The geometry of hyperbolic polynomials

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1. **Introduction & motivation**

2. **Classification results & moduli spaces**

3. **Outlook**

Main references:
"*Properties of the moduli set of complete connected projective special real manifolds*" (DL, 2019), arxiv:1907.06791,
"*Special geometry of quartic curves*" (DL, 2022), arxiv:2206.12524,
"*Special homogeneous curves*" (DL, 2022), arxiv:2208.06890,
"*Special homogeneous surfaces*" (preliminary title, DL & A.S. Swann, 2022)

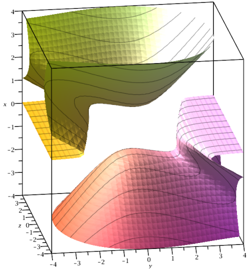
Hyperbolic polynomials

Definition

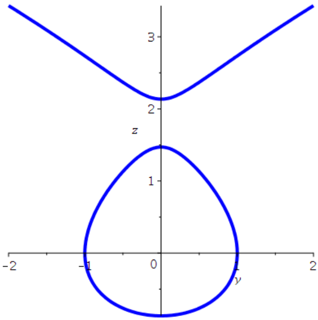
A homogeneous polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is called **hyperbolic** if $\exists\, p \in \{h > 0\}$, such that $-\partial^2 h_p$ has **Minkowski signature**. Such a point p is called **hyperbolic point** of h.

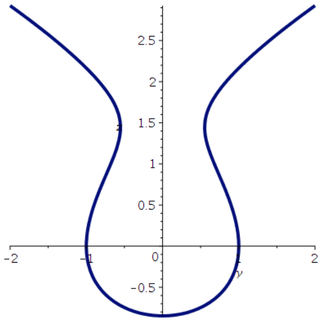
- two hyperbolic polynomials $h, \widetilde{h}$ **equivalent** $:\Leftrightarrow \exists\, A \in \mathrm{GL}(n+1)$, such that $A^* \widetilde{h} = h$
- there is precisely **one** equivalence class of **quadratic** hyperbolic polynomials in each dimension
- there is **no general classification** for higher degree $\mathbf{deg(h) \geq 3}$

Example 1: $h = x^4 - x^2(y^2 + z^2) - \frac{2\sqrt{2}}{3\sqrt{3}} x y^3$, plot of **level set** $\{h = 1\}$



Example 2: Zero set $\{h = 0\}$ of two **Weierstraß cubics** with **positive** and **negative** discriminant





- $\mathrm{hyp}(h)$:= **cone of hyperbolic points** of h

Definition

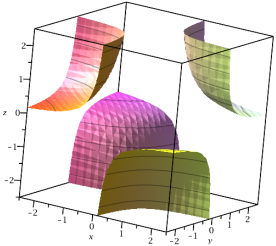
For h a hyperbolic polynomial of degree $\tau \geq 3$, a hypersurface

$$\mathcal{H} \subset \{h = 1\} \cap \mathrm{hyp}(h)$$

is called **projective special real** (**PSR**) manifold for $\tau = 3$, and **generalised PSR** (**GPSR**) manifold for $\tau \geq 4$.

- two (G)PSR mfds. $\mathcal{H}, \widetilde{\mathcal{H}}$ **equivalent** :⇔ $\exists\, A \in \mathrm{GL}(n+1)$, s.t. $A(\mathcal{H}) = \widetilde{\mathcal{H}}$
- $\mathcal{H} \subset \{h = 1\}, \widetilde{\mathcal{H}} \subset \{\widetilde{h} = 1\}$ equivalent ⇒ $h, \widetilde{h}$ equivalent, the **converse** is in general **not true**
- (G)PSR mfds. carry a natural **Riemannian metric** $g = -\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$

Example 3: $h = xyz$, $\{h = 1\}$ is a **homogeneous** & **flat** PSR manifold



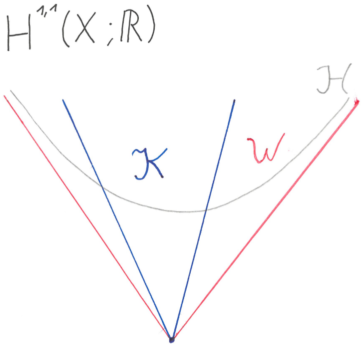
Why study hyperbolic polynomials?

