Lagrangian Floer homology

Lagrangian Floer homology was built to detect intersections of pairs of Lagrangian submanifolds of a given symplectic manifold in a robust way.

Before we get into the details, we review a few examples of and facts about Lagrangian submanifolds.
Recall that $L \subseteq (M, \omega)$ is called Lagrangian if it is maximally isotropic, i.e. $\omega|_L \equiv 0$ and the dimension of L is the largest possible for this condition:

$$\dim L = \frac{1}{2} \dim M.$$

Examples:

(1) Any embedded curve $C \subseteq (\mathbb{R}^2, \omega_{st})$ is a Lagrangian submanifold

(2) If $L_1 \subseteq (M_1, \omega_1)$ and $L_2 \subseteq (M_2, \omega_2)$ a Lagrangian, then $L_1 \times L_2 \subseteq (M_1 \times M_2, \omega_1 + \omega_2)$ is Lagrangian.
In particular,

$$T^n = (S^1)^n \subseteq (\mathbb{R}^{2n}, \omega_{st})$$

is a Lagrangian submanifold

(3) Suppose (M, ω, J) is a Kähler manifold and

$$\varphi : M \to M$$

is an anti-holomorphic involution, i.e. $\varphi_* \circ J = -J$.
If the fixpoint set of φ is not empty, then it is a Lagrangian submanifold.

As a specific instance of this, we can consider $M = \mathbb{CP}^n$ and φ = complex conjugation.
Then $L = \mathrm{Fix}(\varphi) = \mathbb{RP}^n \subseteq \mathbb{CP}^n$.

(4) Let (M, ω) be symplectic, and consider
$N := (M \times M, \pi_1^* \omega - \pi_2^* \omega)$.
Any diffeomorphism $f : M \to M$ defines a submanifold graph $f \subseteq N$, and this is a

Lagrangian submanifold if and only if f is a symplectomorphism.

5 Let Q be a ~~closed~~ smooth manifold and consider the symplectic manifold $(M, \omega) = (T^*Q, \omega_{can})$.
Any 1-form $\alpha \in \Omega^1(Q)$ gives rise to a submanifold graph $\alpha \in T^*Q$. This submanifold is Lagrangian if and only if $d\alpha = 0$ (as follows from the defining property of the Liouville form λ_{can} with $\omega_{can} = d\lambda_{can}$).

For a submanifold $S \subseteq Q$ we define its conormal bundle as

$$N^*S = \{ \alpha \in T^*Q \ : \pi(\alpha) \in S \text{ and } \alpha|_{T_{\pi(\alpha)}S} = 0 \}$$

If in local coordinates $(q_1, \dots, q_n)$ on Q the submanifold S is given as the set $\{q_1 = q_2 = \dots = q_r = 0\}$, then in these coordinates (or more precisely, the corresponding canonical coordinates) N^*S is given as the set

$$\{ q_1 = \dots = q_r = 0, \ p_{r+1} = p_{r+2} = \dots = p_n = 0 \}.$$

The following theorem of Weinstein asserts that a nbhd of any Lagrangian submanifold $L \subseteq (M, \omega)$ inside a symplectic manifold is completely determined by L:

<u>Thm</u> (Lagrangian nbhd theorem)

Every Lagrangian submanifold $L \subseteq (M, \omega)$ has an open nbhd which is symplectomorphic to an open nbhd of the zero section in T^*L by a map which is "the identity" on L.

The construction of Lagrangian Floer homology was motivated by another one of Arnold's conjectures, which can be stated somewhat informally as

<u>Conjecture</u> (Arnold)

> If $L \subseteq (M,\omega)$ is a closed Lagrangian submanifold, satisfying suitable conditions, then the image of L under any Hamiltonian diffeomorphism $\varphi: M \to M$ intersects L in at least as many points as a smooth function $f: L \to \mathbb{R}$ must have critical points.

<u>Rem:</u> ① Clearly, some additional conditions on the pair (M,L) are necessary, as any closed Lagrangian submanifold $L \subseteq (\mathbb{R}^{2n},\omega_{st})$ can be displaced by a Hamiltonian isotopy (and so the same is true for "sufficiently small" Lagrangian submanifolds contained in a Darboux chart in any closed symplectic manifold (M,ω)).

② One specific instance of the conjecture was formulated by Arnold for the zero section $Q \subseteq T^*Q$, where Q is any closed manifold.

