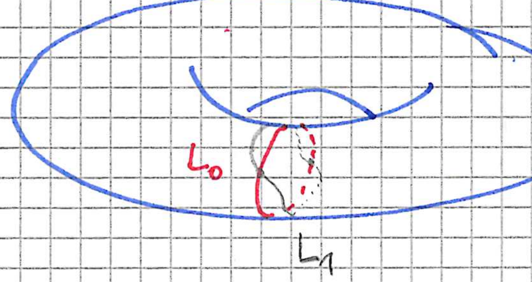
Today I will outline the construction of Lagrangian Floer homology in the setting of Floer's original paper:

Assumptions:
* (M,ω) is a closed symplectic manifold
* $L_0 \subseteq (M,\omega)$ is a closed Lagrangian submanifold with $\omega|_{\pi_2(M,L_0)} = 0$.
* L_1 is the image of L_0 under some Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(M,\omega)$
* L_0 and L_1 intersect transversely

Example: The simplest example of this situation is the following:

$(M,\omega) = (\mathbb{R}^2, \omega_{st}) / \mathbb{Z}^2$ is the standard 2-torus

$L_0 \subseteq M$ is a simple noncontractible closed curve



Pick $x_0 \in L_0$ and consider $\gamma_0(t) := \varphi_t(x_0)$.

Lemma 1: In this situation, the action functional is well-defined on $\mathcal{P}_{\gamma_0}(L_0,L_1)$ itself, i.e. for the ~~each~~ base point $\gamma_0 \in \mathcal{P}(L_0,L_1)$ the value of

$$\mathcal{A}(\gamma) = \iint_{[0,1]^2} h^*\omega$$

is independent of the homotopy h from γ_0 to γ.

Pf: Fix γ_0 and γ and consider two homotopies

$$h, h' : [0,1] \times [0,1] \to M$$

through paths in $\mathcal{P}(L_0,L_1)$.

We then obtain a map

$$g : S^1 \times [0,1] \to M$$

with $g(S^1 \times \{0\}) \subseteq L_0$ and $g(S^1 \times \{1\}) \subseteq L_1$

by viewing S^1 as $[0,1]/_{0\sim 1}$ and setting

$$g(s,t) = \begin{cases} h(2s,t) & \text{for } s \le \frac{1}{2} \\ h'(2-2s,t) & \text{for } s \ge \frac{1}{2}. \end{cases}$$

Note that

$$\int_{S^1\times[0,1]} g^*\omega = \int_{[0,1]^2} h^*\omega - h'^*\omega ,$$

so it suffices to prove that this integral vanishes.

For that, we consider the map

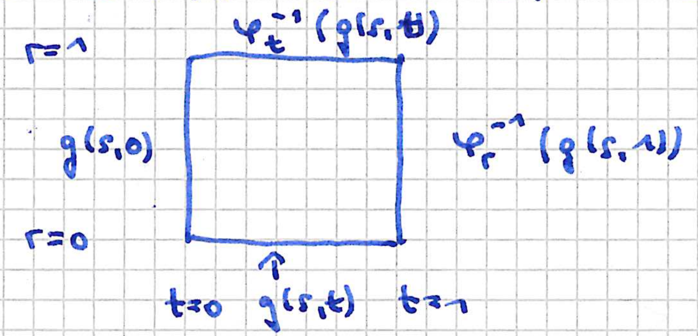
$$G : S^1 \times [0,1] \times [0,1] \longrightarrow M \quad \text{given by}$$

$$G(s,t,r) = \varphi_{tr}^{-1}(g(s,t))$$

By Stokes theorem, we know

$$\int_{\partial(S^1\times[0,1]^2)} G^*\omega = 0 .$$

Now in the (t,r)-coordinates, we know



<u>Claim:</u> If $\gamma : S^1 \to M$ is a closed curve and $\{\psi_r\}_{r\in[0,1]}$ is a path of Hamiltonian diffeomorphisms, then the map

$$\Gamma : S^1 \times [0,1] \to M$$
$$\Gamma(s,r) = \psi_r(\gamma(s))$$

satisfies
$$\int_{S^1\times[0,1]} \Gamma^*\omega = 0 .$$

<u>Pf:</u> If $\{H_r\}_{r\in[0,1]}$ is the family of Hamiltonians generating $\{\psi_r\}$, then

$$\Gamma^*\omega = d(H_r \circ \gamma) \wedge dr$$

and clearly the integral of $d(H_t \circ \gamma)$ over S^1 vanishes.