Geometry of Kähler cones [DP'04, W'04, TW'11]:

- for X a compact Kähler τ-fold, the homogeneous polynomial

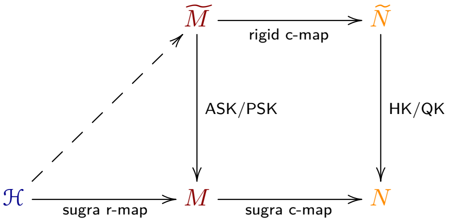
$$h : H^{1,1}(X;\mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_X \omega^\tau,$$

  is **hyperbolic** since every point in the **Kähler cone** $\mathcal{K} \subset H^{1,1}(X;\mathbb{R})$ is hyperbolic by the **Hodge-Riemann bilinear relations**
- $\mathcal{H} := \{h = 1\} \cap \mathcal{K}$ is a **(G)PSR manifold** for $\tau \geq 3$
- in general, $\mathcal{H}$ is not a **connected component** of $\{h = 1\} \cap \mathrm{hyp}(h)$



Explicit constructions of special Kähler and quaternionic Kähler manifolds:

- **supergravity r-map** constructs from given **PSR manifold** $\mathcal{H}$ a **projective special Kähler** (**PSK**) **manifold** $M \cong \mathbb{R}^{n+1} + i\,\mathbb{R}_{>0} \cdot \mathcal{H}$ [DV'92, CHM'12]
- **supergravity c-map** constructs from given **PSK manifold** M a (non-compact) **quaternionic Kähler manifold** $N \cong M \times \mathbb{R}^{2n+5} \times \mathbb{R}_{>0}$ [FS'90]
- above constructions **preserve geodesic completeness**



Why study hyperbolic polynomials?

Real algebraic geometry:

- study of **real polynomials** one of the **defining** problems of **classical** algebraic geometry, study of cubics goes back to Newton [N]
- real polynomials h only classified up to **degree 2**
- **Example:** homogeneous **quadratic** polynomials in n variables $\overset{1:1}{\leftrightarrow}$ bilinear forms on $\mathbb{R}^n$ equivalent to precisely one of

$$x_1^2 + \ldots + x_\ell^2 - x_{\ell+1}^2 - \ldots - x_m^2, \quad 0 \leq \ell \leq m \leq n$$

- even when restricting to **hyperbolic** polynomials **and** restricting **dimension** $\boldsymbol{n}$ or **degree** $\mathbf{deg}(\boldsymbol{h})$, **no general classification** in **almost** all cases

$\rightsquigarrow$ need restrictions based on the **geometry** of associated **(G)PSR manifolds**

Main tasks:

Classifying hyperbolic polynomials

Goals:

- find **canonical representatives** for hyperbolic polynomials under linear coordinate change
- understand the **symmetry groups** of hyperbolic polynomials
- **count** (inequivalent) **c.c.'s** of associated (G)PSR mfds. $\{h = 1\} \cap \mathrm{hyp}(h)$

Moduli spaces & global geometry

Goals:

- understand the **topology** and **local properties** of moduli spaces

$$\mathcal{M}_\tau := \mathrm{Sym}^\tau_{\mathrm{hyp}}(\mathbb{R}^{n+1})^* / \mathrm{GL}(n+1)$$

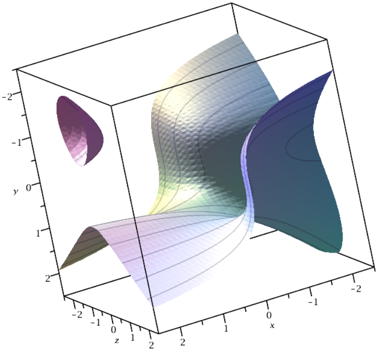
- analyse (local) **differential properties**, e.g. dimension of tangent spaces, to describe **strata** of $\mathcal{M}_\tau$
- study **asymptotic behaviour** of (G)PSR manifolds
- understand **curvature properties**, in particular of **homogeneous (G)PSRs**

Notation: $\tau := \deg(h)$

- set of hyperbolic polynomials is **open** in $\mathrm{Sym}^{\tau}(\mathbb{R}^{n+1})^*$
- $\mathrm{GL}(n+1)$, acting via linear change of coordinates, is **non-compact**
- $\dim(\mathrm{Sym}^{\tau}(\mathbb{R}^{n+1})^*)$ growths with **power τ** in n while $\dim(\mathrm{GL}(n+1))$ growth only **quadratically** in n
- in general **polynomial equivalence** $\not\Rightarrow$ **(G)PSR equivalence**:

Example

$\{h = x(y^2 - z^2) + y^3 = 1\}$ has **four** hyperbolic connected components, **two** of which are equivalent [CDL'14, Thm. 2,5)].



