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Floer Theory

Problem Set 2

1. Let $m \geq 1$ be an integer, and consider the function $f_m : \mathbb{C}P^1 \to \mathbb{R}$ given as

$$f_m([z_0 : z_1]) = \frac{|z_0^m + z_1^m|^2}{(|z_0|^2 + |z_1|^2)^m} = \frac{|z^m + 1|^2}{(|z|^2 + 1)^m}$$

in homogeneous coordinates and in the inhomogeneous coordinate $z = \frac{z_1}{z_0}$ on the open subset $U_0 := \{z_0 \neq 0\} \cong \mathbb{C}$, respectively.

a) Find the critical points of f_m and prove that for $m \geq 2$ it is a Morse function. Compute the indices of the critical points. What happens for $m = 1$?

b) Give a qualitative picture of the gradient flow of f_m (for low values of $m \geq 2$) with respect to the usual (Fubini-Study) metric on $\mathbb{C}P^1$ in the open subset U_0.

c) Determine the boundary operator in the Morse complex of f_m and compute the Morse homology.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with a Morse critical point of index $k \in (0, n)$ at $p = 0$, and let g be any metric on $\mathbb{R}^n$. For sufficiently small $r > 0$ the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ will intersect the sphere $S_r(0)$ of radius r around 0 in smoothly embedded spheres S and U of dimensions $n - k - 1$ and $k - 1$, respectively. There is also a subset $T \subset S_r(0)$ where the gradient of f with respect to g is tangent to $S_r(0)$ (for the standard metric, T would be diffeomorphic to $S \times U$).
Find elementary proofs of the following statements:

a) For every $\tau > 0$ there is some $\delta > 0$ such that the flow line of any point $x \in S_r(0)$ of distance less than δ to S stays in the ball $B_r(0)$ for at least time τ.

b) For every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in S_r(0)$ has distance less than δ to S, then the point $y \in S_r(0)$ where the flow line of x exits the ball $B_r(0)$ has distance less than ϵ to U.

Please turn!

3. Let $f : M \to \mathbb{R}$ be a Morse function and g a metric such that (f, g) is a Morse-Smale pair, and so the Morse complex is defined. For $a \in \mathbb{R}$ we set

$$M^{\leq a} := \{x \in M \, : \, f(x) \leq a\}.$$

a) Prove that if the interval $[a, b]$ contains no critical values, then $M^{\leq a}$ and $M^{\leq b}$ are diffeomorphic.

b) Let b be a regular value of f. Observe that the critical points $p \in \mathrm{Crit}(f)$ with $f(p) < b$ form a subcomplex $CM^{\leq b}(f, g)$ of the Morse complex. What does the homology of this subcomplex compute?

c) Now given regular values $a < b$, one can form the quotient complex

$$CM^{[a,b]} := CM^{\leq b}/CM^{\leq a}.$$

What does its homology compute?

Remark: The answers to these questions use more than what we have discussed in class. If you have difficulties getting started, look at simple examples first. Also, once you have tried a few things and have a conjecture for the answers, feel free to look at the literature for clues/proofs.