Morse theory

References

J. Milnor "Morse theory" (a classic!)

M. Audin, M. Damian "Morse theory and Floer homology"

M. Schwarz "Morse homology"

A. Banyaga, D. Hurtubise "Lectures on Morse homology"

also: R. Bott : Morse theory indomitable

Lectures on Morse theory, old and new

Morse theory studies the relation between the topology of a space and the behaviour of generic functions on it.

Fundamental theorem:

> M closed manifold, $f: M \to \mathbb{R}$ Morse function
> Then M has the homotopy type of a CW complex with one cell of dimension k for each critical point of f of index k.

This has no analogue in Floer theory. Instead, we emulate a weaker statement:

> In the setup above, after choosing auxiliary data (a generic gradient-like vector field), one can build a chain complex that is generated by the critical points of f which computes the homology of M.

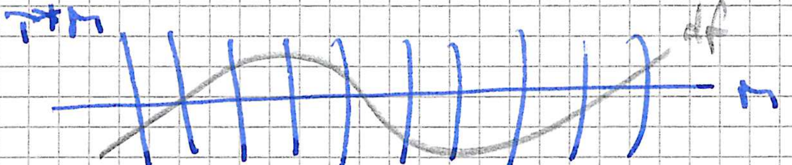
Rem: Many objects from classical algebraic topology (e.g. cup product, Steenrod operations, ...) can be described Morse theoretically. Most of them have analogues in Floer theory.

Basic definitions:

M smooth, $f: M \to \mathbb{R}$ smooth

$p \in M$ is critical for f if $df_p = 0$

One can view df as a section of T^*M.



Along $M \subseteq T^*M$, we have a canonical splitting

$$T(T^*M) \cong TM \oplus T^*M$$

If p is a critical point, we can project the linearization

$$D_p df : T_pM \to T_{(p,0)}(T^*M) \cong T_pM \oplus T_p^*M$$

onto the fibers of T^*M to get

$$\pi_2 D_p df : T_pM \to T_p^*M$$

This can be reinterpreted as a bilinear form

$$\mathrm{Hess}_p f : T_pM \times T_pM \to \mathbb{R}$$

Def: A critical point p of f is called nondegenerate if $\mathrm{Hess}_p f$ is nondegenerate.

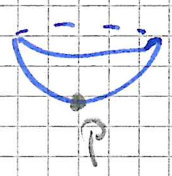
Remarks / Exercises:

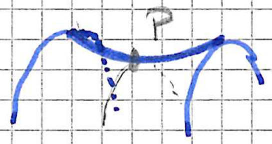
1. $\mathrm{Hess}_p f$ is symmetric, so diagonalizable. p nondegenerate $\Leftrightarrow$ no eigenvalue $= 0$. The *index* of p is the number of negative eigenvalues of $\mathrm{Hess}_p f$.
2. In local coordinates $x_1, \dots, x_n$ on M near p, $\mathrm{Hess}_p f$ is given by the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ of second derivatives
3. p is nondegenerate $\Leftrightarrow$ $df_p \pitchfork M$ at p as submanifolds of T^*M

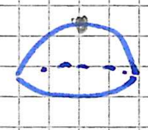
Consequence: nondegenerate critical points are *isolated*

In dimension 2, there are 3 possibilities:

local minimum	saddle	local maximum
$f(x,y) = x^2+y^2$	$f(x,y) = xy$	$f(x,y) = -x^2-y^2$
index 0	index 1	index 2







Def: A function $f: M \to \mathbb{R}$ is called Morse if all its critical points are nondegenerate.

Morse Lemma: Near a critical point $p \in M$ of a Morse function $f: M \to \mathbb{R}$, we can find coordinates s.t.

$$f(x_1, \dots, x_n) = f(p) - \sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{n} x_j^2$$

Here k is the index of p.

Given a Riemannian metric g on M we get a gradient vector field ∇f defined via

$$g(\nabla f, \cdot) = df$$

It is customary to use the negative gradient flow in Morse theory, so we set $X := -\nabla f$.