Theorem [CHM'12]

Up to equivalence, there exist **3** hyperbolic cubics in **2 variables**:

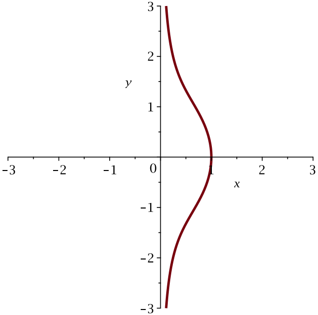
(i) $h = x^2 y$, PSR curve **homogeneous** & **closed**

(ii) $h = x(x^2 - y^2)$, PSR curve **inhomogeneous** & **closed**

(iii) $h = x(x^2 + y^2)$, PSR curve **inhomogeneous** & **not closed**

- in each ob the above cases, $\{h = 1\} \cap \mathrm{hyp}(h)$ has **2 connected components**

Example: $h = x(x^2 + y^2)$, plot of $\{h = 1\}$:



Theorem [CDL'14]

In **3 variables** there are, to equivalence,

- **5** + a **1-parameter family** of hyperbolic cubics with at least one **closed** connected component of $\{h = 1\} \cap \mathrm{hyp}(h)$
- **2** + a **1-parameter family** of hyperbolic cubics with **no closed** connected component of $\{h = 1\} \cap \mathrm{hyp}(h)$

- **two** of the above PSR surfaces are **homogeneous spaces**
- **corresponding cubics:** $h = xyz$ (flat) & $h = x(xy - z^2)$ ($\mathcal{H} \cong$ hyperbolic plane)

Theorem [DV'92]

Homogeneous PSR manifolds and their corresponding cubics have been classified in [DV'92].

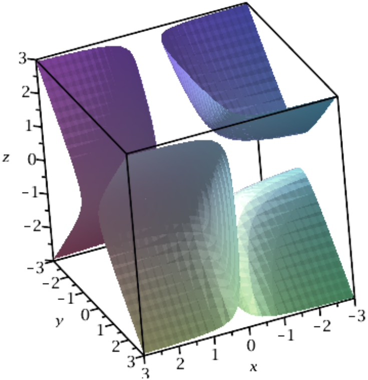
- in [DV'92], the corresponding **homogeneous quaternionic Kähler manifolds** obtained via the **supergravity c∘r=q-map** are also studied

$\rightsquigarrow$ **reducible cubics** can be comparatively easily be controlled, allowing to obtain the following:

Theorem [CDJL'17]

In $n + 1 \geq 3$ **real variables**, there exist up to equivalence **four** reducible hyperbolic cubics that define a **closed** PSR manifold, and **one** reducible hyperbolic cubic that does not.

Example: $h = x(y^2 - z^2)$, plot of $\{h = 1\}$:



Known classification results: $\deg(h) = 4$

$\rightsquigarrow$ for **hyperbolic quartics**, already **considerably fewer** known results!

Theorem [KW]

The **isotopy types** of **all** affine quartic curves $\{h = 0\}$, $h : \mathbb{R}^3 \to \mathbb{R}$, have been classified in [KW].

- **note:** this is unsurprisingly **difficult**!

