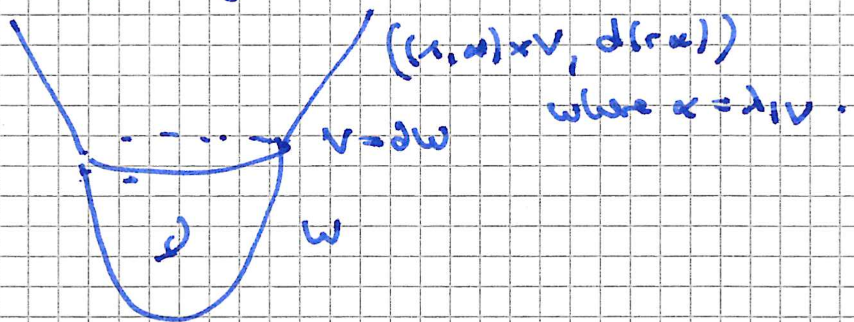
Wrapped Lagrangian Floer theory

So far, when talking about Lagrangian Floer homology, we have always assumed that our Lagrangians are closed submanifolds in the ambient symplectic manifold.

We now consider the following situation:

* our symplectic manifold $(\hat{W}, d\lambda)$ is the completion of a Liouville domain $(W, d\lambda)$. Recall that this means that the vector field Y dual to the primitive λ of the symplectic form is transverse to ∂W pointing outward:



* $L \subseteq (\hat{W}, d\lambda)$ is an exact Lagrangian submanifold such that the intersection of L with the end is of the form $(1,\infty)\times\Lambda$ for a Legendrian submanifold $\Lambda \subseteq (V, \ker\alpha)$, meaning that $\alpha_{|\Lambda} = 0$ and $\dim\Lambda$ is maximal with this property, i.e. $\dim\Lambda = n-1$ when $\dim V = 2n-1$.

Examples: ① If $(W, d\lambda)$ is a compact surface with boundary and a suitable Liouville form, we can start from any curve $L_0 \subseteq W$ with boundary on $\partial W = V$ and complete it to a Lagrangian L as above by adding the cone $(1,\infty)\times(L_0\cap V)$.

② If $(\hat{W}, d\lambda) = (T^*Q, d\lambda_{can})$, we can pick $L = T_q^*Q$ to be a cotangent fiber.

To define the Floer homology of L with itself, we need to choose a Hamiltonian perturbation which produces another copy $L' \subseteq \hat{W}$ of L with $L \pitchfork L'$.

Because of the noncompactness of L, different choices of such a perturbation may lead to different theories. The two most common ones are

(a) Pick a Hamiltonian function $H: \hat{W} \to \mathbb{R}$ s.t.

$$H(r,x) = \varepsilon \cdot r \quad \text{for } \varepsilon > 0 \text{ small.}$$

This has the effect that on the end we just move $(1,\infty) \times \Lambda$ slightly by pushing off Λ in the Reeb direction. This is called "infinitesimal wrapping".

(b) Pick a Hamiltonian function $H: \hat{W} \to \mathbb{R}$ s.t.

$$H(r,x) = c \cdot r^2 \quad \text{for } c > 0,$$

Then we have $X_H = 2rcR$ on the end, where R is the Reeb field of $\alpha = \lambda|_V$.

In case (a), the Lagrangians L and $\varphi_1^H(L)$ have no intersections in the end $(1,\infty) \times V$.
In case (b), the intersection points of L and $\varphi_1(L)$ are in bijective correspondence with the <u>Reeb chords</u> of Λ, i.e. solutions of $\gamma : [0,T] \to V$ of

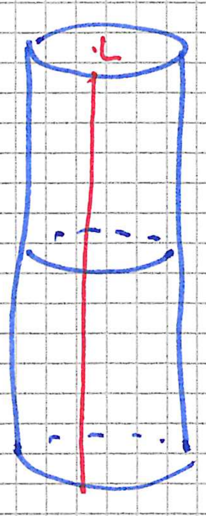
$$\dot{\gamma}(t) = R(\gamma(t))$$

with $\gamma(0) \in \Lambda$ and $\gamma(T) \in \Lambda$ for some $T > 0$.

Ex: Suppose $(\hat{W}, d\lambda) = (\mathbb{R} \times S^1, d(s\,dt))$ and $L = \mathbb{R} \times \{p\}$.

(a)

(b)



The theory in case (b) is called wrapped Floer homology.