Assuming X is complete (e.g. true if M is closed) we get the negative gradient flow

$$\varphi: \mathbb{R} \times M \to M$$

determined by
$$\varphi(0,x) = x$$
$$\frac{d}{dt} \varphi(t,x) = (\nabla f)_{\varphi(t,x)}$$

Now for each $p \in \mathrm{Crit}(f)$ we have

$W^-(p) := \{ x \in M \mid \lim_{t \to -\infty} \varphi_t(x) = p \}$ unstable set of p

$W^+(p) := \{ x \in M \mid \lim_{t \to \infty} \varphi_t(x) = p \}$ stable set of p

For general dynamical systems, or even for general gradient flows, these can be quite complicated. But if f is Morse, then $W^\pm(p)$ are injectively immersed copies of $\mathbb{R}^{n-\mathrm{ind}(p)}$ and $\mathbb{R}^{\mathrm{ind}(p)}$, respectively.

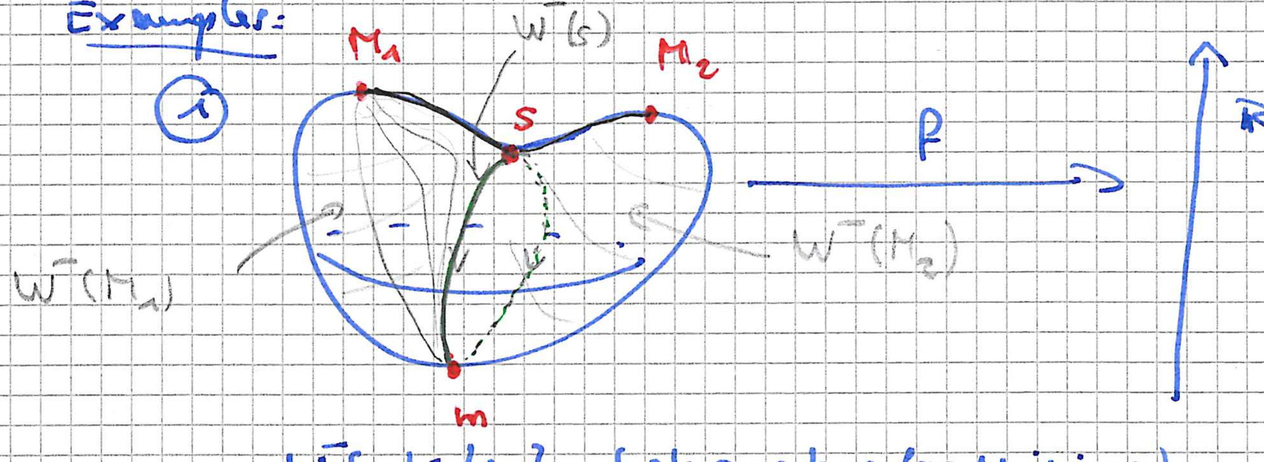
Using the metric g, we can represent $\mathrm{Hess}_p f$ as a linear map

$$A_p : T_pM \to T_pM,$$

and $W^\pm(p)$ are tangent to the direct sum of positive/negative eigenspaces of A_p at p.

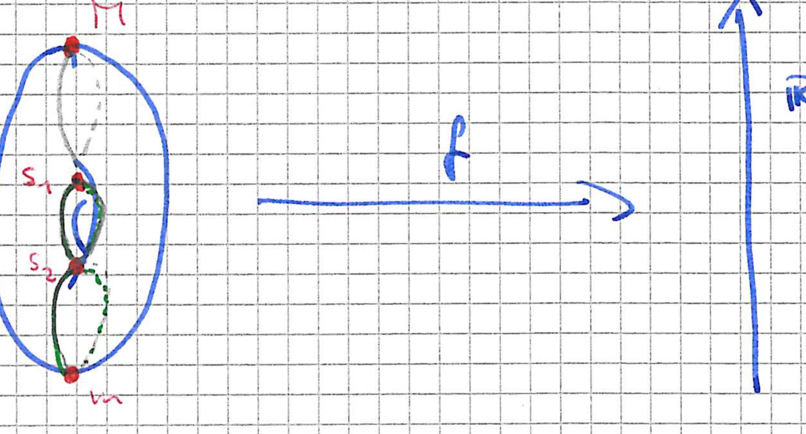
Examples:

