Last time we briefly mentioned that symplectic homology is a ring with unit with respect to the pair-of-pants product.
The construction of the product follows the blueprint used in the closed case, with ~~two~~ one modifications:

(1) To define the product on the chain level, we run into compactness questions similar to those for continuation maps. One solves this by defining the product ~~only~~

$$\mu_s : CF(H_1, J_1) \times CF(H_2, J_2) \to CF(H_0, J_0)$$

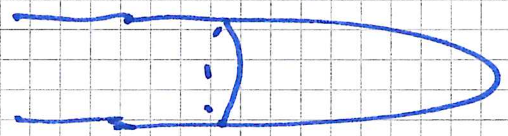
only under the condition

$$b(H_1) + b(H_2) \leq b(H_0)$$

for the slopes at infinity.

(2) To get something well-defined on the direct limit, one needs to discuss compatibility with continuation maps.

(3) The unit in $HF(H, J)$ can be obtained by counting elements in moduli spaces of "caps"



These can be viewed as maps $u : \mathbb{R} \times S^1 \to \hat{W}$ satisfying Floer's equation for an s-dependent family of Hamiltonian functions $H_s : \hat{W} \times \mathbb{R} \times S^1 \to \mathbb{R}$ which agree with H near $s = -\infty$ and vanish near $s = \infty$. The maps are required to have a removable singularity at $s = \infty$.
Again, one needs to make sure these units are compatible with continuation maps.

For details, see e.g. Abouzaid's monograph.

Remark: One could avoid the direct limit in the definition of symplectic homology and work with (time-dependent perturbations of) a Hamiltonian H which agrees with $h(r)$ on the end, where $h''(r) > 0$ and $h'(r) \to \infty$ as $r \to \infty$.

This has 2 drawbacks:

ⓐ the invariance proof is not easy in this case, as two such Hamiltonians might not have continuation maps in either direction.

ⓑ the following result seems impossible to prove with this definition.

Thm: Let $\phi : (W_0, d\lambda_0) \to (W_1, d\lambda_1)$ be an exact embedding of Liouville domains. Then ϕ induces a transfer map

$$\phi^! : SH_*(W_1, \lambda_1) \to SH_*(W_0, \lambda_0)$$

which is a morphism of unital rings.

Def: An <u>exact Liouville embedding</u> of $(W_0, d\lambda_0)$ into $(W_1, d\lambda_1)$ is a smooth embedding

$$\phi : W_0 \to W_1$$

such that $\phi^* \lambda_1 = \lambda_0 + df$ for some function $f : W_0 \to \mathbb{R}$.

Ex: ⓐ If $H^1(W_0, \mathbb{R}) = 0$, then any symplectic embedding

$(W_0, d\lambda_0) \to (W_1, d\lambda_1)$ is an exact Liouville embedding, because

$$0 = \omega_0 - \phi^* \omega_1 = d(\underbrace{\lambda_0 - \phi^* \lambda_1}),$$

and the cohomological condition ensures that this closed 1-form is exact.

(b) The standard example of a nonexact Liouville embedding is the following:

Consider the Lagrangian embedding $S^1 \subseteq (\mathbb{R}^2, \omega_{std})$. By the Lagrangian nbhd theorem there is a symplectic embedding

$$\varphi : (D^*_\varepsilon S^1, d\lambda_{can}) \to (\mathbb{R}^2, \omega_{st})$$

Note that λ_{can} vanishes over the zero section of the cotangent bundle, and so

$$\int_{S^1} \lambda_{can} = 0 .$$

On the other hand, for the primitive

$$\lambda_{st} = \frac{1}{2}(x\,dy - y\,dx)$$

of $\omega_{st} = dx \wedge dy$ we have

$$\int_{S^1} \varphi^* \lambda_{st} = \int_{B^2(0,1)} d\lambda_{st} = \pi$$

Now for any function $f : S^1 \to \mathbb{R}$ we have

$$\int_{S^1} df = 0 ,$$

and so it is impossible that $\varphi^*\lambda_{st} = \lambda_{can} + df$.

The transfer map was originally constructed by Viterbo. Its compatibility with the pair-of-pants product was first proven by McLean.

As a consequence of the theorem we find:

Cor: Suppose $SH_*(W_1, \lambda_1) = 0$ and $SH_*(W_0, \lambda_0) \neq 0$. Then there does not exist an exact Liouville embedding of (W_0, λ_0) into (W_1, λ_1).

