Again, we conclude

$$HF_*(H,J) \cong \bigoplus_{k \equiv * \bmod 2N} H_k(M;\mathbb{Z}_2).$$

---

3 symplectic Calabi-Yau manifolds

Next we consider the case $\omega|_{\pi_2(M)} \neq 0$, but $c_1|_{\pi_2(M)} = 0$.

This generalizes monotonicity in the sense that now we have to deal with the borderline case $\tau = 0$.

A dimension count similar to the one in the monotone case shows that again no sphere bubbling occurs for moduli spaces of Floer cylinders of index 1 or 2: Now all simple J-holomorphic spheres (not multiply covered) come in moduli spaces of dimension $2n-6$, and so the union of their images forms a subset of M of codimension at least 4. So generically they will not meet the ≤ 3-dimensional image of Floer cylinders of index ≤ 2.
The grading of periodic orbits is well-defined, but now that index and energy are unrelated, there can (and generally will) be Floer cylinders connecting a given pair of orbits $x^\pm$ of index difference 1 in infinitely many homotopy classes.
Our compactness argument still works, so below any given energy bound we have finitely many (up to shift in the s-parameter), but the total count may well be infinite.
To deal with this, we define Floer homology with coefficients in the Novikov ring.
Consider

$$\Gamma := \operatorname{im}\left(\pi_2(M) \longrightarrow H_2(M)/\text{torsion}\right)$$

and define

$$\Lambda = \Lambda_\omega := \Big\{ \lambda = \sum_{A\in\Gamma} \lambda_A e^A \; : \; \lambda_A \in \mathbb{Z}_2 \text{ and}$$
$$\forall c \in \mathbb{R}: \#\{A \in \Gamma : \lambda_A \neq 0,\ \omega(A) \leq c\} \text{ is finite}$$

This is a certain completion of the group ring of Γ (which would correspond to finite sums), called the Novikov ring.
Since the integral of ω is additive on homology, the finiteness condition is preserved under the obvious multiplication

$$\lambda \cdot \mu = \sum_{A,B} \lambda_A \mu_B \, e^{A+B},$$

so Λ is indeed a ring.
Let $\widehat{\mathcal{L}_0 M}$ be the covering space of $\mathcal{L}_0 M$ with group of deck transformations equal to Γ. In other words, we take the quotient of the universal covering $\widetilde{\mathcal{L}_0 M}$ by the kernel of the map $\pi_2(M) \to H_2(M)/\text{torsion}$,

i.e. we identify $(x, [v_1])$ and $(x, [v_2])$ if $v_1 - v_2$ represents a torsion class in homology.

Since ω vanishes on torsion classes, we get a well-defined action functional

$$\mathcal{A}_H : \widehat{\mathcal{L}_0 M} \to \mathbb{R}$$

$$\mathcal{A}_H([x,v]) = \int_{D^2} v^* \omega - \int_0^1 H_t(x(t))\, dt$$

It satisfies

$$\mathcal{A}_H([x, v \# u]) = \mathcal{A}_H([x,v]) + \int_{S^2} u^* \omega$$

whenever $u : S^2 \to M$ represents some element of Γ.
We denote lifts $[x,v]$ of a periodic orbit $x \in \mathcal{P}(H)$ by $\hat{x}$, and define

$$CF_k(H) := \Big\{ \sum_{\substack{\hat{x} \\ |\hat{x}| = k}} \xi_{\hat{x}} e^{\hat{x}} \ : \ \xi_{\hat{x}} \in \mathbb{Z}_2, \ \#\{\hat{x} \in \widehat{\mathcal{P}}(H) : \xi_{\hat{x}} \neq 0, \mathcal{A}_H(\hat{x}) \leq c\} \text{ is finite} \Big\}$$

The Novikov ring Λ_ω acts on $CF_k(H)$ by

$$\lambda \xi = \sum_{A \in \Gamma} \sum_{\hat{x}} \lambda_A \xi_{\hat{x}} \, e^{\hat{x} \# A}$$

Finally, the boundary operator is defined as

$$\partial \hat{x}^+ = \sum_{|x^+| = |x^-|+1} \#_2 \{ u \in \mathcal{M}(x^+, x^-) : \hat{x}^+ = \hat{x}^- \# u \} \cdot \hat{x}^-$$

Then one proves that

$HF_*(M)$ is well defined and isomorphic to $H_*(M; \Lambda_\omega)$.

---

(14) Semi-positive symplectic manifolds

Recall: (M,ω) is called semi-positive if for $A \in \pi_2(M)$
$\omega(A) > 0$ and $c_1(A) \geq 3-n$ implies $c_1(A) \geq 0$.

These conditions are designed so that the dimension count arguments still suffice to exclude sphere bubbling. One needs to work over the Novikov ring and the grading is only well-defined modulo twice the minimal Chern number.

This case is discussed in chapter 12 of the book "J-holomorphic curves and symplectic topology" by D. McDuff and D. Salamon.

What we did not cover:

* most of the details
* action filtrations
  $\rightsquigarrow$ can extract invariants of individual Hamiltonian diffeos, used to study the geometry of $\mathrm{Ham}(M,\omega)$

A glimpse of algebraic structures

We want to discuss the simplest algebraic operation on Floer homology, the pair-of-pants-product.