1



$W^-(m) = \{m\}$ (always true for minima)

2



Here $W^-(s_1)$ and $W^+(s_2)$ almost ~~coin~~ coincide (one contains s_1, the other s_2)

Def: The pair (f,g) is called <u>Morse-Smale</u> if f is Morse and for all $p,q \in \mathrm{Crit}(f)$ the negative gradient flow w.r.t. g satisfies

$$W^-(p) \pitchfork W^+(q)$$

So example 1 is Morse-Smale, example 2 is not.

Thm: M closed, $f: M \to \mathbb{R}$ smooth, g any metric

(a) If f is Morse, then the pair (f,g) can be made Morse-Smale by an arbitrarily small perturbation of g (which can be done away from $\mathrm{Crit}(f)$ if desired).

(b) The pair can always be made Morse-Smale by an arbitrarily small perturbation of f.

Pf: (a) is Thm 2.2.5 in Audin-Damian

(b) the fact that Morse functions are dense in all functions is an exercise with hints on p. 20 of Audin-Damian. The above statement is a slight variation

In example 2,

- following (a), we could break the symmetry by changing the metric to force $W^-(s_1)$ to avoid $W^+(s_2)$
- following (b), we can keep the metric but change f (e.g. by tilting the torus slightly)

Suppose (f,g) is a Morse-Smale pair, and $p \neq q$ are critical points of f.

If $W^-(p) \cap W^+(q)$ is nonempty, then it is a submanifold of M of dimension

$$\begin{aligned} &\dim W^-(p) + \dim W^+(q) - n \\ = \;& \mathrm{ind}(p) + n - \mathrm{ind}(q) - n \\ = \;& \mathrm{ind}(p) - \mathrm{ind}(q) \end{aligned}$$

This space comes with a free $\mathbb{R}$-action (shifting along the flow lines), and so we must have

$$\mathrm{ind}(p) - \mathrm{ind}(q) \geq 1$$

Taking the quotient by the $\mathbb{R}$-action we get the "space of flow lines from p to q",

$$\mathcal{F}(p,q) := W^-(p) \cap W^+(q) \big/ \mathbb{R}$$

<u>Unfortunately:</u> In Floer theory, the analogues of $W^{\pm}(p)$ will <u>not</u> exist.

So we give an alternative definition of $\mathcal{F}(p,q)$, which <u>will</u> generalize to Floer theory.

Consider

$$\widetilde{\mathcal{F}}(p,q) := \{ \gamma : \mathbb{R} \to M \text{ smooth}, \; \dot{\gamma}(t) = X(\gamma(t)) \text{ and } \lim_{t \to -\infty} \gamma(t) = p, \; \lim_{t \to +\infty} \gamma(t) = q \}$$

The evaluation at 0 gives an injective map

$$ev_0 : \widetilde{\mathcal{F}}(p,q) \to M$$

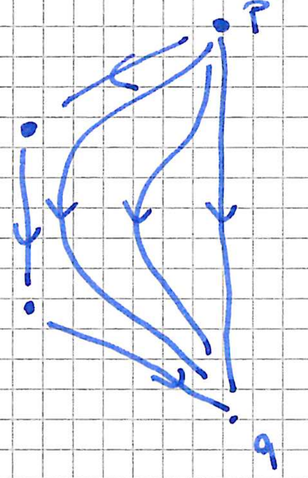
whose image is $W^-(p) \cap W^+(q)$.