The relevant Floer complex

$$CW(L, L; H)$$

is generated by orbits $x: [0,1] \to \hat{W}$ of X_H such that $x(0), x(1) \in L$. The boundary operator

$$\partial : CW(L, L; H) \to CW(L, L; H)$$

counts solutions of Floer's equation $u: \mathbb{R} \times [0,1] \to \hat{W}$

$$\partial_s u + J(\partial_t u - X_H(u)) = 0$$

of finite energy s.t. $u(\mathbb{R} \times \{0\}) \subset L$, $u(\mathbb{R} \times \{1\}) \subseteq L$.

Our exactness assumption excludes disk or sphere bubbles, and one checks that for a given intersection point $x \in L \cap \varphi_1^{-1}(L)$ appearing as an asymptotic limit at $s \to -\infty$ there are only finitely many choices of asymptotic limit as $s \to \infty$, and for each of them the count of rigid strips is finite. Moreover, considering breaking of strips of index 2, one can prove that $\partial^2 = 0$. In this way, one obtains the wrapped Floer (co)homology

$$HW(L, L).$$

Rem: * In the same way, one can define $HW(L_0, L_1)$ for two Lagrangians satisfying our assumptions.

* In general, the theory makes sense over $\mathbb{Z}_2$ with a $\mathbb{Z}_2$-grading. To get a well-defined $\mathbb{Z}$-grading, one needs e.g. $2c_1(T\hat{W}) = 0$ and the vanishing of the Maslov index of both Lagrangians.

Thm (Abbondandolo-Schwarz)

If $(\hat{W}, d\lambda) = (T^*Q, d\lambda_{can})$ and $L = T^*_q Q$, then

$$HW(L, L) \cong H(\Omega_q Q).$$

Abouzaid has improved this result to a chain level equivalence of A_∞ structures.

To understand what this means, we need to understand the construction of algebraic structures on both sides.

(A) Chains on the based loops space

We consider cubical chains, i.e.

$$C_k(\Omega_q Q) = \{ c : [0,1]^k \to \Omega_q Q \}$$

Since loops at q can be concatenated, we get a product

$$C_k(\Omega_q Q) \times C_\ell(\Omega_q Q) \to C_{k+\ell}(\Omega_q Q)$$

$$(\sigma \,,\, \tau) \longmapsto \sigma * \tau$$

where

$$\sigma * \tau(t_1, \ldots, t_{k+\ell}) = \sigma(t_1, \ldots, t_k) \underset{\substack{\uparrow \\ \text{concatenation in } \Omega_q Q}}{*} \tau(t_{k+1}, \ldots, t_{k+\ell})$$

In this way $(C_*(\Omega_q Q), \partial)$ turns into an algebra over the ground ring (this works over $\mathbb{Z}$, but we will use $\mathbb{Z}_2$)

Remark: ~~For coefficients ...~~ The homology of the cubical chain complex is not isomorphic to the usual singular theory. This can be rectified by taking a quotient by degenerate chains, which form an ideal with respect to the above multiplication.

Ex: For $Q = S^1$, we have $H_*(\Omega_q Q) \cong \mathbb{Z}_2[t, t^{-1}]$, where the exponent corresponds to the winding number of the loop.

3 Operations in Floer theory

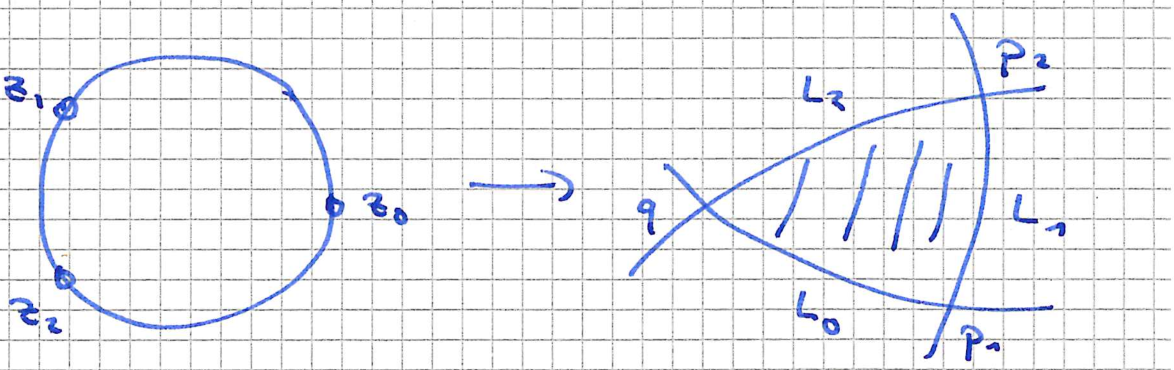
Suppose we are given 3 Lagrangians with cylindrical ends L_0, L_1 and L_2. Using the Lagrangian Floer theory formulation with a Hamiltonian term, we make choices of Hamiltonian functions H_{ij} and a.c. structures J_{ij} such that the pairwise Floer theory

$$HF(L_i, L_j; H_{ij}, J_{ij})$$ is well-defined.