$\rightsquigarrow$ an **example** of a **quartic GPSR surface** has been studied in [T], motivated by the results of [W'04]

Theorem [L'22 (1)]

Quartic GPSR curves & corresponding quartics have been classified. There are, up to equivalence,

- **3** + **one 1-parameter family** of closed quartic GPSR curves
- **2** + **two 1-parameter families** of non-closed maximal quartic GPSR curves

- **maximal** := coincides with a connected component of $\{h = 1\} \cap \mathrm{hyp}(h)$
- in the above, parameter families defined on an **open interval**
- **that's it for quartics!** (modulo ε)

Known classification results: $\deg(h) \geq 5$ and special cases

- there are to this date **NO** classification results for hyperbolic polynomials of **degree** ≥ 5 in **any number of variables**
- **BUT:** when restricting not only to **curves**, but also requiring **homogeneity** of the (G)PSR mfds., we have:

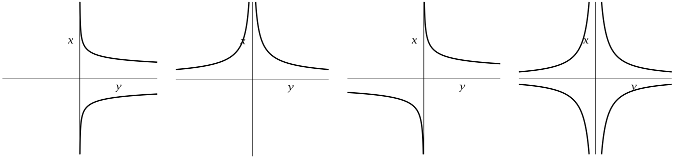
Theorem [L'22 (2)]

Homogeneous (G)PSR curves are classified. For $\deg(h) = \tau$, $\{h = 1\}$ contains such a curve **iff** h is equivalent to

$$h = x^{\tau-k}y^k, \quad k \in \left\{1, \ldots, \left\lfloor \frac{\tau}{2} \right\rfloor \right\}.$$

- in any of the above cases, the **symmetry group** G^h of h is either $\mathbb{R} \times \mathbb{Z}_2$, $\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $(\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_2$

Example: plots for $(\tau = 5, k = 1)$, $(\tau = 5, k = 2)$, $(\tau = 4, k = 1)$, $(\tau = 4, k = 2)$



Known global results:

- **moduli spaces** and **global geometric properties** of (G)PSR manifolds even **less understood**
- **but:** have some nice results for **cubics** by requiring that one of the c.c.'s of $\{h = 1\} \cap \mathrm{hyp}(h)$ is **closed** in the ambient space
- **Note:** geometrically, a PSR manifold **being closed** is equivalent to its **geodesic completeness** w.r.t. the **Riemannian metric** $-\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$ [CNS'16]

$\rightsquigarrow$ we need a **technical result**:

Proposition [L'19]

For **any** hyperbolic polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$, $\deg(h) = \tau$, and **all** $p \in \{h = 1\} \cap \mathrm{hyp}(h)$, $\exists\ A \in \mathrm{GL}(n+1)$, s.t.

(i) $Ap = (1, 0, \ldots, 0)^{\mathrm{T}}$,

(ii) $A^* h = x^\tau - x^{\tau-2}\langle y, y\rangle + \sum_{k=3}^{\tau} x^{\tau-k} P_k(y)$,

$(x, y_1, \ldots, y_n) = (x, y)$ **linear coordinates** on $\mathbb{R}^{n+1}$, $\langle\cdot,\cdot\rangle$ induced **Euclidean scalar product**, P_k's homogeneous polynomials of **degree** k in y.

- the form of h in (ii) is called **standard form**
- **warning:** might not be **ideal** for every problem

Theorem [L'19]

If one of the c.c.'s of $\{h = 1\} \cap \mathrm{hyp}(h)$, h **hyperbolic cubic**, is **closed**, then h has a **representative** in

$$\mathcal{C}_n = \left\{ x^3 - x\langle y, y\rangle + P_3(y) \;\middle|\; \max_{\|y\|=1} |P_3(y)| \le \frac{2}{3\sqrt{3}} \right\}.$$

- the proof of the above theorem relies mainly on **reduction to 2-dim. case** & using available classification

⇝ the moduli space of **closed PSR mfds.**, respectively their defining cubics, is **generated** by the **compact convex set** $\mathcal{C}_n \subset \mathrm{Sym}^3(\mathbb{R}^{n+1})^*$

Corollary [L'18]

For **closed PSR manifolds** there exist **curvature bounds** depending **ONLY** on the dimension n.

$\rightsquigarrow$ in the case of **surfaces**, we know **optimal curvature bounds:**

Proposition [L'18]

The **scalar curvature S** of **PSR surfaces** is contained in $\left[-\frac{9}{4}, 0\right]$. The **two homogeneous** PSR surfaces **maximise**, respectively **minimise**, S.

- **note:** the proof is **explicit** (a.k.a. brute force), difficult to generalise...

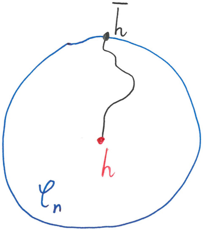
$\rightsquigarrow$ **standard form** well suited to study **asymptotics**:

Theorem [L'20]

Asymptotically, closed PSR manifolds admit an **action** of $\mathbb{R}$ with non-compact orbits.