$\square$

It follows that for $\tilde{g} : S^1 \times [0,1] \to M$ defined as

$$\tilde{g}(s,t) = \varphi_t^{-1}(g(s,t))$$

we have

$$\int_{S^1 \times [0,1]} g^* \omega = \int_{S^1 \times [0,1]} \tilde{g}^* \omega$$

Now $\tilde{g}$ is an annulus in M with <u>both</u> boundary components on L_0. Moreover, since $g(0,t) = \varphi_t(x_0)$, we see that $\tilde{g}(0,t) \equiv x_0$. This allows us to reinterpret $\tilde{g}$ as a disk in M with boundary on L_0, and now the claim of the lemma follows from our assumption $\omega|_{\pi_2(M,L_0)} = 0$.

$\square$

It follows from the lemma that the energy of a holomorphic strip

$$u : \mathbb{R} \times [0,1] \to M$$

with $u(\mathbb{R} \times \{0\}) \subseteq L_0$, $u(\mathbb{R} \times \{1\}) \subseteq L_1$ and

$\lim_{s \to -\infty} u(s,t) = p$ $\quad \lim_{s \to \infty} u(s,t) = q$

is given by

$$E(u) = \mathcal{A}(q) - \mathcal{A}(p),$$

so in particular it is uniformly bounded.

We next consider the moduli spaces

$$\widetilde{\mathcal{M}}(p,q) := \{ u \text{ as above} \}.$$

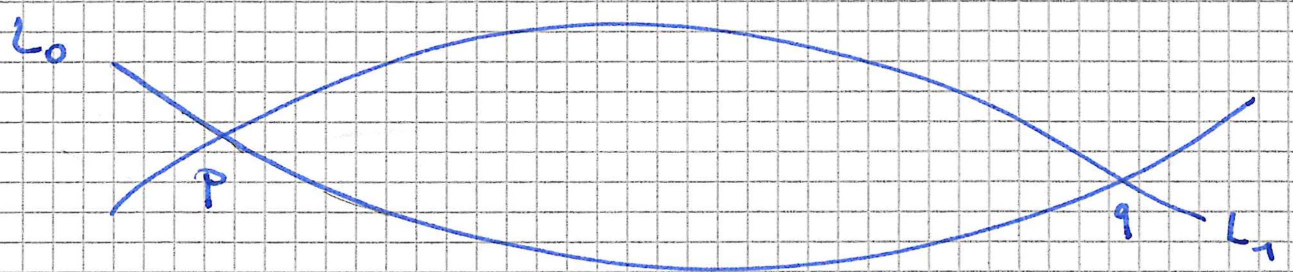
One can prove that this is the zero set of a section of a suitable Banach space bundle over a suitable Banach manifold of maps from the strip to M whose linearization is Fredholm.
The Fredholm index of this section, which is the expected dimension of the moduli space

whenever it is transversally cut out (i.e. the linearization is surjective), can be described in terms of any element $u \in \widetilde{\mathcal{M}}(p,q)$ as follows:

Pick a symplectic trivialization of (u^*TM, ω) over the strip, ~~such that the path $s \mapsto T_{u(s,0)}L_0$ is constant~~ As $\mathbb{R}\times[0,1]$ is contractible, this is unique up to homotopy. Now we get two paths of Lagrangian subspaces connecting T_pL_0 and T_qL_0 (more precisely, their images in $\mathbb{R}^{2n}$) as follows:

$c_1 : [0,1] \to \Lambda(n) = \{$ Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{st})\}$
is obtained by reparametrizing the path of subspaces $T_{u(s,0)}L_0$ for $s \in [-\infty,\infty]$

$c_2 : [0,1] \to \Lambda(n)$ is uniquely specified up to homotopy by the requirement that it be (the reparametrization of) a path of Lagrangian subspaces starting at T_pL_0, ending at T_qL_0, and transverse to $T_{u(s,1)}L_1$ for all $s \in (-\infty,\infty)$



Then we set $\mu(u) := \mu(c_2^{-1} * c_1)$

Example: In the above 2-dimensional picture, we find $\mu(u) = 1$.

Under the standing assumptions of this lecture, Floer proved that this index is actually constant over all of $\widetilde{\mathcal{M}}(p,q)$, and there exists an integer-valued grading of the intersection points such that

$$\operatorname{ind}(D\bar{\partial}(u)) = |q| - |p|$$

Moreover, one has

<u>Prop:</u> In our situation, for a generic family $\{J_t\}_{t\in[0,1]}$ of ω-compatible almost complex structures, all moduli spaces $\widetilde{\mathcal{M}}(p,q)$ are smooth manifolds of the expected dimension.