To define a product operation

$$CF(L_1, L_2; H_{12}, J_{12}) \otimes CF(L_0, L_1; H_{01}, J_{01}) \to CF(L_0, L_2; H_{02}, J_{02})$$

we want to consider maps from a disk with 3 boundary punctures satisfying a suitable equation and mapping the three components of the boundary to the 3 Lagrangians:



For that, we need to fix biholomorphisms of nbhds of the punctures with strips

$$(0, \infty) \times [0,1] \quad \text{for } z_1 \text{ and } z_2$$
$$\text{and} \quad (-\infty, 0] \times [0,1] \quad \text{for } z_0$$

Then we extend the forms $H_{ij}\,dt$ on these strip-like ends to a global form β on the disk and pick a domain-dependent family $H(z)$ of Hamiltonian functions which agree with the H_{ij} in the ends. Then we consider maps

$$u: D^2 \setminus \{z_0, z_1, z_2\} \to \hat{W}$$

satisfying
$$\left(du - X_H \otimes \beta\right)^{0,1} = 0$$
and the correct boundary conditions.

As we have seen in our discussion of symplectic homology, the noncompactness of $\hat{W}$ poses restrictions on the slopes of our Hamiltonians at infinity needed to ensure the compactness (or more precisely compactification) of our spaces of Floer triangles. This has the effect that one is naturally lead to define a product map
$$CW(L_1, L_2; H) \otimes CW(L_0, L_1; H) \to CW(L_0, L_2; 2H).$$
Now we have a problem, as there is no well-behaved continuation map back to $CW(L_0, L_2; H)$ in this setting.

The solution, going back to an idea of Fukaya, Seidel and Smith and developed by Abouzaid, is to use a "rescaling trick".
The flow ψ_ρ of the Liouville vector field Y associated to $(\hat{W}, \lambda)$ is conformally symplectic, meaning that
$$\psi_\rho^* d\lambda = e^\rho d\lambda.$$
On the conical end, we have $Y = r\frac{\partial}{\partial r}$, and by our assumptions this leaves the Lagrangian submanifolds L_0, L_1 and L_2 invariant there, only moving ~~their~~ their intersection with W (the compact part of our Liouville manifold). In this way, one gets an isomorphism
$$(*) \qquad CW(L_0, L_1; H, J) \cong CW(\psi_\rho L_0, \psi_\rho L_1; e^{-\rho} H \circ \psi_\rho)$$
One also checks that each L_i is exact Lagrangian isotopic to $\psi_\rho L_i$ by a compactly supported isotopy.

Moreover, on the end we have
$$e^{-\rho} H \circ \psi_\rho = e^\rho H.$$

Abouzaid shows that in this way one gets a well-defined product map

$$\mu_2 : CW(L_1, L_2; H, J) \otimes CW(L_0, L_1; H, J) \to CW(\psi_{\log 2}(L_0), \psi_{\log 2}(L_2); \tfrac{1}{2} H \circ \psi_{\log 2}, (\psi_{\log 2})_* J)$$

$$\circledast \quad \cong$$

$$CW(L_0, L_2; H, J)$$

More generally, one gets higher product maps

$$\mu_k : CW(L_{k-1}, L_k; H) \otimes \ldots \otimes CW(L_0, L_1; H) \to CW(L_0, L_k; H)$$

counting Floer "polygons", i.e. maps of disks with $k+1$ boundary punctures.

In order to ensure that these satisfy the A_∞-relations

$$\sum_{a+b=k+1} \sum_{i=1}^{a} \mu_a \circ_i \mu_b = 0 \qquad \text{for all } k \geq 1$$

One has to make coherent choices of the functions H and forms β over the whole (compactified) moduli spaces of disks with boundary punctures.

The <u>wrapped Fukaya category</u> of $(\hat{W}, d\lambda)$ has as its objects exact Lagrangians with cylindrical ends, and as morphism spaces intersection points of some L with $\varphi_1^H(L')$ for a suitable H. The above operation μ_2 is then interpreted as a composition rule for morphisms, and the collection $\{\mu_k\}$ turns it into an A_∞ category.
For the special case where all Lagrangians agree with a fixed L, we get the structure of an A_∞ algebra on $CW(L, L)$.

<u>Thm (Abouzaid)</u>

There is an A_∞ equivalence

$$CW(T_q^* Q) \simeq C(\Omega_q Q) .$$