To discuss Floer's original construction, we introduce the following condition:

Let $L \subseteq (M,\omega)$ be a Lagrangian submanifold. We say that $\omega_{|\pi_2(M,L)} = 0$ if for every smooth map

$$u: (D^2, S^1) \to (M,L)$$

we have

$$\int_{D^2} u^*\omega = 0.$$

<u>Exercise:</u> If a Lagrangian submanifold $L \subseteq (M,\omega)$ has this property, then the ambient symplectic manifold satisfies $\omega_{|\pi_2(M)} = 0$.

Examples: ① $Q \subseteq (T^*Q, \omega_{can})$ satisfies the above condition.

② $T^n \subseteq (\mathbb{R}^{2n}, \omega_{std})$ does not satisfy this condition.

In fact, no closed Lagrangian submanifold $L \subseteq (\mathbb{R}^{2n}, \omega_{st})$ satisfies the condition, as one can show using the theorem that there are no exact closed Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{st})$.

Exercise: Suppose (M, ω) is a symplectic manifold with $\omega|_{\pi_2(M)} = 0$. Then the diagonal

$$\Delta \subseteq (N, \Omega) = (M \times M, \pi_1^* \omega - \pi_2^* \omega)$$

has the property

$$\omega|_{\pi_2(N, \Delta)} = 0 \quad .$$

We now review the setting in which Floer first developed the version of his homology for Lagrangian submanifolds.

So fix a symplectic manifold (M, ω) and two closed Lagrangian submanifolds $L_0, L_1 \subseteq M$.

Set

$$\mathcal{P}(L_0, L_1) := \{ \gamma : [0,1] \to M \mid \gamma(0) \in L_0, \gamma(1) \in L_1 \}.$$

The symplectic form ω gives rise to a 1-form on $\mathcal{P}(L_0, L_1)$ as follows:

A tangent vector to $\mathcal{P}(L_0, L_1)$ at a curve γ is a vector field $\xi \in \gamma^* TM$ along γ. We define a 1-form $\alpha \in \Omega^1(\mathcal{P}(L_0, L_1))$ by

$$\alpha_\gamma(\xi) := \int_0^1 \omega(\dot{\gamma}(t), \xi(t))\, dt$$

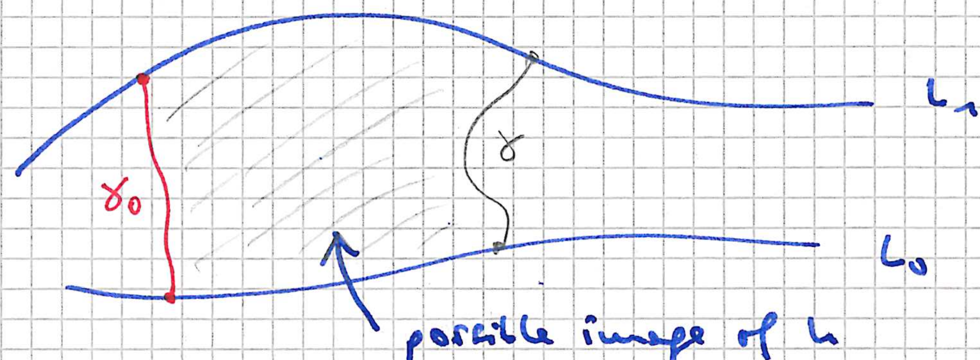
It is clear that $\alpha_\gamma = 0$ if and only if γ is a constant path, which then must correspond to an intersection point of L_0 and L_1.

Fix a reference ~~loop~~ path $\gamma_0 \in \mathcal{P}(L_0, L_1)$. Any other path γ in the component $\mathcal{P}_{\gamma_0}(L_0, L_1) \subseteq \mathcal{P}(L_0, L_1)$ of γ_0 is connected to γ_0 by some homotopy

$$h: [0,1] \times [0,1] \to M$$

with

$$\begin{aligned} h(0,t) &= \gamma_0(t) && \text{for all } t \in [0,1] \\ h(1,t) &= \gamma(t) && \text{for all } t \in [0,1] \\ h(s,0) &\in L_0 && \text{for all } s \in [0,1] \\ h(s,1) &\in L_1 && \text{for all } s \in [0,1] \end{aligned}$$



The universal cover of $\mathcal{P}_{\gamma_0}(L_0, L_1)$ is given by

$$\widetilde{\mathcal{P}}_{\gamma_0}(L_0, L_1) = \{ (\gamma, [h]) \ : \ \gamma \in \mathcal{P}_{\gamma_0}(L_0, L_1),\ [h] \text{ is a homotopy class of homotopies as above} \}.$$