Explanation:

- "**asymptotically**" means the **geometry** of a PSR manifold contained in a limit $\overline{h}$ of the **standard form** of **initial** h along **lines centrally projected** to $\{h = 1\} \cap \mathrm{hyp}(h)$
- w.r.t. the **generating set**, corresponds to curves in $\mathcal{C}_n$:



$\rightsquigarrow$ surprisingly, have the following result for **limit geometries**:

Proposition [L'20]

If $h \in \mathring{\mathcal{C}}_n$, **any** limit geometry $\overline{h}$ defines a **homogeneous PSR manifold**, and that one is **always the same**.

Example: For $n = 2$, $\overline{h} \cong x(xy - z^2)$.
Question: What about $\deg(h) \geq 4$?

Lemma [L'22 (1)]

- **Closed quartic GPSR curves** are **not** compactly generated.
- But **ALL quartic GPSR curves** have well understood asymptotic behaviour: If it **exists**, the limit polynomial defines a **homogeneous curve**.

That's more or less it for $\deg(h) \geq 4$, though we have **one more result** that holds for **cubics and quartics**:

Definition

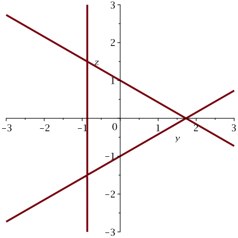
A (G)PSR manifold $\mathcal{H}$ is called **singular at infinity** if $\mathrm{d}h$ **vanishes** along a ray in $\partial(\mathbb{R}_{>0} \cdot \mathcal{H})$.

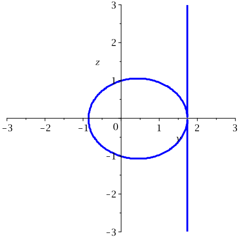
- the above definition is equivalent to a fitting part of $\{h = 0\}$ being **singular** as a real algebraic variety

Theorem [L'19, L'22 (1)]

Homogeneous PSR & **homogeneous quartic GPSR manifolds** are **singular at infinity**.

Example: projective curves of $h \cong xyz$ and $h \cong x(xy - z^2)$ are singular:





Outlook

$\rightsquigarrow$ Which **open problems** are **realistically doable**?

Current project 1 [LS'22]

Classify all **homogeneous GPSR surfaces**.

Advantages:

- can use **homogeneous (G)PSR curves** classification
- necessary **Lie subalgebras** of $\mathrm{GL}(3)$ well understood

Possible problems:

- **strategy** employed for curves not helpful
- have run multiple times into **combinatorial nightmares**, this **WILL** happen again
- **calculation heavy**, leading to potential human error

Verdict: expect a **positive outcome**!

Current project 2

Describe the **asymptotic behaviour** of **maximal non-closed** PSR manifolds.

Advantages:

- expect similar formulas as in the **closed PSR** case
- even for **surfaces** a result could be published

Possible problems:

- already the **closed PSR** case was a extremely **calculation-heavy**, will probably be even worse for **maximal non-closed PSRs**
- cannot really expect **convergence** of standard forms
- $\exists$ explicit example with **no** well-defined asymptotic geometry in our sense, in that case $\overline{\mathrm{hyp}(h)} \cap \{h = 0\}$ contains **only** the origin

Other open questions:

Problem 1

Are **closed** GPSR manifolds **geodesically complete** w.r.t. $-\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$?

- for **PSR manifolds**, three different proofs of the above are known [CNS'16, L'19]
- **none** of these can be generalised to higher degree polynomials
- **reasonable attempt:** quartic GPSR surfaces

Problem 2

Can one find a meaningful **generalisation** of the **supergravity r-map** to quartic GPSR manifolds?

- **probably!**

Problem 3

Relate **asymptotic geometry** of (G)PSR manifolds to limits of the **volume-preserving Kähler-Ricci flow**.

- motivated by the fact that the volume-preserving Kähler-Ricci flow on the level of cohomology is an integral curve in a (G)PSR manifold $\mathcal{H}$, obtained by **projecting** $c_1(X)$**, viewed as constant vector field, centrally to** $\mathcal{H}$

Thank you for your attention!

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