Next we want to understand the compactness properties of the moduli spaces

$$\mathcal{M}(p,q) = \widetilde{\mathcal{M}}(p,q)/\mathbb{R}$$

where we take the quotient w.r.t. the shift in the s-parameter.

For this, one performs a bubbling argument, and discovers two potential types of bubbles:

1. sphere bubbles, i.e. limiting objects of the rescaling which are J-holomorphic spheres
2. disk bubbles, i.e. limiting objects of the rescaling which are J-holomorphic ~~spheres~~ disks with boundary on either L_0 or L_1

Case 1 would happen if the points where the gradient blows up converge (after possible shift in the s-parameter) to an interior point, or they converge to a boundary point <u>slowly</u> compared to the blow up.

Case 2 would happen if they converge to a boundary point quickly compared to the gradient blow up.

As bubbles are nonconstant by construction and so they have positive symplectic area, they are excluded by our assumption $\omega|_{\pi_2(M, L_0)} = 0$.

The only possible type of noncompactness that remains is strips breaking into several strips with chain-matching asymptotics.

A gluing theorem asserts that in the case that all moduli spaces are transversally cut out, such configurations of strips with matching asymptotics can indeed be glued, and we eventually arrive at the following

Thm: In our situation, for a generic family $\{J_t\}_{t\in[0,1]}$ of t-dependent ω-compatible a.c. structures, the following holds:

ⓐ If $|q|-|p|=1$ then $\mathcal{M}(p,q)$ is a compact 0-dimensional manifold, i.e. a finite set of points.

ⓑ If $|q|-|p|=2$, then $\mathcal{M}(p,q)$ is a 1-manifold which admits a compactification to a manifold with boundary

$$\partial\overline{\mathcal{M}}(p,q) = \bigsqcup_{|p|>|r|>|q|} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$$

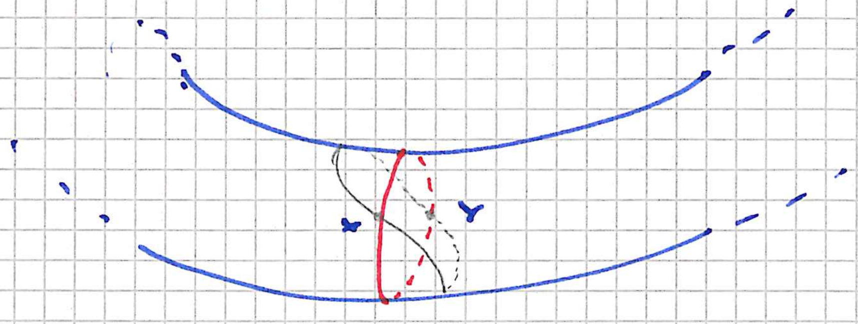
Now we proceed as usual:

Define
$$CF_k(L_0,L_1) := \bigoplus_{|x|=k} \mathbb{Z}_2\cdot x$$

and set
$$\partial y := \sum_{|x|=|y|-1} \#_2\mathcal{M}(x,y)\cdot x$$

The Lagrangian Floer homology of L_0 and L_1 is now defined as
$$HF_*(L_0,L_1;\{J_t\}) := H_*(CF_*(L_0,L_1),\partial).$$

Example : We return to the example at the beginning of this lecture:



We see 2 intersection points x and y, and 2 lunes which can be parametrized as holomorphic strips.
It turns out that one has one intersection point of index 0 and one of index 1,

We get $HF_*(L_0, L_1) \cong H_*(S^1)$.

This is no coincidence. One can set up continuation maps from the holomorphic curve version of the theory to the version with Hamiltonian term.
As the latter can be reinterpreted as a holomorphic curve version as well (for a pair of Lagrangians where one has been moved by a Hamiltonian isotopy), this can be used to prove invariance of $HF_*(L_0, L_1)$ under Hamiltonian isotopy.
To compute the result, one can now use a Hamiltonian image L_1 of L_0 which is C^2-close. Then one argues that all holomorphic strips must stay near L_0 and L_1 and an argument analogous to the Hamiltonian case identifies the Floer complex with the Morse complex of $-H$.
So we obtain:

Thm: In our situation, $HF(L_0, L_1)$ is invariant under Hamiltonian isotopy of the Lagrangians and independent of the choice of a.s. structure up to natural isomorphism.
Moreover, it is isomorphic to $H_*(L_0; \mathbb{Z}_2)$.