The pullback $\pi^*\alpha$ is exact, in fact we have

$$\pi^*\alpha = -d\widetilde{\mathcal{A}},$$

where

$$\widetilde{\mathcal{A}} : \widetilde{\mathcal{P}}_{\gamma_0}(L_0, L_1) \to \mathbb{R} \quad \text{is defined as}$$

$$\widetilde{\mathcal{A}}(\gamma, [h]) = \int_{[0,1]^2} h^*\omega$$

Exercise: Prove that $\widetilde{\mathcal{A}}$ is well-defined and

$$d\widetilde{\mathcal{A}} = -\pi^*\alpha$$

A computation completely analogous to the case of Hamiltonian Floer theory shows that the L^2 gradient of $\tilde{\mathcal{A}}$ has the form

$$(\operatorname{grad} \tilde{\mathcal{A}})_\gamma = -J\dot{\gamma}$$

(where the metric on $\mathcal{P}(L_0, L_1)$ is induced from the Riemannian metric $\omega(\cdot, J\cdot)$ on M for some ω-compatible a.c. structure J). It follows that the formal gradient flow equation for a curve in $\mathcal{P}(L_0, L_1)$, i.e. for

$$u : \mathbb{R} \times [0,1] \to M \text{ with}$$
$$u(s,0) \in L_0, \quad u(s,1) \in L_1$$

is the Cauchy–Riemann equation

$$\boxed{\partial_s u + J \partial_t u = 0}$$

<u>Remark:</u> For many reasons, one is interested in a theory which is invariant under moving one or both of the Lagrangian submanifolds by a Hamiltonian isotopy. So we might want to incorporate a Hamiltonian $H : M \times [0,1] \to \mathbb{R}$ and consider the action functional

$$\tilde{\mathcal{A}}_H : \tilde{\mathcal{P}}_{\gamma_0}(L_0, L_1) \to \mathbb{R}$$
$$(\gamma, [u]) \mapsto \int_0^1\int_0^1 u^*\omega - \int_0^1 H(\gamma(t))\,dt$$

Instead of intersections of L_0 and L_1, the critical points now correspond to intersections of $\varphi_1(L_0)$ and L_1 (or equivalently Hamiltonian orbits which start at L_0 and end at L_1).
The corresponding gradient flow equation is Floer's equation

$$\partial_s u + J(\partial_t u - X_H(u)) = 0.$$

On the one hand, this is more general, as the above case corresponds to $H \equiv 0$.

On the other hand, one can use the Hamiltonian flow to set up a bijective correspondence between Floer strips for L_0 and L_1 and holomorphic strips for $\varphi_1(L_0)$ and L_1.

Exercise: Try to make the above precise.

As usual, we define the energy of a holomorphic strip as

$$E(u) = \frac{1}{2} \iint_{\mathbb{R}\times[0,1]} |\partial_s u|^2 + |\partial_t u|^2 \, ds\, dt .$$

For Floer strips, the second term is modified by subtracting $X_H \circ u$.

We then have the following result:

Prop: Let (M,ω) be a symplectic manifold, J an ω-compatible a.c. structure (possibly depending on $t\in[0,1]$), and L_0 and L_1 two closed Lagrangian submanifolds of M.

~~Then the following are equivalent:~~

Let $u: \mathbb{R}\times[0,1] \to M$ be a J-holomorphic map s.t. $u(\mathbb{R}\times\{0\}) \subseteq L_0$ and $u(\mathbb{R}\times\{1\}) \subseteq L_1$.

Then the following are equivalent:

1. $E(u) < \infty$

2. The limits
$$p := \lim_{s\to\infty} u(s,t) \quad \text{and} \quad q := \lim_{s\to-\infty} u(s,t)$$
exist uniformly in $t\in[0,1]$.

3. $\partial_s u$ decays exponentially in the C^∞-topology, in the sense that there are constants $\delta > 0$ and $c_1, c_2, c_3, \ldots$ such that

$$\| \partial_s^k u(s,t) \| \leq c_k \cdot e^{-\delta |s|} .$$

To set up Floer theory, we now need to address the following points:

* a grading for critical points of $\mathcal{A}$, i.e. intersection points of L_0 and L_1
* a Fredholm theory for moduli spaces of holomorphic strips with given asymptotics
* the compactness properties of moduli spaces of holomorphic strips with finite energy
* gluing of holomorphic strips.