Pf: There is no unital ring map from the trivial ring 0 to a nontrivial ring. $\square$

By the results we discussed last time, this corollary applies in particular if $(W_1, d\lambda_1)$ is a subcritical ~~Stein~~ Weinstein domain and (W_0, λ_0) is a nbhd of the zero section in the cotangent bundle of a closed manifold Q.

So we get:

Cor 2: There is no exact Lagrangian embedding of a closed manifold Q into a subcritical Weinstein domain $(W, d\lambda)$ (or more generally into a Liouville domain with $SH_*(W, \lambda) = 0$).

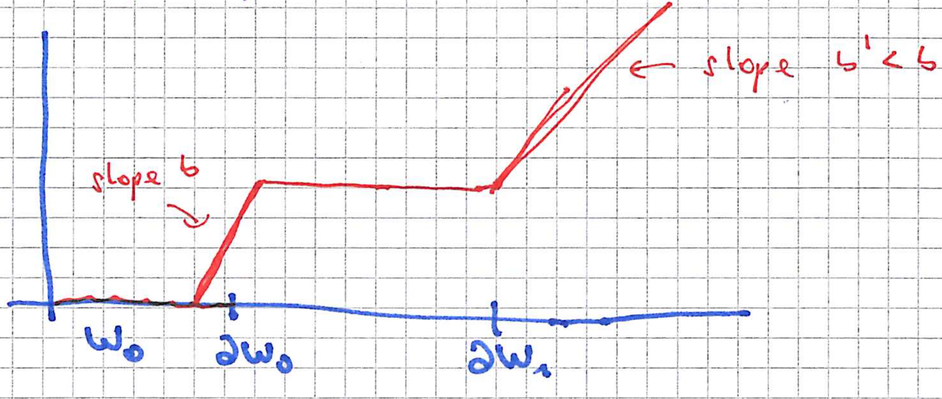
Rem: • A Lagrangian embedding $L \overset{\varphi}{\hookrightarrow} (W, d\lambda)$ gives rise to a closed 1-form $\varphi^*\lambda$ on L, and the embedding is called exact if this form is exact.

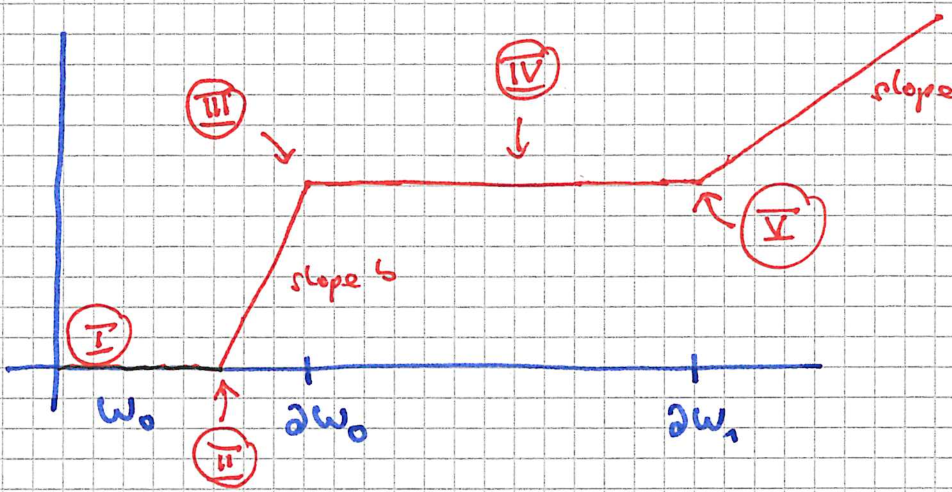
• Cor 2 was originally proven by M. Gromov with a completely different argument.

I want to briefly discuss the idea of the construction of Viterbo's map:

Wlog. we assume $\phi : (W_0, d\lambda_0) \to (W_1, d\lambda_1)$ is a Liouville embedding with $\phi^*\lambda_1 = \lambda_0$ (this can always be achieved by modifying λ_1 in a nbhd of $\phi(W_0)$).

Now one uses a family of Hamiltonian functions of the following shape:





One can now arrange, for given $0 < b' < b$:

* all 1-periodic orbits of type I have action very close to 0
* all 1-periodic orbits of type II have action in ~~some~~ $[\varepsilon, b)$ for some $\varepsilon > 0$.
* all 1-periodic orbits of type III have action < -1
* all 1-periodic orbits of type IV have action close to $-b$
* all 1-periodic orbits of type V have action $< b' - b < 0$.

