that $\frac{1}{6}\partial^3 h_{\Sigma}(U,V,W) = [UVW]$ and that $h_{\Sigma'}$ is **hyperbolic**
- the second point (ii) follows from the fact that for all planes $E \subset \mathbb{R}^{m-2}$, such that $E \not\subset \{x_{m+1} = 0\}$, $h_{\Sigma'}|_E$ is **equivalent** to $x^3 + y^3$
- $\mathrm{hyp}_1(x^3 + y^3)$ has two **isometric**, **non-closed** connected components

$\rightsquigarrow$ next, blowing up along a ***curve***

Blowup in along a curve

Blowing up X_Σ along a **curve** correspond to

- choosing two 3-d. cones $\mathrm{C}(\eta_i, \eta_j, \eta_k), \mathrm{C}(\eta_i, \eta_\ell, \eta_k)$ in $\boldsymbol{\Sigma}$, so that $\mathrm{C}(\eta_i, \eta_k) \in \boldsymbol{\Sigma}$, and $\eta_\ell = -\eta_j + A\eta_i + B\eta_k$
- constructing a **new ray** $\eta_{m+1} = \eta_i + \eta_k$
- building a **new fan** $\boldsymbol{\Sigma}'$ via

$$\text{3-d. cones of } \boldsymbol{\Sigma}' = \text{3-d. cones of } \boldsymbol{\Sigma} \setminus \{\mathrm{C}(\eta_i, \eta_j, \eta_k), \mathrm{C}(\eta_i, \eta_\ell, \eta_k)\} \cup \{\mathrm{C}(\eta_i, \eta_j, \eta_{m+1}), \mathrm{C}(\eta_j, \eta_k, \eta_{m+1}), \mathrm{C}(\eta_i, \eta_\ell, \eta_{m+1}), \mathrm{C}(\eta_k, \eta_\ell, \eta_{m+1})\}$$

- this **completely** determines $\boldsymbol{\Sigma}'$
- $\boldsymbol{\Sigma}'$ is **complete & simplicial**, hence $X_{\Sigma'}$ is a **smooth projective toric 3-fold**

$\rightsquigarrow$ the above is **another type** of **star subdivision**

$\rightsquigarrow$ unfortunately, the situation is more complicated when looking at $h_{\Sigma'}$ compared to the one-point blowup:

Proposition (DL,AS)

Let $\mathbf{\Sigma}$ be a complete simplicial fan. Suppose $\mathbf{\Sigma}'$ is obtained via a one-point blowup (in the tmmp). **Wlog** assume that the **new ray** corresponds to the two 3-d. cones

$$\mathrm{C}(e_1, e_2, e_3), \quad \mathrm{C}(e_1, -e_2 + ae_1 + ce_3, e_3), \quad \eta_{m+1} = e_1 + e_3.$$

Let further $\overline{N} = (\eta_1 | \dots | \eta_{m-4})$, $\widetilde{v} = ([v_1], \dots, [v_{m-4}])^{\mathrm{T}})$. Then

$$\begin{aligned} h_{\Sigma'} = h_{\Sigma} + \Bigg(& -\frac{3(a+c+1)}{ac} x_{m-3}^2 x_{m+1} + 3x_{m-3} x_{m+1}^2 + x_{m+1}^3 \Bigg) \\ & \cdot \Bigg(\frac{ac}{a^2 + ac + c^2 + a + c} \big([e_1^*(\overline{N}\widetilde{v}) e_3^*(\overline{N}\widetilde{v}) v_{m-3}] \\ & \quad + [(a e_3^*(\overline{N}\widetilde{v}) + c e_1^*(\overline{N}\widetilde{v})) v_{m-3}^2] \big) \\ & \quad + \frac{a^2 c^2}{a^2 + ac + c^2 + a + c} [v_{m-3}^3] \Bigg). \end{aligned}$$

$\rightsquigarrow$ no **easy to see general conclusion** (for now)

Since there is no nice **general result** yet, we consider two examples:

Blowup of $\mathbb{C}P^3$ along a curve

We have

$$\text{3-d. cones of } \boldsymbol{\Sigma} = \{\mathrm{C}(e_1, e_2, e_3), \mathrm{C}(e_1, e_2, \eta), \mathrm{C}(e_1, \eta, e_3), \mathrm{C}(\eta, e_2, e_3)\},$$

and

$$\text{3-d. cones of } \boldsymbol{\Sigma}' = \text{3-d. cones of } \boldsymbol{\Sigma} \smallsetminus \{\mathrm{C}(e_1, e_2, e_3), \mathrm{C}(e_1, \eta, e_3), \mathrm{C}(e_1, e_3)\}$$
$$\cup \{\mathrm{C}(e_1, e_2, \mu), \mathrm{C}(e_1, \eta, \mu), \mathrm{C}(e_2, e_3, \mu), \mathrm{C}(e_3, \eta, \mu)\}$$

where $\mu = e_1 + e_3$. With

$$I_\Sigma = (v_1 v_2 v_3 v_4), J_\Sigma = (v_2 - v_1, v_3 - v_1, v_4 - v_1),$$
$$I_{\Sigma'} = (v_2 v_4, v_1 v_3 v_5), J_{\Sigma'} = (v_2 - v_1 + v_5, v_3 - v_1, v_4 - v_1 + v_5).$$

we obtain

$$h_{\Sigma'} = h_\Sigma + (-3x_1 x_5^2 - 2x_5^3)[v_1^3].$$

- $h_{\Sigma'} \cong x^3 - xy^2 + \frac{2}{3\sqrt{3}} y^3$
- $\mathrm{hyp}_1(h_{\Sigma'})$ is a **homogeneous space**.