First some linear algebra preliminaries:

Suppose (s,t) are local complex coordinates on a Riemann surface Σ, i.e. $z = s+it$ is a holomorphic map to $\mathbb{C}$.

A 1-form α can be written in these coordinates as

$$\alpha = a(s,t)\,ds + b(s,t)\,dt$$

If the 1-form takes values in a complex vector space (V,J), we can also write it as a sum of complex linear and antilinear parts

$$\alpha = \alpha^{1,0} + \alpha^{0,1},$$

where
$$\alpha^{1,0} = \tfrac{1}{2}(\alpha - J\circ\alpha\circ j) \quad \text{and}$$
$$\alpha^{0,1} = \tfrac{1}{2}(\alpha + J\circ\alpha\circ j).$$

Explicitly, we have
$$\alpha^{0,1} = \tfrac{1}{2}\left(a\,ds + b\,dt + J(\underset{\substack{\uparrow\\ \text{because } ds\circ j = -dt\\ dt\circ j = ds}}{-a\,dt + b\,ds})\right)$$
$$= \tfrac{1}{2}\left((a+Jb)\,ds - J(a+Jb)\,dt\right)$$

We see that the Cauchy-Riemann equation for a map $u:(\Sigma,j)\to(M,J)$ can be written as

$$\bar\partial_J u := (du)^{0,1} = 0$$

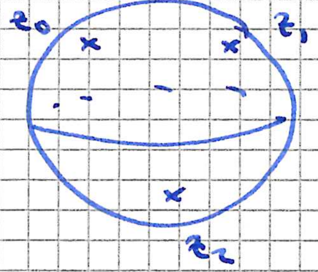
Here we view $\bar\partial_J u$ as a 1-form on Σ with values in the complex vector bundle (u^*TM, J).

<u>Exercise:</u> (a) Verify this by direct computation!

(b) Verify that Floer's equation for a map $u:\mathbb{R}\times S^1\to M$ can similarly be written as

$$\Big(du - \underbrace{dt\otimes X_H\circ u}_{\text{1-form with values in } u^*TM}\Big)^{0,1} = 0.$$

The advantage of this form of the equation is that it can be generalized to situations where we do not have global conformal coordinates.
Specifically, we consider a Riemann surface Σ of genus 0 (i.e. a sphere) with 3 punctures at z_0, z_1 and z_2.



We now fix parametrizations

$$\phi^0 : (-\infty,0)\times S^1 \longrightarrow \Sigma$$
$$(s,t) \longmapsto z_0 + e^{2\pi(s+it)-c_0}$$

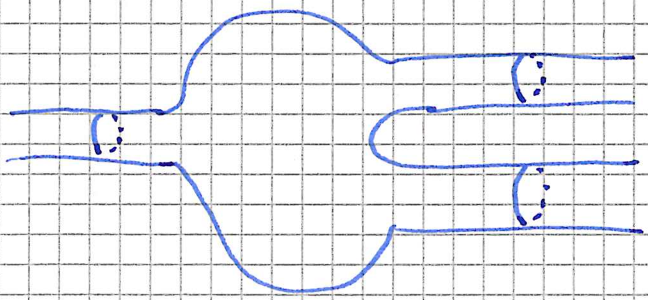
of a punctured nbhd of z_0

as well as

$$\phi^\alpha : (0,\infty)\times S^1 \longrightarrow \Sigma$$
$$(s,t) \longmapsto z_\alpha + e^{-2\pi(s+it)-c_\alpha}$$

of punctured neighborhoods of z_1 and z_2.

<u>Remark:</u> There are many ambiguities in this choice, the main one being a freedom to rotate the complex coordinate near the puncture.
We fix this by requiring that the half-ray $(-\infty,0)\times\{1\}$ should get mapped to a curve asymptotically tangent to the circle determined by z_0, z_1 and z_2, in the direction of z_1, and similar conditions at the other two points.



In these coordinates, the 1-form
$dt \otimes X_H$ on
is well-defined near the punctures for any map $u : \Sigma \setminus \{z_0, z_1, z_2\} \to M$.

Now we pick Hamiltonian functions

$$H^0 : (-\infty,0)\times S^1 \times M \longrightarrow \mathbb{R} \quad \text{with}$$
$$H^0_{s,t} \equiv 0 \quad \text{for } s \geq -\tfrac{1}{2} \quad \text{and}$$
$$H^0_{s,t} = H^0_t \quad \text{for } s \leq -1,$$

and similarly

$$H^{\alpha}_{s,t} : [0,\infty) \times S^1 \times M \to \mathbb{R} \quad \text{for } \alpha \in \{1,2\}$$

with

$$H^{\alpha}_{s,t} \equiv 0 \quad \text{for } s \leq \tfrac{1}{2} \quad \text{and}$$

$$H^{\alpha}_{s,t} = H^{\alpha}_{t} \quad \text{for } s \geq 1.$$

Extending the local 1-forms $dt \otimes X_{H^\alpha}$ by 0 outside the parametrized neighborhoods of the punctures we get a global 1-form β_H on $\cancel{\Sigma \setminus \{z_0, z_1, z_2\}}$ Σ.