Now we set $\mathcal{F}(p,q) = \widetilde{\mathcal{F}}(p,q) \big/ \mathbb{R}$, where the $\mathbb{R}$-action is by shifts as before:

$$(\tau \cdot \gamma)(t) = \gamma(t + \tau).$$

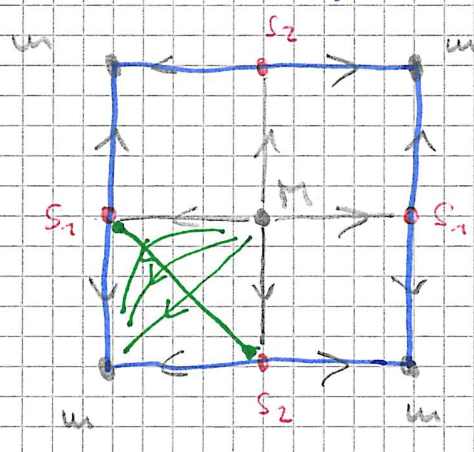
Any reasonable topology on $C^\infty(\mathbb{R}, M)$ gives us a topology on $\widetilde{\mathcal{F}}(p,q)$ for which ev_0 is a homeomorphism onto its image and the $\mathbb{R}$-action is continuous.
In particular, $\mathcal{F}(p,q)$ inherits a (metrizable) topology.

Geometrically, it is "clear" that in general $\mathcal{F}(p,q)$ is not compact, but can be compactified by adding broken flow lines from p to q:



$$d_H(A,B) = \max\left(\sup_{a\in A} \inf_{b\in B} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b) \right)$$

Example: On $T^2 = S^1 \times S^1$ we use
$$f(\varphi,\theta) = -\cos(\varphi) - \cos(\theta)$$



Here $\mathcal{F}(M,m)$ consists of 4 components, each homeomorphic to an open interval

There are 8 broken flow lines from M to m, each corresponding to a unique "end" of $\mathcal{F}(p,q)$.

This is a general phenomenon.

Prop 1: M closed, (f,g) Morse-Smale pair

$p, q \in \mathrm{Crit}(f)$

Then

1. If $\mathrm{ind}(p) - \mathrm{ind}(q) \leq 0$, then $\mathcal{F}(p,q) = \emptyset$

2. If $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, then $\mathcal{F}(p,q)$ is compact, i.e. finite.

3. If $\mathrm{ind}(p) - \mathrm{ind}(q) = 2$, then $\mathcal{F}(p,q)$ has a natural compactification $\overline{\mathcal{F}}(p,q)$ which is a 1-manifold with boundary s.t.

$$\partial \overline{\mathcal{F}}(p,q) \underset{bij}{\cong} \bigcup_{\substack{\mathrm{ind}(p) > \mathrm{ind}(r) \\ > \mathrm{ind}(q)}} \mathcal{F}(p,r) \times \mathcal{F}(r,q)$$

Assuming this proposition, we can construct a chain complex over $\mathbb{Z}_2$ as follows:

For $k \geq 0$, set $\quad CM_k(f,g) := \bigoplus_{\mathrm{ind}(p)=k} \mathbb{Z}_2 \cdot p$

We define the boundary map

$$\partial : CM_k(f,g) \longrightarrow CM_{k-1}(f,g)$$

by $$\partial p := \sum_{\substack{\mathrm{ind}(q) = \\ \mathrm{ind}(p) - 1}} \#_2 \mathcal{F}(p,q) \cdot q$$

This is well-defined by 2.
The fact that $\partial^2 = 0$ follows from 3.

$$\partial^2 p = \partial \Big(\sum_{\substack{\mathrm{ind}(r) = \\ \mathrm{ind}(p) - 1}} \#_2 \mathcal{F}(p,r) \cdot r \Big)$$

$$= \sum_{\substack{\mathrm{ind}(r) = \\ \mathrm{ind}(p) - 1}} \sum_{\substack{\mathrm{ind}(q) = \\ \mathrm{ind}(r) - 1}} \#_2 \mathcal{F}(p,r) \#_2 \mathcal{F}(r,q) \cdot q$$

Fixing q, we get $\displaystyle\sum_{\substack{\mathrm{ind}(p) > \mathrm{ind}(r) \\ > \mathrm{ind}(q)}} \#_2 \mathcal{F}(p,r) \cdot \#_2 \mathcal{F}(r,q)$ as a coefficient. This is the mod 2 count of boundary points of $\overline{\mathcal{F}}(f,g)$, and hence 0.