Blowup of $(\mathbb{C}P^1)^3$ along a curve

Modulo **calculations**, we obtain

$$h_\Sigma = x_1 x_2 x_3 [v_1 v_2 v_3],$$
$$h_{\Sigma'} = 3x_2 \left(x_1 x_3 - x_7^2\right) [v_1 v_2 v_3].$$

- $\mathrm{hyp}_1(h_\Sigma)$ has 4 equivalent **connected components** and is a **homogeneous space** (*flat*) [CDL'14]
- $\mathrm{hyp}_1(h'_\Sigma)$ has 2 equivalent **connected components** and, again, is a **homogeneous space** (*constant negative curvature*) [CDJL'17]

What type of result can we expect in general, including flips?

We conjecture that the following holds:

Conjecture

Let h_Σ be the volume cubic of a **smooth projective toric 3-fold**. Then for **any standard form** $x^3 - x\langle y, y\rangle + P_3(y)$ of h_Σ, such that $(x, y) = (1, 0)$ is a Kähler class, P_3 fulfils **either**

$$\|P_3\| = \frac{2}{3\sqrt{3}}, \quad \mathrm{hyp}_1(h_\Sigma) \text{ is a } \textbf{homogeneous space},$$

or

$$\|P_3\| > \frac{2}{3\sqrt{3}}, \quad \text{c.c. of } \mathrm{hyp}_1(h_\Sigma) \text{ containing a Kähler class is } \textbf{incomplete}.$$

$\rightsquigarrow$ if **true**, the above would disqualify *most* smooth projective toric 3-folds as **toy models** for **supergravity**

Thank you for your attention!

- P. Bränén, J. Huh, *Lorentzian polynomials*, Ann. of Math. **192**, 821–891 (2020).
- V.M. Buchstaber and T.E. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, American Mathematical Soc. (2002).
- V. Cortés, M. Dyckmanns, and D. Lindemann, *Classification of complete projective special real surfaces*, Proc. London Math. Soc. **109** (2014), no. 2, 423–445.
- V. Cortés, M. Dyckmanns, M. Jüngling, and D. Lindemann, *A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one* (2017), arxiv:1701.07882.
- V. Cortés, X. Han, and T. Mohaupt, *Completeness in supergravity constructions*, Commun. Math. Phys. **311** (2012), no. 1, 191–213.
- D.A. Cox, J.B. Little, and H.K. Schenck, *Toric Varieties*, AMS Graduate Studies in Mathematics, Vol. **124** (2011).
- V. Cortés, M. Nardmann, and S. Suhr, *Completeness of hyperbolic centroaffine hypersurfaces*, Comm. Anal. Geom., Vol. **24**, no. 1 (2016), 59–92.

- J.-P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Annals of Mathematics **159** (2004), 1247–1274.
- B. de Wit, A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Comm. Math. Phys. **149** (1992), no. 2, 307–333.
- S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. **B332** (1990), no. 2, 317–332.
- D. Lindemann, *Structure of the class of projective special real manifolds and their generalisations* (2018), PhD thesis.
- A.B. Korchagin and D.A. Weinberg, *The isotopy classification of affine quartic curves*, Rocky Mt. J. Math., Vol. **32** (2002), No. 1, 255–347.
- D. Lindemann, *Properties of the moduli set of complete connected projective special real manifolds* (2019), arxiv:1907.06791.
- D. Lindemann, *Limit geometry of complete projective special real manifolds* (2020), arxiv:2009.12956.
- D. Lindemann, *Special geometry of quartic curves*, arxiv:2206.12524.
- D. Lindemann, *Special homogeneous curves*, arxiv:2208.06890.

- D. Lindemann, A.F. Swann, *Special homogeneous surfaces*, preliminary title, work in progress.
- I. Newton, *Curves*, related entry in Lexicon Technicum Vol. **II** by John Harris (1710).
- B. Totaro, *The curvature of a Hessian metric*, Int. J. Math., **15**, 369 (2004).
- T. Trenner, P.M.H. Wilson, *Asymptotic Curvature of Moduli Spaces for Calabi–Yau Threefolds*, J. Geometric Analysis **21** (2011), no. 2, 409–428.
- P.M.H. Wilson, *Sectional curvatures of Kähler moduli*, Math. Ann. **330** (2004) 631–664.
- H. Wu, *The spherical images of convex hypersurfaces*, J. Differential Geometry **9** (1974), 279–290.