We also pick a map $J: \Sigma \to \mathcal{J}(M,\omega)$

compatible a.c. structures

such that in the cylindrical coordinates near the punctures for $|s| \geq 1$ it agrees with fixed t-dependent a.c. structures J^{α}_t.

Now for 1-periodic orbits x^α, $\alpha \in \{0,1,2\}$ of the Hamiltonians H^α we define

$$\mathcal{M}(x^1, x^2; x^0) := \{ u: \Sigma \to M : (du - \beta)^{0,1} = 0, \lim_{|s| \to \infty} u \circ \phi^\alpha(s,t) = x^\alpha(t) \}.$$

Facts: * The expected dimension of $\mathcal{M}(x^1, x^2; x^0)$ is

$$|x^1| + |x^2| - |x^0| - 2n$$

$$(= \mu_{CZ}(x^1) + \mu_{CZ}(x^2) - \mu_{CZ}(x^0) - n)$$

* In nice cases (e.g. when (M,ω) is symplectically aspherical), the 0-dimensional moduli spaces are compact and the compactification of 1-dimensional moduli spaces is obtained by configurations where one Floer cylinder "breaks off" at one of the ends

Therefore, we can define an operation

$$\mu: CF_k(H^1, J^1) \otimes CF_\ell(H^2, J^2) \to CF_{k+\ell-2n}(H^0, J^0)$$

which descends to homology, yielding

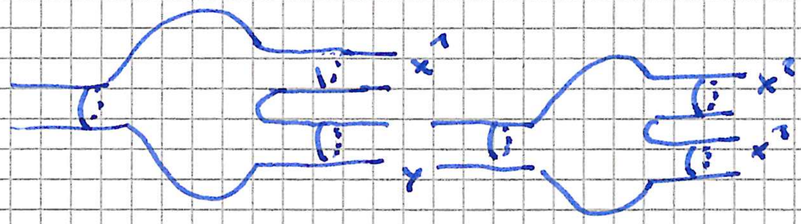
$$\mu : HF_k(H^1, J^1) \otimes HF_l(H^2, J^2) \longrightarrow HF_{k+l-2n}(H^0, J^0)$$

We would like to argue that μ is an associative multiplication, turning HF_* into a ring (with unit).

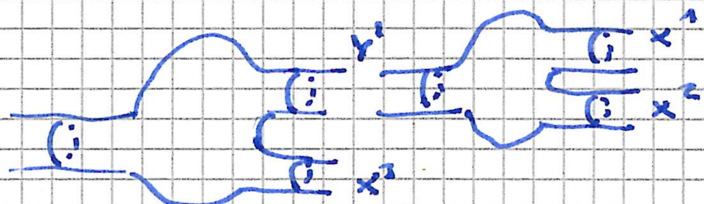
As might be expected, this associativity on homology is proven by constructing a chain homotopy between the two possible triple products

$$\mu(x^1, \mu(x^2, x^3)) \quad \text{and} \quad \mu(\mu(x^1, x^2), x^3).$$

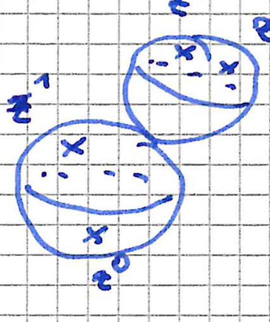
counts configurations looking like



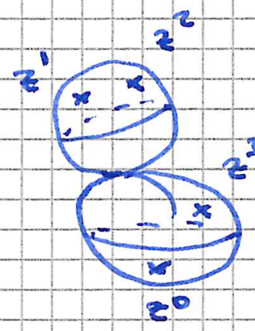
counts configurations looking like



The two corresponding domains



and



are points in the compactification of the space of conformal structures ~~of~~ a sphere with 4 punctures. This compactified space $\overline{\mathcal{M}}_{0,4}$ can be identified with S^2, and we can choose a path in it connecting our two given points.

Along that path of conformal structures on $S^2 \setminus \{z^0, z^1, z^2, z^3\}$ one chooses a path of 1-forms β and a path of a.c.-structures on M which each interpolate between the given data at the end points. Counting solutions to the corresponding Floer-type equation for maps

$$u : S^2 \setminus \{z^0, z^1, z^2, z^3\} \to M$$

of index 0 gives the required chain homotopy.

Rem: A particular choice of path of conformal structures would be obtained by thinking of z^0, z^1 and z^3 as fixed (e.g. by taking the corresponding uniformizing map to a fixed configuration on the sphere) and then "moving z^2 from z^3 to z^1" along the circle determined by z^0, z^1 and z^3.
This circle splits S^2 into two disks, and the cyclic order of z^0, z^1, z^2, z^3 along that circle picks out one of the halves s.t. on the boundary of it they appear in counterclockwise order.
Looking at disks with more marked points on the boundary gives us a second topological model of the A_∞ operad, and one can extend the above construction to get an A_∞ structure on Hamiltonian Floer homology.