It follows that

(a) orbits of type III, IV and V form a subcomplex of the Floer complex

$$CF_*^{\leq -1}(H, J) \subseteq CF(H, J)$$

(b) the quotient complex

$$CF_*(H, J) \big/ CF_*^{\leq -1}(H, J)$$

can be identified with the Floer complex of $H|W_0$ (this also uses the confinement lemma to make sure there are no new Floer cylinders "outside" W_0)

Now the transfer map is obtained from the quotient projections by carefully passing to the limit (for a sequence $b_i \to \infty$ choose an appropriate sequence $b_i' \to \infty$).

$\square$

Here is one other application of symplectic homology:

Thm (M. McLean '09)

For every $k \geq 4$, there is a family of finite type Stein manifolds $\{X_n\}_{n \in \mathbb{N}}$ (i.e. completions of Stein domains) which are all diffeomorphic to $\mathbb{R}^{2k}$ but pairwise not symplectomorphic (and not even Stein deformation equivalent).

Sketch of the proof:

We consider the case $k = 4$.
Let $V := \{x^2 + y^2 + z^2 + w^2 = 0\} \subseteq \mathbb{C}^4$ and let $p \in V$ be a smooth point (e.g. $p = (0,0,1,i)$).

Denote by $H := Bl_p \mathbb{C}^4$ the blow-up of $\mathbb{C}^4$ at p, and let

$$X := H \setminus \tilde{V}$$

where $\tilde{V} = \overline{\pi^{-1}(V \setminus p)} \subseteq H$ is the proper transform of V

Fact 1: $X \cong \mathbb{R}^8$

Fact 2: X inherits a finite type Stein structure from this construction

Fact 3: For a Liouville manifold $(W, d\lambda)$, we define

$$i(W) := \# \text{ idempotents in } SH_*(W, \lambda),$$

i.e. the number of elements x with $x^2 = x$

Then $1 < i(X) < \infty$.

<u>Note:</u> Examples of idempotents are 0 and 1 (which are the same if $SH_*(W)=0$)

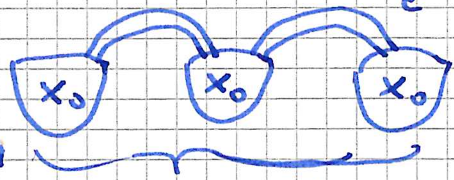
Now one simply defines

$$X_n := \#_{j=1}^{n} X \qquad \text{n-fold "end connected sum"}$$

$\leftarrow$ Weinstein 1-handles

X is the completion of some X_0

X_n is the completion of $\rightarrow$



n copies

<u>Fact 4:</u> $$SH_*(X_n) \cong \prod_{i=1}^{n} SH_*(X) \text{ as rings.}$$

This follows from
$$SH_*(W \amalg W') = SH_*(W) \times SH_*(W')$$
and the fact that subcritical handle attachment does not change SH_*

Now combining Facts 3 and 4, we find
$$\iota(X_n) = \iota(X)^n,$$
and all these numbers are distinct. Invariance of SH_* under Stein deformation now implies the theorem.

$\square$

In a different direction, one can sometimes prove that the "positive part of" symplectic homology" $SH_*^{+}(W, d\lambda)$ is an invariant of the contact manifold $(V = \partial W, \xi = \ker \lambda|_V)$, i.e. independent of the Liouville filling W.

This can then be used to prove results on contact manifolds.

For example, consider the Stein manifolds

$$X_{r,\varepsilon} := \{ z \in \mathbb{C}^4 \; : \; z_1^r + z_2^2 + z_3^2 + z_4^2 = \varepsilon \}$$

for $\varepsilon \neq 0$ (small).

The contact manifolds

$$\Sigma_{r,\varepsilon} := X_{r,\varepsilon} \cap S^7 \qquad \text{with } r \text{ fixed}$$

are all isomorphic for different choices of ε by Gray stability. It is also known that

$$\Sigma_r \underset{\text{diffeo}}{\cong} S^5 \qquad \text{for } r \text{ odd}$$

and $$\Sigma_r \underset{\text{diffeo}}{\cong} S^2 \times S^3 \qquad \text{for } r \text{ even.}$$

By studying positive symplectic homology, P. Uebele proved

Thm (Uebele '15)

The contact manifolds $\Sigma_{2\ell}$ for different $\ell \geq 1$ are pairwise nonisomorphic.