In our application to Floer theory, the construction of the path of symplectic matrices from the linearization of the flow along a periodic orbit x, depended on a choice of trivialization of x^*TM, which in turn depended on a capping disk for x. Different homotopy classes of such disks differ by taking a connected sum with a map $v: S^2 \to M$, and one shows that

$$\mu_{CZ}(x, [u \# v]) = \mu_{CZ}(x, [u]) + 2\langle c_1(TM), [v]\rangle$$

where $c_1(TM) \in H^2(M)$ is the first Chern class of the bundle $(TM, \omega) \to M$, and the term $\langle c_1(TM), [v]\rangle$ can be interpreted as the obstruction to the existence of a symplectic trivialization of $v^*TM \to S^2$.

So if

$$\langle c_1(TM), [v]\rangle = 0 \quad \text{for all } v: S^2 \to M,$$

then the Conley-Zehnder index of any periodic orbit is well-defined.

<u>Def:</u> A symplectic manifold is called <u>symplectically aspherical</u> if

$$\omega|_{\pi_2(M)} = 0 \quad \text{and} \quad c_1|_{\pi_2(M)} = 0.$$

Examples include products of surfaces and cotangent bundles.

On symplectically aspherical manifold (M, ω):

- the action functional $\mathcal{A}_H$ associated to a Hamiltonian $H: \mathbb{R} \times M \to \mathbb{R}$ is well-defined on $\mathcal{L}_0 M$
- every nondegenerate critical point of $\mathcal{A}_H$ (i.e. 1-periodic orbit of the Hamiltonian flow) has a well-defined Conley-Zehnder index.

Symplectic asphericity is a rather restrictive assumption. The theory can still be developed without a massive additional machinery in a more general framework:

Def: A symplectic manifold (M,ω) is called

* monotone if there exists a constant $\tau > 0$ such that
$$\int_{S^2} v^* c_1(M) = \tau \int_{S^2} v^*\omega \quad \text{for all } v: S^2 \to M$$

* weakly monotone if $\tau \geq 0$ in the above condition

* semipositive if for all $A \in \pi_2(M)$
$$\omega(A) > 0 \text{ and } c_1(A) \geq 3-n \implies c_1(A) \geq 0$$

* strongly semipositive if for all $A \in \pi_2(M)$
$$\omega(A) > 0 \text{ and } c_1(A) \geq 2-n \implies c_1(A) \geq 0.$$

Examples:

* ~~For~~ $(\mathbb{CP}^n, \omega_{FS})$ with the normalization
$$\int_{\mathbb{CP}^1} \omega_{FS} = 1 \quad \text{is monotone with } \tau = n+1.$$

* A product $\mathbb{CP}^{n_1} \times \dots \times \mathbb{CP}^{n_r}$ is monotone if and only if all factors are normalized to have the same monotonicity constant τ

* all symplectically aspherical manifolds are (trivially) monotone

* every weakly monotone symplectic manifold is (strongly) semipositive

* Any Calabi-Yau manifold (meaning $c_1(TM) = 0$) is weakly monotone.

Remarks:

- the book of Audin + Damian treats the simplest interesting case of symplectically aspherical manifolds.
- Salamon's lecture notes treat the case of monotone symplectic manifolds
- the semi-positive case is discussed in the book "J-holomorphic curves and symplectic topology" by McDuff and Salamon

  The semipositivity assumption is enough to define Floer homology of a given H. To prove invariance, one needs strong semipositivity.

All of these conditions make sure that one can avoid the following problem:

Suppose (M, ω, J) are such that there exists a map
$$v: S^2 \to M$$
with $\bar{\partial}_J v = 0$ and $\underbrace{\langle c_1(TM), [v]\rangle}_{=: c_1([v])} < 0$

We know from last semester that

$$\dim \mathcal{M}(J; A) = 2n - 6 + 2\langle c_1(TM), A\rangle$$

If the v above is a simple curve ($[v]$ is not an integer multiple of another class), and $c_1([v]) \geq 3-n$, then for generic J the map v could be regular.

But then for this J, all moduli spaces

$$\mathcal{M}(J, kA) \qquad \text{with } k \geq 1$$

are nonempty, because they contain branched covers of maps in $\mathcal{M}(J, A)$ such as v.

On the other hand, for k sufficiently large, the expected dimension

$$\dim \mathcal{M}(J, k[v]) = 2n - 6 + 2k\, c_1([v]) < 0$$

Since $\mathcal{M}(J, [v])$ is regular, perturbing J will not make these $\mathcal{M}(J, k[v])$ transversely cut out.

We will now discuss the symplectically aspherical case first, and comment on changes needed for the other cases later.

Assumption (until further notice)

(M,ω) is closed and symplectically aspherical

Thm 1: Let $H: \mathbb{R}\times M \to \mathbb{R}$ be 1-periodic with all 1-periodic orbits nondegenerate. Then for a generic (possibly time-dependent) a.c. structure $J = \{J_t\}_{t\in S^1}$ on M the moduli space

$$\tilde{\mathcal{M}}(x^+, x^-; H, J) = \{ u: \mathbb{R}\times S^1 \to M : \bar\partial_{J,H} u = 0, \lim_{s\to\pm\infty} u(s,t) = x^\pm(t) \}$$

is a manifold of dimension

$$\mu_{CZ}(x^+) - \mu_{CZ}(x^-).$$

Remarks: 1) As in the Morse case, one views $\bar\partial_{J,H}$, given as

$$\bar\partial_{J,H} u = \partial_s u + J_t(\partial_t u - X_{H_t}(u))$$

as a section of a Banach space bundle $\mathcal{E} \to W^{1,2}(\mathbb{R}\times S^1, M)$ with fiber at u given by $L^2(u^*TM)$.

In a unitary trivialization of u^*TM at a solution u, we get a linearization

$$D_u : W^{1,2}(u^*TM) \to L^2(u^*TM)$$

of the form

$$D_u \xi = \partial_s \xi + J\partial_t \xi + S\xi,$$

which is a lower order perturbation of a Cauchy-Riemann operator.

Using this and the nondegeneracy of the asymptotics, one proves that D_u is Fredholm.
Then the index is computed by a spectral flow argument.

(2) If M is not symplectically aspherical, $\widetilde{\mathcal{M}}(x^+, x^-; H, J)$ can have components of different dimensions.
Still, the formula ~~remains~~ for the local dimension at a map $u \in \widetilde{\mathcal{M}}(x^+, x^-; H, J)$ remains correct if the capping disks used to compute the index are related by

$$\tilde{u}^+ = \tilde{u}^- \# u.$$

(3) Transversality can be achieved either by perturbing $\{H_t\}$ or by perturbing $\{J_t\}$. The latter corresponds to the perturbation of the metric in Morse theory, but note that the metrics obtained from varying J form only a thin subset of all metrics.