There are a few obvious questions:

* How does the Morse complex change when we choose a different pair (f,g)?

* What does the homology of this complex compute?

We postpone the second question and concentrate on the first one.

So ~~f~~ suppose (f^+,g^+) and (f^-,g^-) are two Morse-Smale pairs.

Pick

- a family of functions $\{f_t\}_{t\in\mathbb{R}}$ such that

$$f_t = \begin{cases} f^+ & \text{for } t \gg 0 \\ f^- & \text{for } t \ll 0 \end{cases}$$

- a family of metrics $\{g_t\}_{t\in\mathbb{R}}$ such that

$$g_t = \begin{cases} g^+ & \text{for } t \gg 0 \\ g^- & \text{for } t \ll 0 \end{cases}$$

We get gradient vector fields $X_t = -\underset{\substack{\uparrow \\ \text{gradient w.r.t. } g_t}}{\nabla} f_t$

Now for $p^\pm \in \mathrm{Crit}(f^\pm)$ we define

$$\mathcal{K}(p^-,p^+) := \left\{ \gamma : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \dot\gamma(t) = X_t(\gamma(t)) \\ \lim\limits_{t\to\pm\infty} \gamma(t) = p^\pm \end{array} \right\}$$

Then we have

<u>Prop 2:</u> For a generic choice of the family $\{f_t, g_t\}$ all the spaces $\mathcal{K}(p^-,p^+)$ are manifolds of dimension $\mathrm{ind}(p^-) - \mathrm{ind}(p^+)$.

Moreover:

(a) If $\mathrm{ind}(p^-) = \mathrm{ind}(p^+)$, then $\mathcal{K}(p^-,p^+)$ is compact.

b) If $\mathrm{ind}(p^-) = \mathrm{ind}(p^+) + 1$, then $\mathcal{K}(p^-,p^+)$ has a natural compactification to a 1-manifold with boundary $\overline{\mathcal{K}}(p^-,p^+)$ s.t.

$$\partial \overline{\mathcal{K}}(p^-,p^+) \underset{\text{bij.}}{\cong} \bigcup_{\substack{p' \in \mathrm{Crit}(f^-) \\ \mathrm{ind}(p^-) = \\ \mathrm{ind}(p')+1}} \mathcal{F}(p^-,p') \times \mathcal{K}(p',p^+)$$

$$\cup \bigcup_{\substack{p'' \in \mathrm{Crit}(f^+) \\ \mathrm{ind}(p^-) = \\ \mathrm{ind}(p'')}} \mathcal{K}(p^-,p'') \times \mathcal{F}(p'',p^+)$$

As a consequence of this Proposition, the map

$$c = c^{\{f_t, g_t\}} : CM_*(f^-, g^-) \longrightarrow CM_*(f^+, g^+)$$

$$c(p^-) = \sum_{\substack{\mathrm{ind}(p^+) \\ = \mathrm{ind}(p^-)}} \#_2 \mathcal{K}(p^-,p^+) \cdot p^+$$

is a chain map, i.e.

$$c \circ \partial_- = \partial_+ \circ c .$$

Another question appears:

How does the chain map c depend on the family $\{f_t, g_t\}$?

<u>Example:</u> In the case that $(f^-, g^-) = (f^+, g^+)$, we can look at the constant family $(f_t, g_t) = (f^-, g^-)$. The only connections between critical points of the same index will be the constant flow lines, as

$$\mathcal{K}(p^-,p^+) = \widetilde{\mathcal{F}}(p^-,p^+)$$

in this case.

It follows that $c = \mathrm{id}$ in this case.